Algebra I/Introduction to Algebra V

Due: Please upload solutions to TACT by Tuesday, May 30, 2023.

Problem 1. Let (V, π) be a finite dimensional complex representation of a finite group G, let (V^*, π^*) be the dual representation, and let $\chi_\pi, \chi_{\pi^*} : G \to \mathbb{C}$ be their characters.

(a) Show that for all $g \in G$, $\chi_{\pi^*}(g) = \overline{\chi_\pi(g)}$.
(b) Show that $\pi \simeq \pi^*$ if and only if $\chi_\pi : G \to \mathbb{C}$ takes all its values in $\mathbb{R} \subset \mathbb{C}$.
(c) Recall the two-sided regular representation $(\mathbb{C}[G], \mathrm{Reg})$ of $G \times G$ defined by

$$\mathrm{Reg}(g_1, g_2)(f)(x) = f(g_2^{-1} x g_1).$$

Show that $\mathrm{Reg} \simeq \mathrm{Reg}^*$.