Algebra I/Introduction to Algebra V

Due: Please upload solutions to TACT by Tuesday, June 27, 2023.

Problem 1. Let k be an algebraically closed field, and let G be a finite group, whose order is not divisible by the characteristic of k. Let $H \subset G$ be a subgroup, and let (V, σ) be a finite dimensional k-linear representation of H. Given $a \in G$, we define $H^a \subset G$ be the subgroup $H^a = aHa^{-1}$, and we define (V, σ^a) to be the representation of H^a given by $\sigma^a(g) = \sigma(a^{-1}ga)$ for $g \in H^a$.

Suppose that σ is irreducible. Show that the induced representation $\operatorname{Ind}_H^G(\sigma)$ is irreducible if and only if for all $a \in G$ such that $a \notin H$,

$$\dim_k \operatorname{Hom}(\operatorname{Res}^H_{H \cap H^a}(\sigma), \operatorname{Res}^{H^a}_{H \cap H^a}(\sigma^a)) = 0.$$

[Hint: By Schur's lemma, a finite dimensional k-linear representation π of G is irreducible if and only if $\dim_k \operatorname{Hom}(\pi, \pi) = 1$.]

Problem 2. Let k be a field, let G be a finite group, and let $H \subset G$ be a subgroup. Let σ be a finite dimensional k-linear representation of H, let $\pi = \operatorname{Ind}_H^G(\sigma)$ be the induced k-linear representation of G, and let $\chi_\sigma \colon H \to k$ and $\chi_\pi \colon G \to k$ be their characters. Given $g \in G$, we denote by

$$(G/H)^g = \{aH \in G/H \mid gaH = aH\} \subset G/H$$

the subset fixed by left multiplication by g. Show that

$$\chi_\pi(g) = \sum_{aH \in (G/H)^g} \chi_\sigma(a^{-1}ga).$$

Note that the summand $\chi_\sigma(a^{-1}ga)$ corresponding to $aH \in (G/H)^g$ only depends on aH and not on the choice of $a \in aH$, since $\chi_\sigma \colon H \to k$ is a class function.

[Hint: One possibility is to use that $\operatorname{Ind}_H^G = f_* \simeq p_* \circ i_*$ and that $i_* \simeq r^*$, where $r \colon [G \backslash (G/H)] \to BH$ is a quasi-inverse of $i \colon BH \to [G \backslash (G/H)]$.]