Algebra I/Introduction to Algebra V

Due: Please upload solutions to TACT by Tuesday, July 18, 2023.

Problem 1. Let $G = SU(2)$, and let $T \subset G$ be the subgroup of diagonal matrices.

(1) Show that for every $x \in G$, there exists $t \in T$ and $g \in G$ such that $x = gtg^{-1}$.
[Hint: Use the spectral theorem.]

Let ρ_1 and ρ_2 be two continuous finite dimensional complex representations of G, and let $\mathrm{Res}_T^G(\rho_1)$ and $\mathrm{Res}_T^G(\rho_2)$ be their restrictions to T.

(2) Show that $\rho_1 \simeq \rho_2$ if and only if $\mathrm{Res}_T^G(\rho_1) \simeq \mathrm{Res}_T^G(\rho_2)$.

For every non-negative integer n, let $\pi_n = \mathrm{Sym}_{\mathbb{C}}^n(\pi)$, where $\pi \colon G \to \mathrm{GL}(V)$ is the standard representation on $V = \mathbb{C}^2$.

(3) Determine the dual representation π_n^* for all $n \geq 0$.
(4) Determine the representation $\pi_m \otimes \pi_n$ for all $m, n \geq 0$.
[Hint: Look at some small values of m and n to guess the answer.]