Last time, we define the groupoid of Riemann surfaces. This time, we first extend it to a category by adding non-invertible maps.

Def A map $f: Y \to X$ between (the underlying sets) of Riemann surfaces X and Y is holomorphic if for all $y \in Y$ with image $x = f(y) \in X$, there exist charts

$$x \in U \xrightarrow{h} V \subset \mathbb{C}$$

$$y \in U' \xrightarrow{h'} V' \subset \mathbb{C}$$

s.t. $f(U') \subset U$ and s.t. the unique map that makes the diagram

$$\begin{tikzcd} U' \arrow[r, "f|_{U'}"] \arrow[d, "h'"] & U \arrow[d, "h"] \\ V' \arrow[r, "F"] & V \end{tikzcd}$$

commute is holomorphic.

Rmk 1) For every Riemann surface X, the identity map $id: X \to X$ is holomorphic.

2) If $f: Y \to X$ and $g: Z \to Y$ are composable holomorphic maps btw. Riemann surfaces, then so is their composite $f \circ g: Z \to X$.

So Riemann surfaces and holomorphic maps form a category. The isomorphisms in this category are precisely the isomorphisms of Riemann surfaces that we defined last time.

Def A holomorphic map between Riemann surfaces $f: Y \to X$ is a local isomorphism if for every $y \in Y$, there is an open subset $y \in U \subset Y$ s.t. $f(U) \subset X$ is open and s.t. $f|_U : U \to f(U)$ is an isomorphism.

Rmk 1) If, in the definition of a holomorphic map $f: Y \to X$ we ask that the map F be a biholomorphisms, then we get the definition of a local isom.

2) A holomorphic map $f: Y \to X$ is

a local isomorphism and a bijection (of underlying sets).

Prop Given a commutative diagram of Riemann surfaces

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "q"'] & & X \arrow[dl, "p"] \\ & S & \end{tikzcd}$$

the following hold:

1) If p and f are local isom., then so is q.

2) If p and q are local isom., then so is f.

3) If q and f are local isom. and if f is surjective, then also p is a local isom.

Pf Immediate from def.

Ex The exponential map

$$\mathbb{C} \xrightarrow{\exp} \mathbb{C} \setminus \{0\}$$

is a local isom.

Thm (Inverse function theorem)
If $f: Y \to X$ is a holomorphic map between Riemann surfaces (or, more generally, between complex manifolds), then the following are equivalent:

1) The map $f: Y \to X$ is a local isomorphism.

2) For every $y \in Y$ with image $x = f(y) \in X$, the derivative

$$T_{Y,y} \xrightarrow{df_y} T_{X,x}$$

is an isomorphism. //

Def. A holomorphic map between complex mfds, $f: Y \to X$, is a submersion (resp. an immersion) if for every $y \in Y$ with image $x = f(y) \in X$, the derivative

$$T_{Y,y} \xrightarrow{df_y} T_{X,x}$$

is surjective (resp. injective). //

Cor (Implicit function theorem)
Let $f : Y \to X$ be a submersion of complex mfds, and let $g : X' \to X$ be any holomorphic map of complex mfd. In this situation, the base-change of f along g,

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array},$$

exists and is a submersion. //

Ex Taking $X' = 1$ to be a point, the implicit function theorem shows that the fibers of a submersion,

$$\begin{array}{ccc} Y_x & \xrightarrow{x'} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ 1 & \xrightarrow{x} & X \end{array},$$

are complex manifolds. /

Ex More generally, if $f: Y \to X$ is a holomorphic map between complex manifolds, then we say that $x \in X$ is a regular value of f if for all $y \in Y_x$, the derivative

$$T_{Y,y} \xrightarrow{df_y} T_{X,x}$$

is surjective. In this situation, there exist $x \in U \subset X$ and $Y_x \subset V \subset Y$ open s.t. $f|_V : V \to U$ is a submersion. So

$$\begin{tikzcd} Y_x \arrow[r] \arrow[d] & V \arrow[r, hook] \arrow[d, "f|_V"] & Y \arrow[d, "f"] \\ 1 \arrow[r, "x"] & U \arrow[r, hook] & X \end{tikzcd}$$

shows that Y_x is a complex manifold. //

Let $P(z,w) \in \mathbb{C}(z)[w]$, and let $S \subset \mathbb{C}$ be the finite set of roots of the denominators of the coefficients. So $P(z,w)$ defines a holomorphic map of complex manifolds

$$Y = (\mathbb{C} \setminus S) \times \mathbb{C} \xrightarrow{f} \mathbb{C}$$

$$(z, w) \longmapsto P(z, w)$$

The point $0 \in \mathbb{C}$ may not be a regular point. However, there are at most finitely many points in the fiber over $0 \in \mathbb{C}$, where the diff. df_y is zero ($=$ not surj.). So we choose $0 \in U \subset \mathbb{C}$ (small) open and let $V \subset Y$ be the largest open subset s.t. $f|_V : V \to U$ is a submersion. Explicitly,

$$V = V' \cup V'' \subset f^{-1}(U) \subset Y$$

where

$$V' = \left\{ (z, w) \in f^{-1}(U) \;\middle|\; \frac{\partial P}{\partial w}(z, w) \neq 0 \right\}$$

$$V'' = \left\{ (z, w) \in f^{-1}(U) \;\middle|\; \frac{\partial P}{\partial z}(z, w) \neq 0 \right\}$$

Let $C' = V' \cap Y_0$, $C'' = V'' \cap Y_0$. Now, by the implicit fct. Thm.,

$$X = C' \cup C'' \subset V$$

promotes uniquely, up to unique isomorphism, to a Riemann surface. Moreover, the maps

$$C' \xrightarrow{p_1} \mathbb{C}$$

$$(z,w) \longmapsto z$$

$$C'' \xrightarrow{p_2} \mathbb{C}$$

$$(z,w) \longmapsto w$$

are local isomorphisms.

Ex Let $P(z,w) = w^n - z$. Since

$$\frac{\partial P}{\partial z}(z,w) = -1$$

is non-zero, the map

$$Y = \mathbb{C}^2 \longrightarrow \mathbb{C}$$

$$(z,w) \longmapsto P(z,w)$$

is a submersion, so the fiber $X = Y_0 \subset Y$ promotes to a Riemann surface. The projection to the w-axis

$$X \xrightarrow{p_2} \mathbb{C}$$

$$(z,w) \longmapsto w$$

is a local isom. In fact, it is a global isom. with inverse

$$\begin{aligned} \mathbb{C} &\longrightarrow X \\ w &\longmapsto (w^n, w) \end{aligned}$$

The projection to the z-axis

$$\begin{aligned} X &\xrightarrow{p_1} \mathbb{C} \\ (z,w) &\longmapsto z \end{aligned}$$

is a local isom. on

$$X' = \{(z,w) \in X \mid w \neq 0\}.$$

Given a Riemann surface X, write

$$\mathrm{Aut}(X)$$

for its group of automorphisms.

<u>Ex</u> Will show that

$$\mathrm{Aut}(\mathbb{C}) \simeq GL_1(\mathbb{C}) \ltimes \mathbb{C}$$

is the group of maps of the form $z \mapsto a \cdot z + b$, $a \in \mathbb{C} \setminus \{0\}$, $b \in \mathbb{C}$.

Def Let X be a Riemann surface. A subgroup

$$\Gamma \subset \mathrm{Aut}(X)$$

acts properly discontinuously on X if for all $x \in X$, there exists $x \in U \subset X$ open s.t.

$$U \cap \gamma(U) = \emptyset$$

for all $\gamma \in \Gamma \setminus \{\mathrm{id}\}$. /

Ex The subgroup

$$\Gamma = 2\pi i \mathbb{Z} \subset \mathrm{Aut}(\mathbb{C})$$

acts properly discont. on $\mathbb{C}$. /

In general, given a left action of a group Γ on a set X, we write

$$X \xrightarrow{p} \Gamma \backslash X$$

for the quotient by the equiv. rel. on X given by the image

$R \subset X \times X$ of the map

$$X \times \Gamma \longrightarrow X \times X$$

$$(x, \gamma) \longmapsto (x, \gamma(x))$$

Thm Let X be a Riemann surface. If $\Gamma \subset \mathrm{Aut}(X)$ acts properly discontinuously on X, then, up to unique isomorphism, X/Γ admits a unique structure of Riemann surface s.t.

$$X \xrightarrow{p} \Gamma \backslash X$$

is holomorphic. In fact, in this situation, the map p is a local isomorphism.

Pf The set $|\Gamma \backslash X|$ is the set of subsets of X of the form

$$\Gamma \cdot x = \{ \gamma(x) \in X \mid \gamma \in \Gamma \} \subset X.$$

We say that $\Gamma \cdot x$ is the orbit through $x \in X$ of the left action by Γ on X. The map

$$|X| \xrightarrow{p} |\Gamma \backslash X|$$

is given by $p(x) = \Gamma \cdot x$. It is surjective, but not injective, unless $\Gamma = \{id\}$.

We fix $x \in |X|$ and define a chart around $p(x) \in |\Gamma \backslash X|$ as follows. Since Γ acts properly discontinuously on X, we can find $x \in U \subset X$ open s.t.

$$U \cap \gamma(U) = \emptyset$$

for all $\gamma \in \Gamma \setminus \{id\}$. Hence,

$$U \xrightarrow{p|U} p(U)$$

is a bijection. Replacing U by a smaller open subset if necessary, we can further choose a chart $h : U \longrightarrow V$ on X. We now define

$$p(U) \xrightarrow{(p|U)^{-1}} U \xrightarrow{h} V$$

to be a chart on $\Gamma \backslash X$. This family of charts, indexed by $x \in X$, defines a $\mathcal{G}$-surface strucon $|\Gamma \backslash X|$, where $\mathcal{G}$ is the groupoid of biholomorphisms of open subsets of $\mathbb{C}$. So it only remains to check that the topological space $|\Gamma \backslash X|$ is Hausdorff. This (also) uses that the action by Γ on X is properly discontinuous; see notes. //

Ex We let

$$\Gamma = 2\pi i \mathbb{Z} < \mathrm{Aut}(\mathbb{C})$$

and consider the diagram

$$\begin{tikzcd} \mathbb{C} \arrow[rr, "p"] \arrow[dr, "\exp"'] & & \Gamma \backslash \mathbb{C} \arrow[dl, "\bar{p}"] \\ & \mathbb{C} \setminus \{0\} & \end{tikzcd}$$

Since exp and p are local isom., and since p is surjective, also $\bar{p}$ is a local isom. But $\bar{p}$ is a bijection, so it is an isom. Note that the Euclidean geometry on $\Gamma \backslash \mathbb{C}$ defines (via $\bar{p}$) a new geometry on $\mathbb{C} \setminus \{0\}$ s.t. mult. by $a \in \mathbb{C} \setminus \{0\}$ is an isometry.