Recall that, last time, we showed that the exponential map induces an isom. $\overline{\exp}$ of Riemann surfaces

$$\begin{tikzcd} \mathbb{C} \arrow[r, "p"] \arrow[dr, "\exp"'] & \Gamma\backslash\mathbb{C} \arrow[d, "\overline{\exp}"] \\ & \mathbb{C}\setminus\{0\} \end{tikzcd}$$

where $\Gamma = 2\pi i\mathbb{Z} \subset \mathrm{Aut}(\mathbb{C})$. In fact, $\Gamma \subset \mathrm{Isom}^+(\mathbb{C})$ acts (properly discontinuously) through (orientation-preserving) isometries for the Euclidean geometry on $\mathbb{C}$. Hence, $\Gamma\backslash\mathbb{C}$, and thus $\mathbb{C}\setminus\{0\}$, acquires a structure of Euclidean surface, and automorphisms of the Riemann surface $\mathbb{C}\setminus\{0\}$ are (orientation-preserving) isometries w.r.t. this geometry. This is an example of "uniformization" or "geometrization."

Today, we state the uniformization theorem, which says that every connected Riemann surface can be "uniformized" by one of three model geometries:

1) The unit disc $Y = \mathbb{D}$ with its hyperbolic geometry

$$ds^2 = \frac{4}{(1-|z|^2)^2}|dz|^2.$$

2) The complex plane $Y = \mathbb{C}$ with its Euclidean geometry

$$ds^2 = |dz|^2.$$

3) The Riemann sphere $Y = \mathbb{P}^1$ with its spherical geometry

$$ds^2 = \frac{4}{(1+|z|^2)^2}|dz|^2.$$

Recall that a topological space X is connected if for all $U, V \subset X$ open s.t. $U \cap V = \emptyset$ and $U \cup V = X$, either $U = \emptyset$ or $V = \emptyset$ or both. (An open subset $U \subset \mathbb{C}$ is connected if and only if it is path-connected.) If $f : Y \to X$ is a continuous map between topological spaces, and if Y is connected, then also $f(Y) \subset X$ is connected. Since $\mathbb{D}$, $\mathbb{C}$, and $\mathbb{P}^1$

are all connected, so is any quotient $p : Y \to \Gamma \backslash Y = X$. The uniformization theorem states that every non-empty connected Riemann surface arises in this way:

<u>Thm</u> (Uniformization theorem) Let X be a non-empty connected Riemann surface. There exists a subgroup

$$\Gamma \subset \mathrm{Aut}(Y),$$

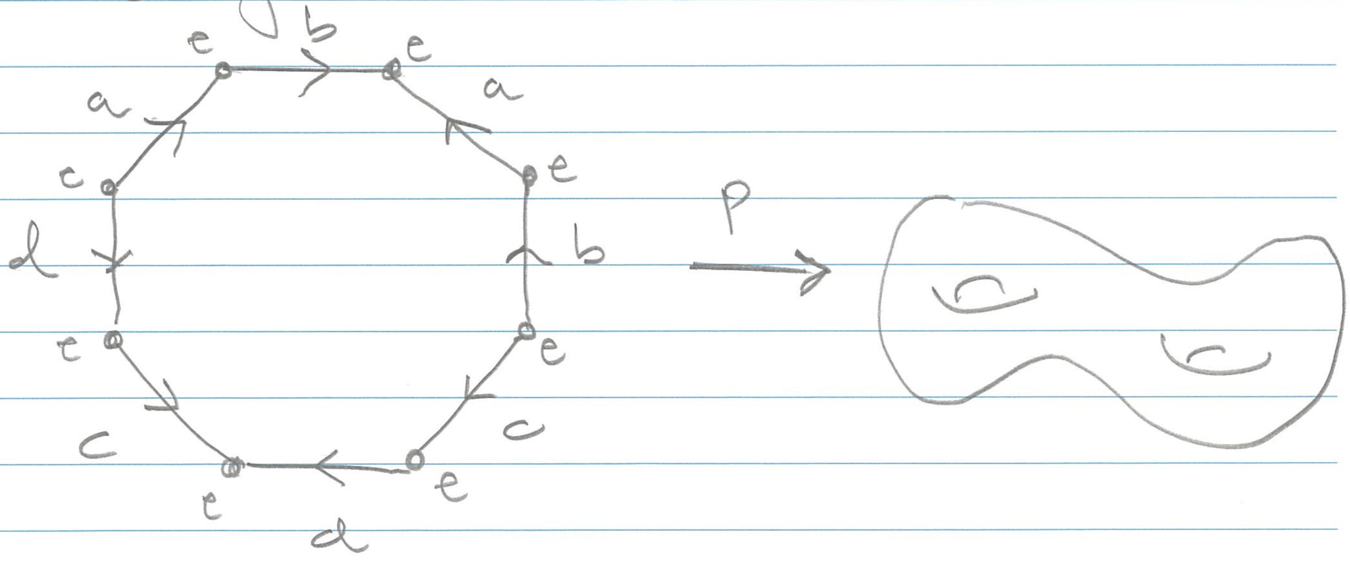
where $Y = \mathbb{D}, \mathbb{C}$, or $\mathbb{P}^1$ that acts properly discontinuously on Y and an isomorphism

$$\Gamma \backslash Y \xrightarrow{f} X$$

of Riemann surfaces. /

<u>Rmk</u> 1) If $Y = \mathbb{P}^1$, then $\Gamma = \{id\}$ is the only possibility, and if $Y = \mathbb{C}$ then $\Gamma \subset \mathrm{Aut}(\mathbb{C})$ is generated by translations along the vectors in an $\mathbb{R}$-linearly independent family of 0, 1, or 2 vectors in $\mathbb{C}$. But for $Y = \mathbb{D}$, there are very

many possible $\Gamma \subset \mathrm{Aut}(\mathbb{D})$. These correspond to tesselations. For example, the hyperbolic disc $\mathbb{D}$ admits a tesselation by regular octagons with 8 octagons meeting at each vertex. This gives a Riemann surface (and hyperbolic geometry) structure to a genus 2 surface:



A priori $\gamma \in \Gamma \subset \mathrm{Aut}(Y)$ only preserves oriented angles, but it turns out that it automatically preserves distances, too. So $\Gamma \subset \mathrm{Isom}^+(Y)$. Hence:

Cor Every connected Riemann surface admits a structure of $\mathcal{G}$-surface for $\mathcal{G}$ one of hyperbolic Euclidean, or spherical (orientation-pres.) isometries.

In general, given two objects X and X' in a category $\mathcal{C}$, the set $\mathrm{Isom}(X', X)$ of isomorphisms $f: X' \to X$ is either empty or is simultaneously a torsor for the groups of automorphisms $\mathrm{Aut}(X)$ and $\mathrm{Aut}(X')$. This means that if $f: X' \to X$ is an isomorphism (so $\mathrm{Isom}(X', X) \neq \emptyset$), then the maps

$$\mathrm{Aut}(X') \longrightarrow \mathrm{Isom}(X', X)$$
$$g \longmapsto f \circ g$$

$$\mathrm{Aut}(X) \longrightarrow \mathrm{Isom}(X', X)$$
$$h \longmapsto h \circ f$$

are bijections: There is no preferred isom. $f: X' \to X$, but if one exists, then every other isom. is given by composing it with automorphisms of either X or X'. The proof is easy.

Prop Let Y be one of $\mathbb{D}$, $\mathbb{C}$, or $\mathbb{P}^1$, let $\Gamma, \Gamma' \subset \mathrm{Aut}(Y)$ act properly discontinuously on Y, and let $X = \Gamma \backslash Y$ and $X' = \Gamma' \backslash Y$. Let

$$\mathrm{Aut}(Y, \Gamma, \Gamma') \subset \mathrm{Aut}(Y)$$

be the subset consisting of the autom. $f : Y \to Y$ s.t.

$$f \Gamma' f^{-1} = \Gamma .$$

In this situation, the map

$$\mathrm{Aut}(Y, \Gamma, \Gamma') \longrightarrow \mathrm{Isom}(X', X)$$

that to $f : Y \to Y$ assigns the unique map $\bar{f} : X' \to X$ that makes the diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "p'"] & Y \arrow[d, "p"] \\ X' \arrow[r, "\bar{f}"] & X \end{tikzcd}$$

commute is a bijection. /

Rmk 1) $\mathrm{Aut}(Y, \Gamma, \Gamma') \subset \mathrm{Aut}(Y)$ is not a subgroup, unless $\Gamma = \Gamma'$!

2) Unless $\Gamma = \{id\}$ and $Y = \mathbb{C}$ or $Y = \mathbb{P}^1$ every $f : Y \to Y$ as in Prop. is an (orientation-preserving) isometry. So isom. between conn. Riemann surfaces can (largely) be understood in terms of geometry. /

Next, we discuss the algebraization theorem. First:

Thm As a differentiable surface, every non-empty, connected, compact Riemann surface is determined, up to non-unique isomorphism, by its genus g given by

$$\chi(|X|) = 2 - 2g.$$

/

Def Let X be a non-empty, connected Riemann surface. A meromorphic function on X is a holomorphic map

$$X \xrightarrow{f} \mathbb{P}^1$$

that is not the constant map with value $\infty \in \mathbb{P}^1$.

A holomorphic fct. on X is a meromorphic fct. on X that takes values in $\mathbb{C} \subset \mathbb{P}^1$.

The set $\mathcal{M}(X)$ of meromorphic fct. on X promotes to a field with addition and multiplication defined pointwise. Moreover, it contains $\mathbb{C}$ as constant holomorphic functions. (Every nonzero elem. in $\mathcal{M}(X)$ is invertible, since $z \mapsto 1/z$ is holom. $\mathbb{P}^1 \to \mathbb{P}^1$.) In general, $\mathcal{M}(X)$ is enormous. But:

<u>Thm</u> If X is a compact non-empty connected Riemann surface, then $\mathcal{M}(X)/\mathbb{C}$ is finitely generated of transcendence degree 1.

This means: There exist $z, f \in \mathcal{M}(X)$ such that z is transcendental over $\mathbb{C}$, f is algebraic over $\mathbb{C}(z)$, and every $g \in \mathcal{M}(X)$ can be written as a rational fct. of z and f.

Ex 1) $\mathcal{M}(\mathbb{P}^1) = \mathbb{C}(z)$. In particular, $\exp : \mathbb{C} \to \mathbb{P}^1$ does not extend along $\mathbb{C} \hookrightarrow \mathbb{P}^1$, since it is not meromorphic at ∞.

2) Given $P(z,w) \in \mathbb{C}(z)[w]$, we have defined a structure of Riemann surface on the subset

$$X \subset (\mathbb{C} \setminus S) \times \mathbb{C}$$

of those (z_0, w_0) s.t. $P(z_0, w_0) = 0$ and s.t. $(\partial P/\partial w)(z_0, w_0) \neq 0$ or $(\partial P/\partial z)(z_0, w_0) \neq 0$, or both. The map

$$X \xrightarrow{z} \mathbb{P}^1$$
$$(z,w) \longmapsto z$$

is meromorphic, and for generic choices of P, it is transcendental over $\mathbb{C}$ in $\mathcal{M}(X)$. The map

$$X \xrightarrow{w} \mathbb{P}^1$$
$$(z,w) \longmapsto w$$

is also meromorphic, and it is algebraic over $\mathbb{C}(z)$, since

$$P(z,w) = 0$$

in $\mathcal{M}(X)$. But z and w do not generate $\mathcal{M}(X)$, because X is not compact. However, X admits a compactification $\bar{X}$ with $\bar{X} \setminus X$ a finite set, and z and w extend to meromorphic fcts. on $\bar{X}$, so

$$\mathcal{M}(\bar{X}) = \mathbb{C}(z)[w]/(P(z,w)),$$

provided that $P(z,w)$ is irred. over $\mathbb{C}(z)$.

A non-constant holomorphic map of non-empty connected Riem. surf.

$$Y \xrightarrow{h} X$$

determines a map of fields

$$\mathcal{M}(X) \xrightarrow{h^*} \mathcal{M}(Y)$$

that to $f : X \to \mathbb{P}^1$ assigns $f \circ g : Y \to \mathbb{P}^1$. This assignment defines a map

$$\mathrm{Map}(Y, X)' \longrightarrow \mathrm{Map}(M(X), M(Y))$$
$$f \longmapsto f^*$$

from the subset

$$\mathrm{Map}(Y, X)' \subset \mathrm{Map}(Y, X)$$

consisting of the non-constant holomorphic maps to the set of field (= ring) homomorphisms from $M(X)$ to $M(Y)$.

<u>Thm</u> (Algebraization theorem). If X and Y are compact, non-empty, connected Riemann surfaces, then the map

$$\mathrm{Map}(Y, X)' \longrightarrow \mathrm{Map}(M(X), M(Y))$$
$$f \longmapsto f^*$$

is a bijection. /

In particular, for X and Y as in the theorem, there exists an isom. of Riemann surfaces $f : Y \to X$ if and only if there exists an isom. of fields $g : \mathcal{M}(X) \to \mathcal{M}(Y)$. Moreover, the map

$$\mathrm{Aut}(X)^{op} \longrightarrow \mathrm{Aut}(\mathcal{M}(X)/\mathbb{C})$$

$$f \longmapsto f^*$$

is an isomorphism of groups.

Conversely, there is a functor, called the Zariski–Riemann surface, that to a f. g. field $F/\mathbb{C}$ of transcendence deg. 1 assigns a Riemann surface

$$X = \mathrm{Spec}(F/\mathbb{C})$$

together with an isom. of fields

$$F \xrightarrow{\sim} \mathcal{M}(X).$$

The Riemann surface X is compact, non-empty and conn.