Algebra I/Introduction to Algebra V

Due: Please upload solutions to TACT by Tuesday, May 16, 2023.

Problem 1. Let $\pi\colon G \to \mathrm{GL}(V)$ be a real representation of finite dimension d. Show that if π is irreducible and if d is odd, then the complexication

$$\pi_{\mathbb{C}}\colon G \to \mathrm{GL}(V_{\mathbb{C}})$$

is an irreducible complex representation.

Problem 2. Let $f\colon k \to k'$ be an extension of fields, let $f^*\colon \mathrm{Vect}_k \to \mathrm{Vect}_{k'}$ be extension of scalars along f, and let $f_*\colon \mathrm{Vect}_{k'} \to \mathrm{Vect}_k$ be restriction of scalars along f. Given k'-vector space V_1' and V_2', we define

$$f_*(V_1') \otimes_k f_*(V_2') \xrightarrow{\mu} f_*(V_1' \otimes_{k'} V_2')$$

be the k-linear map given by $\mu(\boldsymbol{x}_1' \otimes_k \boldsymbol{x}_2') = \boldsymbol{x}_1' \otimes_{k'} \boldsymbol{x}_2'$.

(a) Give an example to show that, in general, the map μ is *not* an isomorphism.

Given k-vector spaces V_1 and V_2, the composite k-linear map

$$V_1 \otimes_k V_2 \xrightarrow{\eta\otimes\eta} f_*f^*(V_1) \otimes_k f_*f^*(V_2) \xrightarrow{\mu} f_*(f^*(V_1) \otimes_{k'} f^*(V_2))$$

determines and is determined by a unique k'-linear map

$$f^*(V_1 \otimes_k V_2) \xrightarrow{\tilde{\mu}} f^*(V_1) \otimes_{k'} f^*(V_2).$$

(b) Calculate that for $\boldsymbol{x}_1 \in V_1$, $\boldsymbol{x}_2 \in V_2$, and $b \in k'$,

$$\tilde{\mu}((\boldsymbol{x}_1 \otimes_k \boldsymbol{x}_2) \otimes_k b) = (\boldsymbol{x}_1 \otimes_k b) \otimes_{k'} (\boldsymbol{x}_2 \otimes_k 1) = (\boldsymbol{x}_1 \otimes_k 1) \otimes_{k'} (\boldsymbol{x}_2 \otimes_k b).$$

(c) Show that the map $\tilde{\mu}$ is an isomorphism. [Hint: Write down a k'-linear map

$$f^*(V_1) \otimes_{k'} f^*(V_2) \longrightarrow f^*(V_1 \otimes_k V_2)$$

and show that it is inverse to $\tilde{\mu}$.]