

Algebra I/Introduction to Algebra V

Due: Please upload solutions to TACT by Tuesday, May 23, 2023.

Problem 1. Let G be an abelian group, and let (V, π) be a finite dimensional irreducible real representation of G . Show that either $\dim_{\mathbb{R}}(V) = 1$ or $\dim_{\mathbb{R}}(V) = 2$.

Problem 2. Let $k = \bar{k}$ be an algebraically closed field,¹ let $\pi: G \rightarrow \mathrm{GL}(V)$ be a finite dimensional irreducible k -linear representation, and let

$$A = \mathrm{span}(\pi(g) \mid g \in G) \subset \mathrm{End}_k(V).$$

Show that $A = \mathrm{End}_k(V)$.

[Hint: Consider the representation $\rho: G \times G \rightarrow \mathrm{GL}(\mathrm{End}_k(V))$ defined by

$$\rho(g_1, g_2)(f)(\mathbf{x}) = \pi(g_1)(f(\pi(g_2^{-1})(\mathbf{x}))).$$

Show that $A \subset \mathrm{End}_k(V)$ is ρ -invariant and that $\rho \simeq \pi \boxtimes \pi^*$. Then use a theorem.]

You are welcome to hand in the next problem, but you do not have to do so.

Problem 3 (Optional). We consider the additive group $G = (\mathbb{R}, +)$ of real numbers and the representation $\pi: G \rightarrow \mathrm{GL}(V)$ on $V = \mathbb{R}^2$ defined by

$$\pi(t)\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Since π is irreducible, the ring $D = \mathrm{End}(\pi) \subset \mathrm{End}_{\mathbb{R}}(V)$ is a real division algebra, so either $D \simeq \mathbb{R}$, \mathbb{C} , or \mathbb{H} . Determine which one it is.

¹You are welcome to assume that $k = \mathbb{C}$.