Last time, we proved that for a holomorphic map $f: Y \to X$, not constant in a neighborhood of $y \in Y$, there exists open subsets $y \in V \subset Y$ and $x = f(y) \in U \subset X$ and charts $\varphi: U \to U' \subset \mathbb{C}$ and $\psi: V \to V' \subset \mathbb{C}$ with $\varphi(x) = 0 = \psi(y)$ and $f(V) \subset U$ and an integer $n \geq 1$ s.t.

$$\begin{tikzcd} V \arrow[r, "\psi"] \arrow[d, "f|_V"'] & V' \arrow[d, "z \mapsto z^n"] \\ U \arrow[r, "\varphi"] & U' \end{tikzcd}$$

commutes. (We concluded that if Y is connected and non-constant, then it is non-constant in a neighborhood of every point $y \in Y$.) We defined the multiplicity of $f: Y \to X$ at $y \in Y$ to be

$$m_y(f) = n$$

and noted that it is the integer s.t. f is n-to-1 in a punctured neighborhood of y, hence, it is independent of the choices made.

This time, we assign to a nonzero meromorphic function

$$X \xrightarrow{f} \mathbb{P}^1$$

a map $\nu(f): X \to \mathbb{Z}$ that to $x \in X$ assigns an integer $\nu_x(f) \in \mathbb{Z}$ that we call the order of vanishing of f at x. The integer can be negative, which means that f has a pole at x.

Given $x \in X$, we choose $x \in U \subset X$ open and a chart $\varphi : U \to U' \subset \mathbb{C}$ s.t. $\varphi(x) = 0$. In this situation, the unique map $F : U' \to \mathbb{P}^1$ that makes

$$\begin{tikzcd} U' \arrow[rr, "F"] & & \mathbb{P}^1 \\ & U \arrow[ul, "\varphi"] \arrow[ur, "f|U"'] & \end{tikzcd}$$

commute is a meromorphic fct. on U'. Writing $F = p/q$ with $p, q : U' \to \mathbb{C}$ holomorphic and $q \not\equiv 0$ and considering the power series expansions of p and q at $0 \in U'$, we conclude that there exists an integer $\nu \in \mathbb{Z}$ and a holomorphic fct. $G : U' \to \mathbb{C}$, which is nonzero in a neighborhood of $0 \in U'$ s.t.

$$F(z) = z^{\nu} \cdot G(z).$$

We define the order of vanishing of $f : X \to \mathbb{P}^1$ at $x \in X$ to be

$$\nu_x(f) = \nu.$$

To see that this integer does not depend on the choices made, we prove:

<u>Lemma</u> Let $f : X \to \mathbb{P}^1$ be a nonzero meromorphic function with X connected, and let $x \in X$.

1) If $f(x) \notin \{0, \infty\}$, then $\nu_x(f) = 0$.

2) If $f(x) = 0$, then $\nu_x(f) = m_x(f)$.

3) If $f(x) = \infty$, then $\nu_x(f) = -m_x(f)$.

<u>Pf</u> If $f(x) \notin \{0, \infty\}$, the we cannot write $F(z) = z^{\nu} \cdot G(z)$ with $G(0) \neq 0$

unless $\nu = 0$. If $f(x) = 0$ and if we write $F(z) = z^{\nu} \cdot G(z)$ with $G(0) \neq 0$, then the proof of the first thm. from last shows that $m_x(f) = \nu$. So $\nu_x(f) = m_x(f)$, as claimed. Finally, if $f(x) = \infty$, then, to find $m_x(f)$, we use the chart

$$\mathbb{P}^1 \setminus \{0\} \longrightarrow \mathbb{C}$$

$$z \longmapsto 1/z$$

around $\infty \in \mathbb{P}^1$. The same argument as in 2) shows that $m_x(f) = -\nu_x(f)$. //

It is helpful to also define $\nu_x(f) = \infty$ if $f = 0 \in \mathcal{M}(X)$ is the zero meromorphic function. So for every $x \in X$, we have a map

$$\mathcal{M}(X) \xrightarrow{\nu_x} \mathbb{Z} \cup \{\infty\}$$

which satisfies

$$\nu_x(f+g) = \min\{\nu_x(f), \nu_x(g)\}$$

$$\nu_x(f \cdot g) = \nu_x(f) \cdot \nu_x(g) .$$

In general, if K is a field, then a map $\nu : K \to \mathbb{Z} \cup \{\infty\}$ with these properties is called a valuation. If we choose any real number $0 < r < 1$, then the map

$$K \xrightarrow{|\cdot|} [0, \infty)$$

$$f \longmapsto r^{\nu(f)}$$

satisfies

a) $|f+g| \geq \max\{|f|, |g|\}$

b) $|f \cdot g| = |f| \cdot |g|$

c) $|f| = 0 \iff f = 0.$

We say that $|\cdot|$ is a nonarchimedean absolute value on K. Note that a) implies that all triangles are isoceles.

<u>Prop</u> Let $f : X \to \mathbb{P}^1$ be a nonzero meromorphic function on a compact connected Riemann surface. In this situation, $\nu_x(f)$ is

nonzero for only finitely many $x \in X$, and moreover,

$$\Sigma_{x \in X} \nu_x(f) = 0.$$

Pf If f is constant, then $\nu_x(f) = 0$ for all $x \in X$, since we have excluded $f = 0$ and $f = \infty$. So we may assume that f is non-constant. Since X is compact, f is a non-constant proper map between connected Riemann surfaces. So by the second theorem from last time, f has finite fibers. In particular, the fibers $f^{-1}(0) \subset X$ and $f^{-1}(\infty) \subset X$ are finite, so the first claim follows from the lemma. But we also have

$$\deg_0(f) = \deg_\infty(f),$$

or equivalently,

$$\begin{array}{ccc} \Sigma_{x \in f^{-1}(0)} m_x(f) & = & \Sigma_{x \in f^{-1}(\infty)} m_x(f) \\ \parallel & & \parallel \\ \Sigma_{x \in f^{-1}(0)} \nu_x(f) & = & -\Sigma_{x \in f^{-1}(\infty)} \nu_x(f). \end{array}$$

//

Ex If $f : \mathbb{C} \to \mathbb{C}$ is a non-zero polynomial function, and if $\bar{f} : \mathbb{P}^1 \to \mathbb{P}^1$ is its extension to a meromorphic fct. on $\mathbb{P}^1$, then

$$\nu_\infty(\bar{f}) = -\deg(f),$$

whereas, for $a \in \mathbb{C}$, the integer $\nu_a(\bar{f})$ is the multiplicity of $z = a$ as a root of f. So

$$\Sigma_{x \in \mathbb{P}^1} \nu_x(\bar{f}) = 0$$

again tells us that the number of roots of f counted with multiplicity is equal to the degree of f. //

Let X be a compact, connected Riemann surface. The abelian group $\mathrm{Div}(X)$ of maps $a : |X| \to \mathbb{Z}$ that are zero for all but finitely many $x \in X$ is called the group of Weil divisors on X. We have the map

$$\mathcal{M}(X)^* \xrightarrow{\mathrm{div}} \mathrm{Div}(X)$$

defined by

$$\mathrm{div}(f)(x) = v_x(f).$$

It takes values in the kernel of the group homomorphism

$$\mathrm{Div}(X) \xrightarrow{\deg} \mathbb{Z}$$

$$\Sigma_{x\in|X|} a_x x \longmapsto \Sigma_{x\in|X|} a_x.$$

We can ask if the sequence

$$M(X)^* \xrightarrow{\mathrm{div}} \mathrm{Div}(X) \xrightarrow{\deg} \mathbb{Z}$$

is exact?

Prop The answer to this question is "Yes" if and only if $X = \emptyset$ or $X \simeq \mathbb{P}^1$.

Pf Suppose then answer is "Yes." If $X \neq \emptyset$, then we can choose two distinct points $x_0, x_\infty \in X$. Let $a: |X| \to \mathbb{Z}$ be the divisor $x_0 - x_\infty$, that is, $a_{x_0} = 1$, $a_{x_\infty} = -1$, and $a_x = 0$ if $x \notin \{x_0, x_\infty\}$. Since

$\deg(a) = 1-1=0$, and since we assume the answer is "Yes," there exists a meromorphic fct.

$$X \xrightarrow{f} \mathbb{P}^1$$

s.t. $v_x(f) = a_x$ for all $x \in X$. We claim that f is an isomorphism. Since $f^{-1}(\infty) = \{x_\infty\}$, which has multiplicity 1, we have

$$\deg_\infty(f) = 1.$$

But then $\deg_s(f) = 1$ for all $s \in \mathbb{P}^1$, so f is bijective. Last time, we proved that a holomorphic bijection is an isomorphism, so f is an isomorphism.

Conversely, if $X = \emptyset$, then the answer is "Yes," so it remain to prove that the answer is "Yes" also for $X = \mathbb{P}^1$. Given $a \in \mathrm{Div}(X)$ with $\deg(a) = 0$, we define $f : \mathbb{C} \to \mathbb{C}$ to be the polynomial function

$$f(z) = \prod_{y \in \mathbb{C}} (z-y)^{a_y}$$

and let $\bar{f}: \mathbb{P}^1 \to \mathbb{P}^1$ be its extension to a meromorphic function on $\mathbb{P}^1$. For all $y \in \mathbb{P}^1 \setminus \{\infty\}$,

$$v_y(\bar{f}) = a_y$$

by the definition of f, so it remains to prove that

$$v_\infty(\bar{f}) = a_\infty .$$

But we have

$$\sum_{y \in \mathbb{P}^1} v_y(\bar{f}) = 0$$

and

$$\sum_{y \in \mathbb{P}^1} a_y = 0,$$

so this follows from what was already proved. //

Prop If X is a non-empty, compact, connected Riemann surface, then

$$0 \to \mathbb{C}^* \to M(X)^* \xrightarrow{\mathrm{div}} \mathrm{Div}(X)$$

is exact.

Pf If $f : X \to \mathbb{P}^1$ is constant, then $\mathrm{div}(f) = 0$. We must prove the converse. If f is non-constant and $\nu_x(f) = 0$ for all $x \in X$, then f is not surjective. But every non-constant map $f : X \to \mathbb{P}^1$ has $\deg(f) \geq 1$, so is surjective. //

Cor Every nonzero meromorphic function $f : \mathbb{P}^1 \to \mathbb{P}^1$ can be written uniquely as

$$f(z) = \lambda \cdot \prod_{y \in \mathbb{C}} (z-y)^{a_y}$$

for arbitrary $a \in \mathrm{Div}(\mathbb{C})$ and arbitrary $\lambda \in \mathbb{C} \setminus \{0\}$. Hence,

$$\mathcal{M}(\mathbb{P}^1) = \mathbb{C}(z)$$

is the field of rational fcts. in one variable over $\mathbb{C}$.

Next, we consider automorphisms of $\mathbb{P}^1$.

<u>Thm</u> If $a, b, c \in \mathbb{P}^1$ are three discint points, then there is a unique automorphism

$$\mathbb{P}^1 \xrightarrow{h} \mathbb{P}^1$$

s.t. $h(0) = a$, $h(1) = b$, and $h(\infty) = c$.

<u>Pf</u> We will construct $g = h^{-1}$, so $g(a) = 0$, $g(b) = 1$, and $g(c) = \infty$. Let $d = a - c \in \mathrm{Div}(\mathbb{P}^1)$. So $d_a = 1$, $d_c = -1$, and $d_x = 0$ for $x \notin \{a, c\}$. By theorem above, there exists a meromorphic fct. $g : \mathbb{P}^1 \to \mathbb{P}^1$ s.t. $\mathrm{div}(g) = d$, and g is an isomorphism. By design, we have $g(a) = 0$ and $g(c) = \infty$. Moreover, g is unique, up to mult. by a nonzero constant. But $g(b) \in \mathbb{P}^1 \setminus \{0, \infty\}$, so by fixing the constant, we can arrange, uniquely, that $g(b) = 1$. //

Cor Every automorphism of $\mathbb{P}^1$ is of the form

$$h(z) = \frac{az+b}{cz+d}$$

with $ad-bc \neq 0$. Moreover, a, b, c and d are unique, up to a common nonzero factor.

Cor Every automorphism of $\mathbb{P}^1$ has a fixed point. In particular, the only subgroup

$$\Gamma \subset \mathrm{Aut}(\mathbb{P}^1)$$

that acts properly discontinuously on $\mathbb{P}^1$ is $\Gamma = \{id\}$.

To prove the first corollary, we write $|\mathbb{P}^1|$ as the set of complex lines (or hyperplanes) in $\mathbb{C} \oplus \mathbb{C}$. The gives an obvious map

$$PGL_2(\mathbb{C}) \longrightarrow \mathrm{Aut}(\mathbb{P}^1)$$

and the corollary states that this map is an isomorphism.