Algebra I/Introduction to Algebra V

Due: Please upload solutions to TACT by Tuesday, June 6, 2023.

Problem 1. Let G be a finite group, and let $\widehat{G}$ be the set of isomorphism classes of irreducible finite dimensional complex representations of G. For every $\sigma \in \widehat{G}$, we choose a representative (V_σ, π_σ) of the class σ. We define the Fourier transform of $f \in \mathbb{C}[G]$ to be the "function"[1] $\widehat{f}$ that to $\sigma \in \widehat{G}$ assigns the endomorphism

$$\widehat{f}(\sigma) = \sum_{g \in G} f(g)\pi_\sigma(g) \in \mathrm{End}_{\mathbb{C}}(V_\sigma).$$

Prove the following statements:

(a) For all $f_1, f_2 \in \mathbb{C}[G]$ and $\sigma \in \widehat{G}$,

$$\widehat{f_1 * f_2}(\sigma) = \widehat{f_1}(\sigma) \circ \widehat{f_2}(\sigma),$$

where $f_1 * f_2 \in \mathbb{C}[G]$ is the convolution product of f_1 and f_2 defined by

$$(f_1 * f_2)(g) = \sum_{hk=g} f_1(h) f_2(k).$$

Here the sum ranges over pairs $(h, k) \in G \times G$ such that $hk = g$.

(b) For all $f \in \mathbb{C}[G]$, the Frobenius inversion formula

$$f(g) = \frac{1}{|G|} \sum_{\sigma \in \widehat{G}} n_\sigma \, \mathrm{tr}(\pi_\sigma(g)^{-1} \circ \widehat{f}(\sigma))$$

holds. Here $n_\sigma = \dim_{\mathbb{C}}(V_\sigma)$ and we recall that $|G| = \sum_{\sigma \in \widehat{G}} n_\sigma^2$.

[1] The Fourier transform $\widehat{f}$ is really not a function, but rather a section of a bundle, where the fiber over σ is $\mathrm{End}_{\mathbb{C}}(V_\sigma)$. Also, to avoid making the choice of (V_σ, π_σ), one should work with the ∞-category of representations instead.