

## Algebra I/Introduction to Algebra V

Due: Please upload solutions to TACT by Tuesday, May 23, 2023.

**Problem 1.** Let  $G$  be an abelian group, and let  $(V, \pi)$  be a finite dimensional irreducible real representation of  $G$ . Show that either  $\dim_{\mathbb{R}}(V) = 1$  or  $\dim_{\mathbb{R}}(V) = 2$ .

**Problem 2.** Let  $k = \bar{k}$  be an algebraically closed field,<sup>1</sup> let  $\pi: G \rightarrow \mathrm{GL}(V)$  be a finite dimensional irreducible  $k$ -linear representation, and let

$$A = \mathrm{span}(\pi(g) \mid g \in G) \subset \mathrm{End}_k(V).$$

Show that  $A = \mathrm{End}_k(V)$ .

[Hint: Consider the representation  $\rho: G \times G \rightarrow \mathrm{GL}(\mathrm{End}_k(V))$  defined by

$$\rho(g_1, g_2)(f)(\mathbf{x}) = \pi(g_1)(f(\pi(g_2^{-1})(\mathbf{x}))).$$

Show that  $A \subset \mathrm{End}_k(V)$  is  $\rho$ -invariant and that  $\rho \simeq \pi \boxtimes \pi^*$ . Then use a theorem.]

You are welcome to hand in the next problem, but you do not have to do so.

**Problem 3 (Optional).** We consider the additive group  $G = (\mathbb{R}, +)$  of real numbers and the representation  $\pi: G \rightarrow \mathrm{GL}(V)$  on  $V = \mathbb{R}^2$  defined by

$$\pi(t) \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Since  $\pi$  is irreducible, the ring  $D = \mathrm{End}(\pi) \subset \mathrm{End}_{\mathbb{R}}(V)$  is a real division algebra, so either  $D \simeq \mathbb{R}$ ,  $\mathbb{C}$ , or  $\mathbb{H}$ . Determine which one it is.

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<sup>1</sup> You are welcome to assume that  $k = \mathbb{C}$ .