Algebra I/Introduction to Algebra V

Due: Please upload solutions to TACT by Tuesday, July 4, 2023.

Problem 1. The purpose of this problem is to show that the Specht representations are self-dual. We recall that if (V, π) is a k-linear representation of a group G, then its dual representation (V^*, π^*) is defined by $V^* = \mathrm{Hom}_k(V, k)$ and

$$\pi^*(g)(\varphi)(\boldsymbol{x}) = \varphi(\pi(g^{-1})(\boldsymbol{x})),$$

where $g \in G$, $\varphi \in V^*$, and $\boldsymbol{x} \in V$.

(1) Let k be a field, let G be a group, and let $H \subset G$ be a subgroup. Show that for every a k-linear representation π of G,

$$\mathrm{Res}_H^G(\pi^*) \simeq \mathrm{Res}_H^G(\pi)^*.$$

(2) Show that if $[G : H] < \infty$, then for every k-linear representation τ of H,

$$\mathrm{Ind}_H^G(\tau^*) \simeq \mathrm{Ind}_H^G(\tau)^*.$$

[Hint: Show that the functors $\mathrm{Hom}(-, \mathrm{Ind}_H^G(\tau^*))$ and $\mathrm{Hom}(-, \mathrm{Ind}_H^G(\tau)^*)$ from $\mathrm{Rep}_k(G)$ to Set are naturally isomorphic and conclude by Yoneda lemma.]

(3) Let $k = \mathbb{C}$ be the field of complex numbers, and let $G = \mathrm{Aut}(X)$ be the group of permuations of a finite set X. Show that for the Specht representation π_S of G associated with a Young diagram S of size $n = \mathrm{card}(X)$,

$$\pi_S^* \simeq \pi_S.$$

This is the end of the problem.

Remark. We proved in the problem set from Week 7 that a finite dimensional complex representation π of a finite group G is self-dual in the sense that $\pi \simeq \pi^*$ if and only if its character $\chi_\pi \colon G \to \mathbb{C}$ takes values in $\mathbb{R} \subset \mathbb{C}$. So we conclude from (3) that the characters of the Specht representations π_S of $G = \mathrm{Aut}(X)$,

$$G = \mathrm{Aut}(X) \xrightarrow{\chi_{\pi_S}} \mathbb{C},$$

take values in $\mathbb{R} \subset \mathbb{C}$. It follows from the Frobenius character formula that more is true: The characters of the Specht representations take values in $\mathbb{Z} \subset \mathbb{C}$.