Recall that if $f : U \to \mathbb{C}$ is holomorphic and if $D(z,r) \subset U$, then the power series expansion of f at z converges on all of $D(z,r)$.

Thm Let $D \subset \mathbb{C}$ be an open disc, and let $f, g : D \to \mathbb{C}$ be holomorphic functions. Suppose that for some smaller open disc $D' \subset D$,

$$f|_{D'} = g|_{D'}.$$

In this situation, $f = g$.

Pf If D' and D are concentric, then this is clear, but we are not assuming that they are. We let r be the radius of D' and choose points $z_0, \dots, z_n$ on the line segment joining the center z_0 of D' and the center z_n of D and such that for all $1 \leq i \leq n$,

$$|z_i - z_{i-1}| < r.$$

So for all $0 \leq i \leq n$, $D(z_i, r) \subset D$, and for all $1 \leq i \leq n$, $D(z_{i-1}, r)$

contains some small disc centered at z_i. We prove by induction on $0 \le i \le n$ that

$$f|_{D(z_i, r)} = g|_{D(z_i, r)}.$$

The case $i=0$ is our assumption that $f|_{D'} = g|_{D'}$. So we assume the statement for $i = m-1$ and prove it for $i = m$. Since

$$f|_{D(z_{m-1}, r)} = g|_{D(z_{m-1}, r)}$$

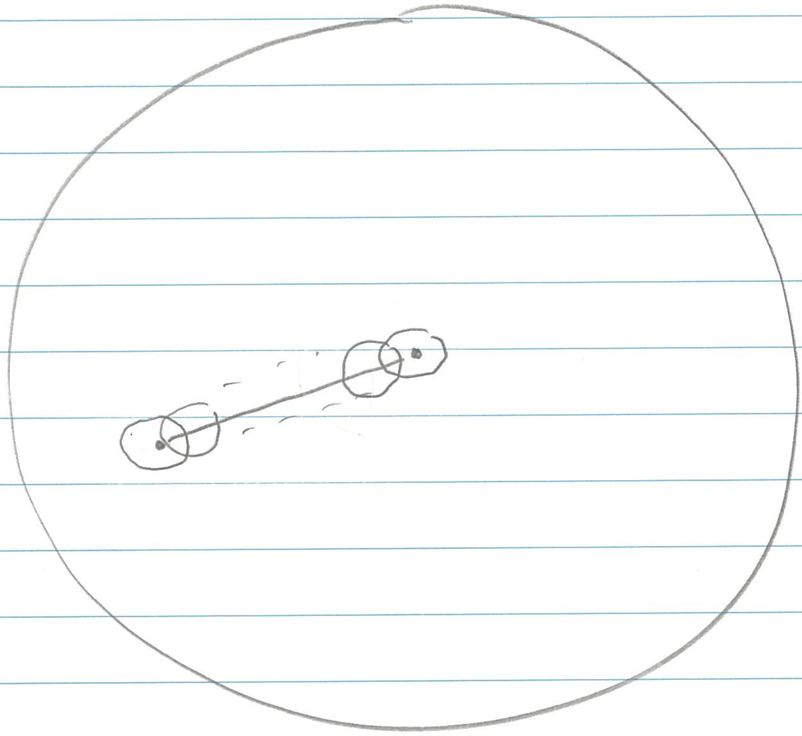
and since for some $s > 0$,

$$D(z_m, s) \subset D(z_{m-1}, r),$$

we conclude that f and g have the same power series expansion at z_m. Since f and g both are defined and holomorphic on $D(z_m, r)$, we conclude that

$$f|_{D(z_m, r)} = g|_{D(z_m, r)}.$$

This proves the induction step. Finally, $D(z_m, r) \subset D$ are concentrue and f and g are defined and holomorphic on the larger disc, so $f = g$ as desired.



//

Def An open subset $U \subset \mathbb{C}$ is path-connected if for all $z, w \in U$, there exists $\gamma : [0,1] \to U$ continuous s.t. $\gamma(0) = z$ and $\gamma(1) = w$. /

Thm Let $U \subset \mathbb{C}$ be a path-connected open subset, and let $f, g : U \to \mathbb{C}$ be holomorphic functions. If there exists an open disc $D \subset U$ s.t.

$$f|_D = g|_D,$$

then $f = g$.

Pf Write $D = D(z, r) \subset U$. We fix $w \in U$ and proceed to prove that $f(w) = g(w)$. Since U is path-connected, we can choose a path $\gamma : [0,1] \longrightarrow U$ s.t. $\gamma(0) = z$ and $\gamma(1) = w$. Moreover, since $U \subset \mathbb{C}$ is open, we can choose for every $t \in [0,1]$, and open disc $D_t = D(\gamma(t), r_t) \subset U$ and and open subset $I_t \subset [0,1]$ that is an interval s.t. $\gamma(I_t) \subset D_t$. The family of intervals

$$(I_t)_{t \in [0,1]}$$

is an open cover of $[0,1]$, and since $[0,1]$ is compact, it admits a finite subcover. So we can choose

$$0 = t_0 < t_1 < \cdots < t_N = 1$$

s.t. $(I_{t_i})_{0 \le i \le N}$ is a cover. We prove by induction on $0 \le i \le N$ that $f|_{D_{t_i}} = g|_{D_{t_i}}$. The case

$i=0$ is our assumption, so we assume the statement for $i=m-1$ and prove it for $i=m$. But

$$D_{t_{m-1}} \cap D_{t_m} \neq \emptyset,$$

and since both discs are open, there exists some open disc

$$D' \subset D_{t_{m-1}} \cap D_{t_m}.$$

By the inductive hypothesis,

$$f|_{D_{t_{m-1}}} = g|_{D_{t_{m-1}}},$$

so

$$f|_{D'} = g|_{D'}.$$

Hence, by the previous theorem,

$$f|_{D_{t_m}} = g|_{D_{t_m}}.$$

This proves the induction step, and hence, the theorem. //

Refer to Thm. as "identity principle" for holom. fcts.

Two obstructions to extending holomorphic fcts. using strategy of proof of identity principle one topological and one analytic. Here, illustrate topological obstruction using algebraic fcts.

Let $\mathbb{C}(z)$ be the field of rational fcts. in one variable over $\mathbb{C}$. It is the quotient field of the ring $\mathbb{C}[z]$ of polynomials in one variable over $\mathbb{C}$. Given

$$f = p/q \in \mathbb{C}(z),$$

we let

$$U_f = \{ z \in \mathbb{C} \mid q(z) \neq 0 \} \subset \mathbb{C}.$$

It is an open subset, the complement of the finite set of roots of q.

Let $\mathbb{C}(z)[w]$ be the ring of pol. in one variable over $\mathbb{C}(z)$. Given

$$P = f_N(z) w^N + \cdots + f_0(z)$$

nonzero, we set

$$U_P = \bigcap_{0 \leq i \leq N} U_{f_i} \subset \mathbb{C}.$$

It is an open subset, the complement of the finite subset $S \subset \mathbb{C}$, where some of the denominators of $f_0, \ldots, f_N \in \mathbb{C}(z)$ have a root. So $P(z,w)$ gives rise to a map

$$U_P \times \mathbb{C} \longrightarrow \mathbb{C},$$

which we also write

$$(z,w) \longmapsto P(z,w)$$

by abuse of notation.

Ex 1) For $P = w^2 - z$, $U_P = \mathbb{C}$.

2) For $P = \frac{z}{z^2 - i} w^3 + z \cdot w + \frac{1}{z}$,

$$U_P = \mathbb{C} \setminus \left\{ 0, \frac{1}{\sqrt{2}} \pm i \frac{1}{\sqrt{2}} \right\} \subset \mathbb{C}.$$

Def Let $U \subset \mathbb{C}$ be a path-connected open subset. A holomorphic fct. $f : U \to \mathbb{C}$ is algebraic, if there exists $P \subset \mathbb{C}(z)[w]$ nonzero with $U \subset U_p$ s.t. for all $z \in U$,

$$P(z, f(z)) = 0.$$

Thm (Local existence and uniqueness) Let $P \in \mathbb{C}(z)[w]$, and let $(z_0, w_0) \in U_p \times \mathbb{C}$ be s.t. $P(z_0, w_0) = 0$ and

$$\frac{\partial P}{\partial w}(z_0, w_0) \neq 0.$$

In this situation, there exists an open disc $z_0 \in D \subset U_p$ s.t. there is a unique hol. fct.

$$D \xrightarrow{f} \mathbb{C}$$

with $f(z_0) = w_0$ and $P(z, f(z)) = 0$ for all $z \in D$.

Pf We will only use that

$$U_p \times \mathbb{C} \xrightarrow{P} \mathbb{C}$$

is a holomorphic fct. of two variables. This means that $P \in C^1(U_p \times \mathbb{C}, \mathbb{C})$ and that for all $(z,w) \in U_p \times \mathbb{C}$, the derivative

$$\mathbb{C} \times \mathbb{C} \xrightarrow{dP_{(z,w)}} \mathbb{C}$$

is $\mathbb{C}$-linear, and not only $\mathbb{R}$-linear.
By assumption,

$$\begin{tikzcd} (0,v) & \mathbb{C} \times \mathbb{C} \arrow[r, "dP_{(z_0,w_0)}"] & \mathbb{C} \\ v \arrow[u, mapsto] & \mathbb{C} \arrow[u, hook] \arrow[ur, "\sim"'] & \end{tikzcd}$$

so the implicit fct. thm. shows that there exist $z_0 \in D \subset U_p$ and $w_0 \in D' \subset \mathbb{C}$ and a C^1-fct. $f : D \to D'$ s.t. $f(z_0) = w_0$ and $P(z, f(z)) = 0$ for all $z \in D$. We must show that f is holomorphic and that it is unique with these properties.

The implicit fct. thm. also shows that, by possibly replacing D by a smaller disc around $z_0 \in U_p$, the derivative

$$\mathbb{C} \xrightarrow{df_z} \mathbb{C}$$

is given by the composition

$$\begin{tikzcd} (u,v) & \mathbb{C}\times\mathbb{C} \arrow[r, "dP_{(z,f(z))}"] & \mathbb{C} \\ u \arrow[u, mapsto] & \mathbb{C} \arrow[u] \arrow[ur, "df_z"'] & \end{tikzcd}$$

for all $z \in D$. So df_z is the composition of two $\mathbb{C}$-linear maps, and hence, $\mathbb{C}$-linear. This shows that $f : D \to \mathbb{C}$ is holomorphic at $z \in D$.

Finally, to prove uniqueness we have $f(z_0) = w_0$ by assumption, and by iteratedly differentiating the equation

$$P(z, f(z)) = 0,$$

we can determine all higher order derivatives of f at z_0. This determines the power series expansion of f at z_0, so by the identity principle, the uniqueness of $f : D \to \mathbb{C}$ follows. //

$$0 = t_0 < t_1 < \dots < t_N = 1,$$

open discs $\gamma(t_i) \in D_i$ with $D_{i-1} \cap D_i \neq \emptyset$ for all $1 \leq i \leq N$, and holomorphic fcts. $f_i : D_i \to \mathbb{C}$ s.t.

$$f_{i-1}|_{D_{i-1} \cap D_i} = f_i|_{D_{i-1} \cap D_i}$$

for all $1 \leq i \leq N$ and s.t.

$$f_0|_{D \cap U} = f|_{D \cap U}.$$

In this situation, $f_N(\gamma(1)) \in \mathbb{C}$ is said to be the value of f at $\gamma(1)$ as determined by analytic continuation along γ. //

Rmk The identity principle shows that $f_N(\gamma(1))$ only depends on $f : U \to \mathbb{C}$ and $\gamma : [0,1] \to \mathbb{C}$. We abuse notation and write $f(\gamma(1))$ instead of $f_N(\gamma(1))$, but note that this value does depend on γ, and not only on f, in general.

We will prove this Thm. later:

Thm Let $U \subset \mathbb{C}$ be a path-connected open subset. A hol. fct. $f : U \to \mathbb{C}$ is algebraic if and only if there exists a finite subset $S \subseteq \mathbb{C}$ s.t. the following holds:

1) For any path $\gamma : [0,1] \longrightarrow \mathbb{C}$ with $\gamma(0) \in U$ and $\gamma(1) \in \mathbb{C} \setminus S$ there exists an analytic continuation of f along γ. Moreover, as γ varies with $\gamma(0)$ and $\gamma(1)$ fixed, the set of possible values of $f(\gamma(1))$ is finite.

2) For every $s \in S$, there exists $s \in D \subseteq \mathbb{C}$, $d \in \mathbb{Z}_{\geq 0}$, and $C \in \mathbb{R}_{>0}$ such that for all $z \in D \setminus \{s\}$,

$$|f(z)| \leq C \cdot |z-s|^{-d}.$$

3) There exists $d \in \mathbb{Z}_{\geq 0}$ and $C, R \in \mathbb{R}_{>0}$ s.t. for all $z \in \mathbb{C}$ with $|z| > R$,

$$|f(z)| \leq C \cdot |z|^{d}.$$