Today, we will consider $\mathbb{P}^1$ as a Riemannian manifold and compare this structure to its structure as a Riemann surface. To define the notion of a Riemannian metric on a smooth manifold (without any hand-waving), we begin with a bit of algebra.

If R is a commutative ring, and if M is an R-module, then for any integer $n \geq 0$, we can form the n-fold tensor product

$$M^{\otimes_R n} = M \otimes_R \cdots \otimes_R M .$$

The symmetric group Σ_n acts on the n-fold tensor product by permuting the tensor factors. We define the n'th divided power of M to be the submod.

$$\Gamma^n_R(M) = (M^{\otimes_R n})^{\Sigma_n} \hookrightarrow M^{\otimes_R n}$$

fixed by the action of Σ_n, and we define the n'th symmetric power of M to be quotient

$$M^{\otimes_R n} \longrightarrow (M^{\otimes_R n})_{\Sigma_n} = \mathrm{Sym}^n_R(M)$$

by the action of Σ_n. There is an R-linear map

$$\begin{tikzcd}
(M^{\otimes_R n})_{\Sigma_n} \arrow[r, "\mathrm{Nm}"] \arrow[d, equal] & (M^{\otimes_R n})^{\Sigma_n} \arrow[d, equal] \\
\mathrm{Sym}^n_R(M) \arrow[r, "\mathrm{Nm}"] & \Gamma^n_R(M)
\end{tikzcd}$$

called the norm map. It is the unique R-linear map that makes the diagram

$$\begin{tikzcd}
(M^{\otimes_R n})_{\Sigma_n} \arrow[r] & (M^{\otimes_R n})^{\Sigma_n} \arrow[d] \\
M^{\otimes_R n} \arrow[u] \arrow[r] & M^{\otimes_R n}
\end{tikzcd}$$

$$x \longmapsto \sum_{g \in \Sigma_n} g \cdot x$$

commute. It is an isomorphism if $n!$ is invertible in R. Let

$$M^{\vee} = \mathrm{Hom}_R(M, R)$$

be the dual R-module of M.

There is a canonical isom.

$$\begin{tikzcd}
\mathrm{Sym}^n_R(M)^\vee \arrow[r, phantom, "="] \arrow[dd, equal] & \mathrm{Hom}_R((M^{\otimes_R n})_{\Sigma_n}, R) \arrow[d, "\sim"] \\
 & \mathrm{Hom}_R(M^{\otimes_R n}, R)^{\Sigma_n} \\
\Gamma^n_R(M^\vee) \arrow[r, phantom, "="] & (\mathrm{Hom}_R(M, R)^{\otimes_R n})^{\Sigma_n} \arrow[u, "\sim"']
\end{tikzcd}$$

Let $R = \mathbb{R}$ and let $M = V$ be a real vector space. A symmetric bilinear form on V is an element

$$(-,-) \in \mathrm{Sym}^2_{\mathbb{R}}(V)^\vee \simeq \Gamma^2_{\mathbb{R}}(V^\vee).$$

It is an inner product if

$$\langle v, v \rangle \geq 0$$

for all $v \in V$ and if

$$\langle v, v \rangle = 0 \iff v = 0.$$

If $\langle -, - \rangle$ is an inner product, then the associated norm is

$$\|v\| = \langle v, v \rangle^{1/2}$$

Moreover, if $v, w \in V$ are nonzero, then the formula

$$\cos(\theta) = \frac{\langle v, w \rangle}{\|v\| \cdot \|w\|}$$

defines the unoriented angle $\theta \in [0, \pi]$ between them.

Now, let

$$X = (|X|, \mathcal{O}_X)$$

be a smooth manifold, and let Ω^1_X be the sheaf of differential 1-forms on X. It is an $\mathcal{O}_X$-module, so we can form the $\mathcal{O}_X$-module

$$\Gamma^2_{\mathcal{O}_X}(\Omega^1_X).$$

Let $\langle -, - \rangle$ be a global section of this $\mathcal{O}_X$-module. Its value at $p \in X$ is an element

$$\langle -,- \rangle_p \in \Gamma^2_{\mathbb{R}}(T_p(X)^\vee)$$
$$\text{\rotatebox{90}{$\subseteq$}}$$
$$\mathrm{Sym}^2_{\mathbb{R}}(T_p(X))^\vee$$

so the following definition is meaningful.

<u>Def</u> A Riemannian metric on a smooth mfd. $X = (|X|, \mathcal{O}_X)$ is a global section

$$\langle -,- \rangle \in \Gamma(|X|, \Gamma^2_{\mathcal{O}_X}(\Omega^1_X))$$

such that the bilinear form

$$\langle -,- \rangle_p \in \mathrm{Sym}^2_{\mathbb{R}}(T_p(X))$$

is an inner product on $T_p(X)$ for all $p \in |X|$. //

<u>Rmk</u> 1) The definition of an hermitian metric on a complex mfd. is completely analogous, but obviously, we instead require that $\langle -,- \rangle_p$

be an hermitian inner product on $T_p(X)$ for all $p \in |X|$!

2) If $X = (|X|, \mathcal{O}_X)$ is a smooth mfd. with $|X|$ paracompact, then, using a partition of unity, one can prove that

$$\Gamma(|X|, \Gamma^2_{\mathcal{O}_X}(\Omega^1_X)) \simeq \Gamma^2_{\Omega^0(X)}(\Omega^1(X)),$$

where $\Omega^0(X) = \Gamma(|X|, \mathcal{O}_X)$ is the ring of smooth functions on X, and $\Omega^1(X) = \Gamma(|X|, \Omega^1_X)$ is the $\Omega^0(X)$-module of 1-forms on X. Hence, it is possible to avoid to use sheaves, but it makes the presentation less clear. //

Def Let $f: Y \to X$ be a map of smooth manifolds, and let $\langle -,- \rangle_X$ and $\langle -,- \rangle_Y$ be Riemannian metrics of X and Y, respectively. The map f is an isometry (resp. a conformal map) if the induced map

$$\Gamma(|X|, \Gamma^2_{\mathcal{O}_X}(\Omega^1_X)) \longrightarrow \Gamma(|Y|, \Gamma^2_{\mathcal{O}_Y}(\Omega^1_Y))$$

maps $\langle -,- \rangle_X$ to $\langle -,- \rangle_Y$ (resp. maps $\langle -,- \rangle_X$ to $\lambda^2 \cdot \langle -,- \rangle$ for some λ in $\Omega^0(Y) = \Gamma(|Y|, \mathcal{O}_Y)$). //

Let us spell this out. If $y \in Y$ with image $x = f(y) \in X$, then the derivative of f at y is an $\mathbb{R}$-linear map

$$T_y(Y) \xrightarrow{df_y} T_x(X).$$

The map f is an isometry if and only if for all $y \in Y$ and all $v, w \in T_y(Y)$,

$$\langle df_y(v), df_y(w) \rangle_{X,x}$$

$$= \langle v, w \rangle_{Y,y}$$

and it is conformal if there exists a smooth fct. $\lambda \in \Omega^0(Y)$ s.t. for all $y \in Y$ and $v, w \in T_y(Y)$,

$$\langle df_y(v), df_y(w) \rangle_{X,x}$$

$$= \lambda(y)^2 \langle v, w \rangle_{Y,y}.$$

Let us now see this in action! Let

$$S^2 = \{ v \in \mathbb{R}^3 \mid \|v\| = 1 \}$$

be the unit sphere. For $p \in \mathbb{R}^3$, we have a canonical identification (by translation)

$$T_p(\mathbb{R}^3) \simeq \mathbb{R}^3,$$

and the standard inner product on $\mathbb{R}^3$, using this identification, define a Riemannian metric on $\mathbb{R}^3$, which we also call the standard Riemannian metric. There is a unique Riemannian metric on S^2 s.t. the incl.

$$S^2 \xhookrightarrow{i} \mathbb{R}^3$$

is an isometry. The derivative

$$T_p(S^2) \xrightarrow{di_p} T_p(\mathbb{R}^3)$$

is injective, and we define the inner product on $T_p(S^2)$ to be the restriction of the inner product on $T_p(\mathbb{R}^3)$. Let

$$N = (0,0,1) \in S^2$$

be the north pole. We define a diffeomorphism

$$S^2 \xrightarrow{\pi} \mathbb{P}^1$$

given by stereographic projection from N. This means that we identify $x+iy \in \mathbb{C}$ with $(x,y,0) \in \mathbb{R}^3$ and define

$$S^2 \setminus \{N\} \xrightarrow{\pi} \mathbb{C}$$

to be the map that to a point $p \in S^2 \setminus \{N\}$ assigns the unique point $\pi(p) \in \mathbb{C}$ s.t. N, p, and $\pi(p)$ are colinear in $\mathbb{R}^3$. Clausen works out that

$$\pi(x,y,z) = \frac{x}{1-z} + i\,\frac{y}{1-z}$$

with inverse

$$\pi^{-1}(a+ib)$$
$$= \left(\frac{2a}{a^2+b^2+1}, \frac{2b}{a^2+b^2+1}, \frac{a^2+b^2-1}{a^2+b^2+1} \right)$$

and makes some interesting remarks about why these maps are given by rational functions.

<u>Thm</u> The map

$$S^2 \setminus \{N\} \xrightarrow{\pi} \mathbb{C}$$

is conformal with respect to the standard Riemannian metrics with conformal factor $\lambda \in \Omega^0(S^2 \setminus \{N\})$ given by

$$\lambda(x,y,z) = \frac{1}{1-z}.$$

<u>Pf</u> The map π extends to

$$\mathbb{R}^2 \times (\mathbb{R} \setminus \{1\}) \xrightarrow{\tilde{\pi}} \mathbb{C}$$

given by the same formula, and

the matrix that represents

$$T_p(\mathbb{R}^2 \times (\mathbb{R} \setminus \{1\})) \xrightarrow{d\tilde{\pi}_p} T_{\tilde{\pi}(p)}(\mathbb{C})$$

with respect to the standard bases is given by

$$\begin{pmatrix} \frac{1}{1-z} & 0 & \frac{x}{(1-z)^2} \\ 0 & \frac{1}{1-z} & \frac{y}{(1-z)^2} \end{pmatrix}.$$

By symmetry, it suffices to consider $p = (0, y, z)$, in which case

$$d\tilde{\pi}_p \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{1-z} \begin{pmatrix} a \\ b + \frac{y}{1-z} c \end{pmatrix}.$$

The subspace

$$T_p(S^2) \subset T_p(\mathbb{R}^2 \times (\mathbb{R} \setminus \{1\}))$$

consists of those vectors s.t.

$$ax + by + cz = 0,$$

and we wish to find $\lambda = \lambda(p)$ s.t. for all $v, w \in T_p(S^2)$,

$$\langle d\pi_p(v), d\pi_p(w)\rangle_{\pi(p)}$$
$$= \lambda(p)^2 \langle v, w\rangle_p .$$

Since both sides are linear in v and w, it suffices to let v and w be members of a basis of $T_p(S^2)$, and choosing a clever basis, we can simplify our calculations. Remember that $p = (0, y, z)$ so $x = 0$. We take

$$e = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 \\ -z \\ y \end{pmatrix}$$

and it suffices to show

$$\langle d\pi_p(e), d\pi_p(f)\rangle = 0$$

$$\| d\pi_p(e) \| = \| d\pi_p(f) \| .$$

In this case, we take $\lambda(p)$ to be the common number. We calc.

$$d\pi_p(e) = \frac{1}{1-z} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$d\pi_p(f) = \frac{1}{1-z}\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and conclude that π is conformal with conformal factor

$$\lambda(p) = \frac{1}{1-z}.$$