

## Algebra I/Introduction to Algebra V

*Due:* Please upload solutions to TACT by Tuesday, July 11, 2023.

**Problem 1.** Let  $k = \mathbb{R}$  (resp.  $\mathbb{C}$ , resp.  $\mathbb{H}$ ), and let  $\sigma: k \rightarrow k^{\text{op}}$  be the identity map (resp. complex conjugation, resp. quaternionic conjugation). We first consider  $M_n(k)$  a right  $k$ -vector space with sum given by matrix sum and with right scalar multiplication given by  $(x_{ij}) \cdot a = (x_{ij} \cdot a)$ . The zero vector in this vector space is given by the zero matrix  $O$ .

(i) Show that the map  $\langle -, - \rangle: M_n(k) \times M_n(k) \rightarrow k$  defined by

$$\langle X, Y \rangle = \text{tr}(X^*Y)$$

is a hermitian inner product on  $M_n(k)$ .

We next consider  $M_n(k)$  a real vector space by restriction of scalars along  $\mathbb{R} \rightarrow k$ , and we consider the norm  $\|X\| = \sqrt{\langle X, X \rangle}$  on this real vector space induced by the hermitian inner product in (i). We give each of the classical groups  $G \subset M_n(k)$ <sup>1</sup> the subspace topology induced by the metric topology on  $M_n(k)$ .

- (ii) Show that each of the classical groups  $G \subset M_n(k)$  is contained in the sphere of radius  $\sqrt{n}$  centered at  $O$ .
- (iii) Show that each of the classical groups  $G \subset M_n(k)$  is a closed subset of  $M_n(k)$ .
- (iv) Conclude that each of the classical groups  $G$  is compact.

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<sup>1</sup>For  $G = O(n)$  or  $SO(n)$ ,  $k = \mathbb{R}$ ; for  $G = U(n)$  or  $SU(n)$ ,  $k = \mathbb{C}$ , and for  $G = \text{Sp}(n)$ ,  $k = \mathbb{H}$ .