Algebra I/Introduction to Algebra V: Representation Theory

Due: Please upload solutions to TACT by Tuesday, May 9, 2023.

Problem 1. If $(V, +, \cdot\,)$ is a (right) complex vector space, then we consider the new (right) complex vector space $(V, +, \star)$, where for $\boldsymbol{x} \in V$ and $z \in \mathbb{C}$,

$$\boldsymbol{x} \star z = \boldsymbol{x} \cdot \overline{z}.$$

Here $\overline{z} = a - ib$ is the complex conjugate of $z = a + ib$. We call $(V, +, \star)$ the conjugate vector space of $(V, +, \cdot\,)$. As usual, we abuse notation and write V instead of $(V, +, \cdot\,)$ and $\overline{V}$ instead of $(V, +, \star)$.

(a) Prove that the groups $\mathrm{GL}(\overline{V})$ and $\mathrm{GL}(V)$ are equal.

Let (V, π) be a complex representation of a group G. The conjugate representation is the complex representation $(\overline{V}, \pi)$. Here we use that $\mathrm{GL}(\overline{V}) = \mathrm{GL}(V)$. It is common to abbreviate (V, π) by π and $(\overline{V}, \pi)$ by $\overline{\pi}$. This is confusing, because the maps $\pi \colon G \to \mathrm{GL}(V)$ and $\overline{\pi} \colon G \to \mathrm{GL}(\overline{V})$ are equal!

Suppose that π is a unitary representation and let $\langle -, - \rangle \colon V \times V \to \mathbb{C}$ be a hermitian inner product[1] such that $\langle \pi(g)(\boldsymbol{x}), \pi(g)(\boldsymbol{y}) \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle$, for all $g \in G$ and $\boldsymbol{x}, \boldsymbol{y} \in V$.

(b) Show that the map $b \colon \overline{V} \to V^*$ defined by $b(\boldsymbol{x})(\boldsymbol{y}) = \langle \boldsymbol{x}, \boldsymbol{y} \rangle$ is intertwining between $\overline{\pi}$ and π^*. Here (V^*, π^*) is the dual representation of (V, π).

We remark that, if V is finite dimensional, then $b \colon \overline{V} \to V^*$ is an isomorphism of complex vector spaces. So it follows from (b) that for every finite dimensional unitary representation π, we have $\overline{\pi} \simeq \pi^*$.

[1] In particular, for $\boldsymbol{x}, \boldsymbol{y} \in V$ and $z, w \in \mathbb{C}$, we have $\langle \boldsymbol{x} \cdot z, \boldsymbol{y} \cdot w \rangle = \overline{z} \langle \boldsymbol{x}, \boldsymbol{y} \rangle w$.