Thm Let $f: Y \to X$ be a holomorphic map between Riemann surfaces. Let $y \in Y$, $x = f(y) \in X$ and suppose that f is not constant in a neighborhood of y. In this situation, there exists an integer $n \geq 1$ and a commutative diagram

$$\begin{tikzcd} V \arrow[r, "f|V"] \arrow[d, "\psi"] & U \arrow[d, "\varphi"] \\ D \arrow[r, "\tilde{f}"] & D \end{tikzcd},$$

where φ and ψ are charts around x and y with $D = D(0,1) \subset \mathbb{C}$ and $\varphi(x) = 0$ and $\psi(y) = 0$, and where

$$\tilde{f}(z) = z^n.$$

Pf We can assume that $X = Y = D$, $x = y = 0$, and that f admits a power series expansion

$$f(z) = c_1 z + c_2 z^2 + \cdots + c_n z^n + \cdots.$$

Since f is not constant, there is a smallest integer $n \geq 1$ s.t. $c_n \neq 0$.

So we can write

$$f(z) = z^n g(z),$$

where $g: D \to D$ is a holomorphic function with $g(0) \neq 0$. Since $g(0) \neq 0$, a holomorphic n'th root function is defined in a neighborhood of $g(0)$, and since g is continuous it will map a neighborhood of $0 \in D$ into this neighborhood. So on a possibly smaller disc around $0 \in D$, we can write

$$f(z) = z^n \cdot h(z)^n,$$

where h is a holomorphic fct. with $h(0) \neq 0$. Now, set

$$\psi(z) = z \cdot h(z).$$

Since $\psi'(0) = h(0) \neq 0$, the inverse fct. thm. shows that ψ restricts to a biholomorphism of an open nbhd. of 0 to an open nbhd. of $\psi(0) = 0$. So we have

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[d, "\psi"] & D \arrow[d, "\varphi = \mathrm{id}"] \\ D \arrow[r, "z \mapsto z^n"] & D \end{tikzcd}$$

as desired. //

<u>Rmk</u> The proof of the theorem also shows us how to determine the integer $n \geq 1$: If we can find charts around $y \in Y$ and $x = f(y) \in X$ s.t. f is repr. by

$$z \longmapsto z^m g(z)$$

with g hol. and $g(0) \neq 0$, then $n = m$. It is not clear that this integer only depends on f and y and not on the choice of charts. However, we also see that, in a neighborhood of $y \in Y$, f is n-to-1, and this integer $n \geq 1$ only depends on f and y. We call it the multiplicity of f at $y \in Y$ and write

$$m_y(f) = n.$$

Cor Let $f: Y \to X$ be a hol. map between Riemann surfaces. If Y is connected and if f is not constant, then there are no points $y \in Y$ s.t. f is constant in a neighborhood around y.

(Hence, if Y is connected and if f is non-constant, then around every point $y \in Y$, f can be represented by $z \mapsto z^n$ for some $n \geq 1$.)

Pf Given $x \in X$, let $S = S_x \subset Y$ be the subset of those $y \in Y$ for which there exists a neighborhood $y \in V \subset Y$ s.t. $f|_V = x$. We wish to prove that $S = \emptyset$, and since Y is connected, it will suffice to show that $S \subset Y$ is both open and closed. Indeed, this will imply that $S = \emptyset$ or $S = Y$, but $S \neq Y$, because f is non-constant.

It is clear from the definition that $S \subset Y$ is open. So let

$S \subset \bar{S} \subset Y$ be its closure. If $y \in \bar{S} \setminus S$, then f is non-constant on every neighborhood of y. So by theorem, f is repr. by $z \mapsto z^n$ with $n \geq 1$ around y. But every such neighborhood intersects S, by the definition closure, so this is impossible. Hence, $S = \bar{S}$. //

Cor If $f: Y \to X$ is a holomorphic map between Riemann surfaces, then the following are equivalent:

1) $f: Y \to X$ is a biholomorphism.

2) $f: |Y| \to |X|$ is a bijection.

Pf It is clear that 1) implies 2). To prove that 2) implies 1), we must show that $g = f^{-1}: |X| \to |Y|$ is holomorphic. In fact, it suffices to show that f is a local isomorphism, because this implies that g is locally holomorphic, hence, holomorphic. Since f is injective, it is not constant in a neighborhood of any point. So locally, f can be represented by $z \longmapsto z^n$ for some $n \geq 1$. But since f is injective, we necessarily have $n=1$, so f is a local isomorphism. //

Rmk The proof shows the following stronger result: If $f: Y \to X$ is holomorphic and injective, then $f(Y) \subset X$ is open and

$$Y \xrightarrow{f} f(X)$$

is an isom. of Riemann surfaces. //

The first theorem described non-constant holomorphic maps locally on the source. The next theorem will concern holomorphic maps locally on the target, but it requires stronger hypotheses.

Def A holomorphic map between Riemann surface $f: Y \to X$ is proper if for every $K \subset X$ compact, $f^{-1}(K) \subseteq Y$ is compact.

Rmk 1) If Y is compact, then every holomorphic map $f: Y \to X$ is proper.

2) Given a cartesian square of Riemann surfaces

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ \downarrow f' & & \downarrow f \\ X' & \xrightarrow{g} & X \end{array},$$

If f is proper, then so is f'. In particular, if $f: Y \to X$ is proper and $U \subset X$ is open, then also $f|f^{-1}(U): f^{-1}(U) \to U$ is so.

3) If $f: Y \to X$ is proper and if $y \in Y$, then $f|_{Y \setminus \{y\}}: Y \setminus \{y\} \to X$ is not proper.

Prop A non-constant polynomial map $f: \mathbb{C} \to \mathbb{C}$ extends (uniquely) to a holomorphic map $\bar{f}: \mathbb{P}^1 \to \mathbb{P}^1$ with $\bar{f}(\infty) = \infty$. In particular, the map $f: \mathbb{C} \to \mathbb{C}$ is proper.

Pf First, every holomorphic map

$\bar{f} : \mathbb{P}^1 \to \mathbb{P}^1$ is proper, since $\mathbb{P}^1$ is compact. Since $f : \mathbb{C} \to \mathbb{C}$ is the restriction of $\bar{f}$ to the open subset $\mathbb{C} \subset \mathbb{P}^1$, the second statement follows from the first. So we wish to show that the map

$$\mathbb{P}^1 \xrightarrow{\bar{f}} \mathbb{P}^1$$

$$\bar{f}(z) = \begin{cases} f(z) & \text{if } z \neq \infty \\ \infty & \text{if } z = \infty \end{cases}$$

is holomorphic. This is clear for $z \neq \infty$, because $f : \mathbb{C} \to \mathbb{C}$ is hol. In the chart $h : \mathbb{P}^1 \setminus \{0\} \to \mathbb{C}$, $h(z) = 1/z$, around $\infty \in \mathbb{P}^1$, $\bar{f}$ is represented by $g : \mathbb{C} \to \mathbb{C}$,

$$g(z) = \begin{cases} f(z^{-1})^{-1} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

It is continuous at $z = 0$, because f is non-constant. To see that g is holomorphic at $z = 0$, we write

$$f(z) = c_n z^n + \cdots + c_1 z + c_0$$

with $n \geq 1$ and $c_n \neq 0$. Thus,

$$g(z) = f(z^{-1})^{-1}$$

$$= c_n z^{-n} + \cdots + c_1 z^{-1} + c_0$$

$$= z^n / (c_n + \cdots + c_1 z^{n-1} + c_0 z^n),$$

which is hol. at $z=0$ as desired. //

Rmk The proof of the proposition shows that the multiplicity of $\bar{f} : \mathbb{P}^1 \to \mathbb{P}^1$ at ∞ is

$$m_\infty(\bar{f}) = n = \deg(f).$$ /

We prove two lemmas about proper maps:

Lemma If $f: Y \to X$ is a non-constant proper map between Riemann surfaces with Y conn., then for all $x \in X$, the fiber $Y_x = f^{-1}(x) \subset Y$ is finite.

Pf Since $\{x\} \subset X$ is compact and $f: Y \to X$ is proper, $Y_x \subset Y$ is compact. So it suffices to show that $Y_x \subset Y$ is discrete. But by the first theorem, for every $y \in Y_x$, f is represented by $z \longmapsto z^n$ with $n \geq 1$ in some chart around $y \in Y$. The fiber of $z \mapsto z^n$ at $z=0$ is $\{0\} \subset D$, so this neighborhood of $y \in Y_x \subset Y$ does not contain any other $y' \in Y_x$. //

<u>Lemma</u> Let $f: Y \to X$ be a proper map between Riemann surfaces, and let $x \in X$. If $Y_x \subset V \subset Y$ is open, then there exists $x \in U \subset X$ open s.t. $f^{-1}(U) \subset V$.

<u>Pf</u> We fix a chart around $x \in X$ and use this to define the notion of a "closed disc in X centered at x." Let $D \subset X$ be such a disc. Since D is compact and since $f: Y \to X$ is proper, $f^{-1}(D) \subset Y$ is compact. Now let $(D_i)_{i \in I}$ be the (self-indexed) family of all closed discs in X centered at x. We have $\bigcap_{i \in I} D_i = \{x\}$, as we can check in the chosen chart.

It follows that the family consisting of V and the $Y \setminus f^{-1}(D_i)$ is an open cover of Y. Hence, it is also an open cover of $f^{-1}(D)$, and since $f^{-1}(D)$ is compact, we can pick a finite subcover consisting of V and $Y \setminus f^{-1}(D_i)$ with $i \in I_0 \subset I$ finite. Now, if $U \subset X$ is any open disc centred at x and contained in D_i for all $i \in I_0$, then $f^{-1}(U) \subset V$ as desired. $\square$

__Thm__ Let $f: Y \to X$ be a proper map between Riemann surfaces which is not constant on any connected component of Y. Let $x \in X$, and let

$$Y_x = \{y_1, \dots, y_k\} \subset Y$$

be the fiber. There exists a chart around $x \in X$ (with x corresponding to $0 \in \mathbb{C}$) and charts around each $y_i \in Y$ (with y_i corresponding to $0 \in \mathbb{C}$) with the following property: The inverse image by f

of the chart around $x \in X$ is the disjoint union of the charts around the y_i, and moreover, in each chart around y_i, the map f is represented by

$$z \longmapsto z^{n_i}$$

for some $n_i \geq 1$. So $m_{y_i}(f) = n_i$.

Pf We use first theorem to choose charts around each y_i. Since Y is Hausdorff, we can shrink the charts if necessary to make them pairwise disjoint. By the second lemma, we can shrink the chart around $x \in X$ to make its inverse image by f be contained in the disjoint union of the charts around the y_i. Shrinking the latter further, we get equality. //

Cor Let $f : Y \to X$ be a non-constant proper map between connected Riemann surfaces. In this situation, the integer

$$\deg_x(f) = \sum_{y \in f^{-1}(x)} m_y(f)$$

is independent of $x \in X$.

(So we write $\deg(f)$ for this integer and call it the degree of $f : Y \to X$.)

Pf See notes.

Ex Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant polynomial function, and let $\bar{f} : \mathbb{P}^1 \to \mathbb{P}^1$ be the holomorphic map defined earlier. We have

$\deg_0(\bar{f})$ = number of roots of f counted with multiplicity

$$\deg_\infty(\bar{f}) = \deg(f).$$

So deduce fundamental thm of algebra!