Topology | Geometry

Study topological spaces | Study ringed spaces

$|X|$ | $X = (|X|, \mathcal{O}_X)$

sheaf of "functions" on $|X|$

Def A ringed space $X = (|X|, \mathcal{O}_X)$ is a smooth manifold of dim. d if, locally on $|X|$, there exists an isom.

$$(|X|, \mathcal{O}_X) \simeq (U, \mathcal{O}_U^{sm})$$

with $U \subset \mathbb{R}^d$ open and $\mathcal{O}_U^{sm}$ the sheaf of $C^\infty$ real functions on $U$.

Thm (Milnor) Up to isomorphism, there exists 28 different smooth mfds. $X = (|X|, \mathcal{O}_X)$ s.t. $|X|$ is homeomorphic to $S^7$.

"The 7-sphere admits 14 distinct smooth structures"

Def A ringed space $X = (|X|, \mathcal{O}_X)$ is a complex mfd. of dim. $d$ if, locally on $|X|$, there exists an isomorphism of ringed spaces

$$(|X|, \mathcal{O}_X) \simeq (U, \mathcal{O}_U^{hol})$$

with $U \subset \mathbb{C}^d$ open and $\mathcal{O}_U^{hol}$ the sheaf of holomorphic complex functions on $U$.

Thm (Riemann) There is a canonical bijection

$$\left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{complex manifolds} \\ X = (|X|, \mathcal{O}_X) \\ \text{of dim. 1 for which} \\ \text{there exists a homeo.} \\ |X| \simeq S^1 \times S^1 \end{array} \right\}$$

$$\xrightarrow[\sim]{j} \left(\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})\right) / GL_2(\mathbb{Z}).$$

So geometry of complex mflds. very different from geometry of smooth (real) mflds.

Def A Riemann surface is a cx. mfld. of dim. $d = 1$.

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Fix notation:

$$\mathbb{C} = \{ a + ib \mid a, b \in \mathbb{R} \}$$

Write $z = a + ib$, $\bar{z} = a - ib$, and

$$|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}.$$

The open disc of radius $r > 0$ centered at $z \in \mathbb{C}$ is

$$D(z, r) = \{ w \in \mathbb{C} \mid |w - z| < r \}$$

and the closed disc of radius $r > 0$ centered at $z \in \mathbb{C}$ is

$$\bar{D}(z, r) = \{ w \in \mathbb{C} \mid |w - z| \leq r \}.$$

Its boundary is

$$\partial \bar{D}(z,r) = \bar{D}(z,r) \setminus D(z,r)$$

$$= \{ w \in \mathbb{C} \mid |w - z| = r \}.$$

A subset $U \subset \mathbb{C}$ is open if for all $z \in U$, there exists $r > 0$ s.t.

$$D(z,r) \subset U.$$

Equivalently, a subset $U \subset \mathbb{C}$ is open if it is a union of open discs. Note that $U = \emptyset$ is open.

Def Let $U \subset \mathbb{C}$ be an open subset. A map $f : U \to \mathbb{C}$ is holomorphic at $z_0 \in U$ if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. /

Compare to differentiability as real function: Write

$$z = x + iy$$

The $\Omega^0(U) = C^\infty(U, \mathbb{R})$ - module

$\Omega^1(U)$ of smooth 1-forms on $U$ is free with basis $(dx, dy)$. Hence, also the complexified module $\Omega^1(U) \otimes_{\mathbb{R}} \mathbb{C}$ over

$$\Omega^0(U) \otimes_{\mathbb{R}} \mathbb{C} \simeq C^\infty(U, \mathbb{C})$$

is free with basis $(dx, dy)$. However we may instead use the basis $(dz, d\bar{z})$, where

$$dz = dx + i\,dy$$

$$d\bar{z} = dx - i\,dy.$$

Given $f \in C^\infty(U, \mathbb{C})$, we have

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$
$$= \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}$$

where

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y}\right)$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y}\right).$$

Now, $f \in C^\infty(U, \mathbb{C})$ is holomorphic at $z_0 \in U$ if and only if

$$T_{U,z_0} \xrightarrow{df_{z_0}} \mathbb{C}$$

is $\mathbb{C}$-linear wrt. the complex structure $\partial/\partial y = i\,\partial/\partial x$ on $T_{U,z_0}$. So $f$ is holomorphic at $z_0 \in U$ if and only if

$$\frac{\partial f}{\partial y}(z_0) = i\,\frac{\partial f}{\partial x}(z_0),$$

or equivalently, if and only if

$$\frac{\partial f}{\partial \bar{z}_0}(z_0) = 0.$$

Also, prove as usual that if $f, g : U \to \mathbb{C}$ are holomorphic at $z_0 \in U$, then so are $f+g$, $f \cdot g$, and, if $g(z_0) \neq 0$, $f/g$.

Say that $f : U \to \mathbb{C}$ is holomorphic if $f$ is holomorphic at $z_0 \in U$ for all $z_0 \in U$.

Thm (Cauchy integral formula)
Let $f: U \to \mathbb{C}$ be holomorphic with $U \subset \mathbb{C}$ open. Suppose

$$z_0 \in \overline{D} \subset U.$$

In this situation
$$f(z_0) = \frac{1}{2\pi i} \int_{\partial \overline{D}} \frac{f(z)}{z - z_0}\, dz,$$
where $\partial \overline{D}$ is given the counter-clockwise orientation.

Pf Later.

Cor If $f: U \to \mathbb{C}$ is holomorphic, then so is $f': U \to \mathbb{C}$.

Cor If $f: U \to \mathbb{C}$ is holomorphic and $D = D(z_0, r) \subset U$, then the power series expansion for $f$ centered at $z_0$ converges for all $z \in D$.

Cor (Removable singularities Thm.)
Let $U \subset \mathbb{C}$ be open, let $z_0 \in U$, and suppose that $f: U \setminus \{z_0\} \to \mathbb{C}$

is holomorphic and bounded on some disc centered at $z_0$. In this situation, $f$ extends uniquely to a holomorphic fct.

$$U \xrightarrow{\hat{f}} \mathbb{C}.$$

Cor Let $(f_n : U \to \mathbb{C})_{n \geq 0}$ be a sequence of holomorphic fcts. on $U \subset \mathbb{C}$ open with pointwise limit $f : U \to \mathbb{C}$. Suppose that for every closed disc $\bar{D} \subset U$,

$$\lim_{n \to \infty} \sup_{z \in \bar{D}} |f(z) - f_n(z)| = 0.$$

In this situation, $f : U \to \mathbb{C}$ is holomorphic.

Def Let $U, V \subset \mathbb{C}$ be open. A map $f : U \to V$ is a biholomorphism if $f$ is a bijection and if

$$U \xrightarrow{f} V \hookrightarrow \mathbb{C}$$

$$V \xrightarrow{f^{-1}} U \hookrightarrow \mathbb{C}$$

are holomorphic.

Thm (Riemann mapping theorem)
Let $U \subset \mathbb{C}$ be an open subset and assume that:

1) $U \neq \emptyset$ and $U \neq \mathbb{C}$
2) $U$ is simply connected.

In this situation, there exists a biholomorphism

$$U \xrightarrow{f} D(0,1).$$

Proof of Cauchy's integral formula:
We wish to prove that

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial \bar{D}(z_0,r)} \frac{f(z)}{z-z_0}\, dz$$

for some $\bar{D}(z_0,r) \subset U$. Since $f$ is holomorphic on $U$, $f(z)/(z-z_0)$ is holomorphic on $U \setminus \{z_0\}$. Also, if $g$ is any holomorphic fct. then $g\,dz$ is a closed form:

$$d(g\,dz) = dg \cdot dz$$
$$= \left(\frac{\partial g}{\partial z} dz + \frac{\partial g}{\partial \bar{z}} d\bar{z}\right) \cdot dz$$

$$= \frac{\partial g}{\partial \bar{z}} \, d\bar{z} \, dz = 0.$$

Hence, by Stokes' formula, we conclude that for all $0 < s < r$,

$$\frac{1}{2\pi i} \int_{\partial \bar{D}(z_0, r)} \frac{f(z)}{z - z_0} \, dz$$

$$= \frac{1}{2\pi i} \int_{\partial \bar{D}(z_0, s)} \frac{f(z)}{z - z_0} \, dz.$$

But

$$\frac{1}{2\pi i} \int_{\partial \bar{D}(z_0, s)} \frac{f(z)}{z - z_0} \, dz$$

$$= \int_0^1 f(z_0 + s e^{2\pi i t}) \, dt,$$

and since $f$ is cts. at $z_0$, the functions $f_s : [0,1] \to \mathbb{C}$,

$$f_s(t) = f(z_0 + s e^{2\pi i t}),$$

converge uniformly to the constant fct. $f(z_0) : [0,1] \to \mathbb{C}$, so

$$\lim_{s \to 0} \frac{1}{2\pi i} \int_{\partial \bar{D}(z_0, s)} \frac{f(z)}{z - z_0} \, dz = f(z_0).$$

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