Algebra I/Introduction to Algebra V

Due: Please upload solutions to TACT by Tuesday, July 25, 2023.

Problem 1. Let $G = ((G, \mathcal{O}_G), \mu, \iota, e)$ be a Lie group, that is, a group object in the category of smooth manifolds and morphisms of smooth manifolds, and let

$$\mathfrak{g} = T(G, \mathcal{O}_G)_e$$

be the real vector space given by the tangent space at the identity element.[1] This problem will show that the fact that the inversion ι is a morphism of smooth manifolds is a consequence of the facts that the multiplication μ is a morphism of smooth manifolds and that (G, μ, ι) is a group.

(a) Let $p_1, p_2 \colon G \times G \to G$ be the two projections, and let $i_1, i_2 \colon G \to G \times G$ be the maps defined by $i_1(g) = (g, e)$ and $i_2(g) = (e, g)$. Show that the diagram

$$T(G, \mathcal{O}_G)_e \underset{di_{1,e}}{\overset{dp_{1,e}}{\rightleftarrows}} T(G \times G, \mathcal{O}_{G\times G})_{(e,e,)} \underset{di_{2,e}}{\overset{dp_{2,e}}{\rightleftarrows}} T(G, \mathcal{O}_G)_e$$

is a biproduct in the category of real vector spaces.[2]

(b) Show that the composite map

$$\mathfrak{g} \oplus \mathfrak{g} \xrightarrow{di_{1,e} + di_{2,e}} T(G \times G, \mathcal{O}_{G\times G})_{(e,e)} \xrightarrow{d\mu_{(e,e)}} \mathfrak{g}$$

maps $(\boldsymbol{v}, \boldsymbol{w})$ to the vector space sum $\boldsymbol{v} + \boldsymbol{w}$.

Hence, to first order at $e \in G$, all group structures on $(G, \mathscr{O}_G)$ look the same! Let

$$(G \times G, \mathcal{O}_{G\times G}) \xrightarrow{\sigma} (G \times G, \mathcal{O}_{G\times G})$$

be the "shearing map" defined by $\sigma(g, h) = (g, \mu(g, h))$. It is a morphism of smooth manifolds, because μ is a morphism of smooth manifolds, and it is a bijection, because (G, μ, ι) is a group.

(c) Show that the composite map

$$\begin{array}{ccc} \mathfrak{g} \oplus \mathfrak{g} \xrightarrow{di_{1,e} + di_{2,e}} & T(G \times G, \mathcal{O}_{G\times G})_{(e,e)} & \\ & \big\downarrow d\sigma_{(e,e)} & \\ & T(G \times G, \mathcal{O}_{G\times G})_{(e,e)} & \xrightarrow{(dp_{1,e}, dp_{2,e})} \mathfrak{g} \oplus \mathfrak{g} \end{array}$$

maps $(\boldsymbol{v}, \boldsymbol{w})$ to $(\boldsymbol{v}, \boldsymbol{v} + \boldsymbol{w})$, and conclude that it is an isomorphism.

(d) Show that σ is both an immersion and submersion, and conclude that it is a diffeomorphism. [Hint: Use left translation and right translation to reduce to what was proved in (c).]

This shows that $\iota = p_2 \circ \sigma^{-1} \circ i_1$ is automatically a morphism of smooth manifolds.

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[1] The group structure on G gives rise to a structure of real Lie on $\mathfrak{g}$, but we will not need this fact here.

[2] If you do not know what this means, then either look up the definition in MacLane's book, Categories for the Working Mathematician, or in Wikipedia.