gluing maps

Def A subset $\mathcal{G}$ of the set of bijections $g : V' \to V$ with $V, V' \subset \mathbb{R}^2$ open is admissible if the following holds:

1) If two composable bijections $g : V' \to V$ and $g' : V'' \to V$ both belong to $\mathcal{G}$, then so does their composition $g \circ g' : V'' \to V$.

2) If $g : V' \to V$ belongs to $\mathcal{G}$, then so does $g^{-1} : V \to V'$.

3) If $g : V' \to V$ belongs to $\mathcal{G}$, and if $W \subset V$ is open, then $W' = g^{-1}(W) \subset V'$ is open and $g|_{W'} : W' \to W$ is in $\mathcal{G}$.

4) If $g : V' \to V$ is a bijection with $V, V' \subset \mathbb{R}^2$ open, and if for every $x \in V$, there exists $x \in W \subset V$ open s.t. $W' = g^{-1}(W) \subset V'$ is open and $g|_{W'} : W' \to W$ is in $\mathcal{G}$, then $g : V' \to V$ is in $\mathcal{G}$.

Rmk We have allowed $\mathcal{G} = \emptyset$. Do we want to do so?

preserving spherical isometries between open subsets of

$$S^2 = \{ x \in \mathbb{R}^3 \mid \|x\| = 1 \} \subset \mathbb{R}^3$$

(almost) satisfies 1) - 4).

Def Let $\mathcal{G}$ be an admissible set of gluing maps. A $\mathcal{G}$-surface is a pair atlas

$$X = (|X|, (f_i : U_i \to V_i)_{i \in I})$$

of a set $|X|$ and a family of bijections $f_i : U_i \to V_i$ with $U_i \subset |X|$ and with $V_i \subset \mathbb{R}^2$ open s.t. the following hold:

1) For every $x \in |X|$, there exists $i \in I$ s.t. $x \in U_i$.

2) For all $(i, j) \in I \times I$, the subsets

$$f_i(U_i \cap U_j) \subset V_i$$

$$f_j(U_i \cap U_j) \subset V_j$$

are open, and the composite

$$f_i(U_i\cap U_j) \xrightarrow{(f_i|U_i\cap U_j)^{-1}} U_i\cap U_j \xrightarrow{f_j|U_i\cap U_j} f_j(U_i\cap U_j)$$

is in $\mathcal{G}$. //

We say that $f_i : U_i \to V_i$ is a (defining) chart on X, and that the gluing map

$$f_i(U_i\cap U_j) \xrightarrow{f_j|U_i\cap U_j \circ (f_i|U_i\cap U_j)^{-1}} f_j(U_i\cap U_j)$$

is the transition map from the chart $f_i : U_i \to V_i$ to the chart $f_j : U_j \to V_j$.

We think of the admissible gluing maps as the maps that preserve some structure on (some) open subsets of $\mathbb{R}^2$. By gluing along these maps, the structure in question (e.g. complex structure) becomes available, more generally, on all $\mathcal{G}$-surfaces.

It is really not the <u>set</u> of $\mathcal{G}$-surfaces that we are interested in; it is the groupoid of $\mathcal{G}$-surfaces that is the interesting mathematical object.

<u>Def</u> Given $\mathcal{G}$-surfaces

$$X = (|X|, (f_i : U_i \to V_i)_{i \in I})$$

$$X' = (|X'|, (f'_j : U'_j \to V'_j)_{j \in J}),$$

a bijection $h : |X'| \to |X|$ is an isomorphism if for all $(i,j) \in I \times J$, the subsets

$$f_i(U_i \cap h(U'_j)) \subset V_i$$

$$f'_j(h^{-1}(U_i) \cap U'_j) \subset V'_j$$

are open and the composite

$$\begin{array}{l} f_i(U_i \cap h(U'_j)) \\ \qquad \xrightarrow{(f_i|_{U_i \cap h(U'_j)})^{-1}} U_i \cap h(U'_j) \\ \qquad\qquad \downarrow h^{-1}|_{U_i \cap h(U'_j)} \\ \qquad\qquad h^{-1}(U_i) \cap U'_j \\ f'_j(h^{-1}(U_i) \cap U'_j) \xleftarrow{f'_j|_{h^{-1}(U_i) \cap U'_j}} \end{array}$$

is in $\mathcal{G}$. //

Prop Let $\mathcal{G}$ be an admissible set of gluing maps.

1) The identity map of any $\mathcal{G}$-surface X is an isomorphism.

2) The composition of two composable isomorphisms of $\mathcal{G}$-surfaces is an isomorphism.

3) The inverse of an isomorphism of $\mathcal{G}$-surfaces is an isomorphism.

Pf Clear. //

Rmk This shows that the category of $\mathcal{G}$-surfaces and isomorphisms of $\mathcal{G}$-surfaces is a groupoid. //

Def Let $\mathcal{G}$ be an admissible set of gluing maps, and let

$$X = (|X|, (f_i : U_i \to V_i)_{i \in I})$$

be a $\mathcal{G}$-surface. A bijection

$$U \xrightarrow{h} V$$

with $U \subset |X|$ and $V \subset \mathbb{R}^2$ open is a chart on X if for every $i \in I$, the subsets

$$f(U \cap U_i) \subset V$$

$$f_i(U \cap U_i) \subset V_i$$

are open and the composite

$$f(U \cap U_i) \xrightarrow{(f|_{U \cap U_i})^{-1}} U \cap U_i \xrightarrow{f_i|_{U \cap U_i}} f_i(U \cap U_i)$$

is in $\mathcal{G}$. //

The notion of a chart on a $\mathcal{G}$-surface is an invariant notion in the sense that the following holds:

Prop Let $h \colon X' \to X$ be an isomorphism between $\mathcal{G}$-surfaces, let $f \colon U \to V$ be a bijection with $U \subset X$ and with $V \subset \mathbb{R}^2$ open, and let

$f' = f \circ h : U' = h^{-1}(U) \longrightarrow V$. In this situation, $f : U \to V$ is a chart on X if and only if $f' : U' \to V$ is a chart on X'.

Pf Immediate from definitions. //

We can use the notion of charts to define a topology on the underlying set $|X|$ of a ξ-surface X.

Prop Let X be a ξ-surface. The subset $\mathcal{B} \subset \mathcal{P}(|X|)$ consisting of the subsets $U \subset |X|$ that are domains of a chart $f : U \to V$ is a basis for a topology on $|X|$. Moreover, with respect to this topology, every chart

$$U \xrightarrow{f} V$$

is a homeomorphism. /

We will now consider $|X|$ as a topological space with this top. Alternatively, we could have assumed $|X|$ to be a topological space to begin with and the bijections $f_i : U_i \to V_i$ to be homeomorphisms. This choice is made in most textbooks, but it is not necessary.

Prop Let X be a $\mathcal{G}$-surface. The following are equivalent:

1) The topological space $|X|$ is Hausdorff.

2) For every $(x, x') \in |X| \times |X|$, there exist charts $f : U \to V$ and $f' : U' \to V'$ with $x \in U$ and $x' \in U'$ s.t. $U \cap U' = \emptyset$.

Pf This is a general fact about topologies defined by a basis. //

Rmk Being Hausdorff is a global property of a topological space.

Def A Riemann surface is a Hausdorff $\mathcal{G}$-surface, where $\mathcal{G}$ is the set of biholomorphisms.

Ex 0) The empty $\mathcal{E}$-surface

$$X = (\emptyset, 1)$$

is a Riemann surface.

1) The complex plane

$$X = (\mathbb{C}, (\mathrm{id} : \mathbb{C} \longrightarrow \mathbb{C})_1)$$

is a Riemann surface. A chart on X is a biholomorphism $f : U \to V$ with $U \subset |X| = \mathbb{C}$ and $V \subset \mathbb{C}$ open.

2) If X is a Riemann surface, and if $S \subset |X|$ is a finite subset, then the pair

$$X \setminus S = (|X| \setminus S, (f : U \to V)),$$

where $(f : U \to V)$ is the self-indexed family of all charts $f : U \to V$ on X s.t. $U \subset |X| \setminus S$, is a Riemann surface.

3) The Riemann sphere is the pair

$$\mathbb{P}^1 = (\mathbb{C} \cup \{\infty\}, (f_i : U_i \to \mathbb{C})_{i \in \{1,2\}})$$

where f_1 is the identity map

$$U_1 = \mathbb{P}^1 \setminus \{\infty\} \xrightarrow{f_1} \mathbb{C}$$

and f_2 is the inversion map

$$U_2 = \mathbb{P}^1 \setminus \{0\} \xrightarrow{f_2} \mathbb{C}$$

$$z \longmapsto 1/z$$

with $1/\infty = 0$. Its underlying topological space $|\mathbb{P}^1|$ is a one-point compactification of $\mathbb{C}$. It is a Riemann surface.

4) The complex plane with zero doubled is the pair

$$X = (\mathbb{C} \cup \{0'\}, (f_i : U_i \to \mathbb{C})_{i \in \{1,2\}})$$

where f_1 is the identity map

$$U_1 = X \setminus \{0'\} \xrightarrow{f_1} \mathbb{C}$$

and f_2 is the map

$$U_2 = X \setminus \{0\} \xrightarrow{f_2} \mathbb{C}$$

defined by

$$f(z) = \begin{cases} 0 & \text{if } z = 0' \\ z & \text{if } z \neq 0'. \end{cases}$$

It is a $\mathcal{G}$-surface, but $|X|$ is not Hausdorff: The point $0, 0'$ cannot be separated!