Last time, we saw that stereographic projection from the north pole,

$$S^2 \xrightarrow{\pi} \mathbb{P}^1$$

is conformal w.r.t. the (Riemannian) metric on S^2 obtained by restriction from the euclidean metric on $\mathbb{R}^3$ and the euclidean metric on $\mathbb{P}^1 \smallsetminus \{\infty\} = \mathbb{C}$. We now give $\mathbb{P}^1$ the "round" metric that makes this map an isometry. On $\mathbb{C} \subset \mathbb{P}^1$, it is given by

$$\langle -, - \rangle_z^{\text{round}} = \left(\frac{2}{1+|z|^2} \right)^2 \cdot \langle -, - \rangle_z^{\text{std}}.$$

As Clausen points out, maps the great circle on S^2 that lies in the (x,z)-plane bijectively onto the circle $\mathbb{R} \cup \{\infty\} \subset \mathbb{C} \cup \{\infty\} = \mathbb{P}^1$. This tells us that

$$\int_{-\infty}^{\infty} \frac{2}{1+t^2} \, dt = 2\pi,$$

which is indeed the case. We will now prove two theorems, exhibiting the group $\mathrm{Isom}^+(\mathbb{P}^1)$ of automorphisms of $\mathbb{P}^1$ as an oriented Riemannian mfd. and comparing it to the group $\mathrm{Aut}(\mathbb{P}^1)$ of automorphisms of $\mathbb{P}^1$ as a Riemann surface. The stereographic projection, being an isometry, induces an isomorphism

$$\mathrm{Isom}^+(S^2) \longrightarrow \mathrm{Isom}^+(\mathbb{P}^1)$$

$$g \longmapsto \pi g \pi^{-1}$$

<u>Thm 1</u> The orientation-preserving isometries of S^2 are exactly the rotations around some line in $\mathbb{R}^3$ through the origin. /

It is not geometrically clear that the composition of a rotation around a line L_1 and a rotation around a line L_2 is a rotation around some line L_3. We prove, algebraically, that this is in fact the case:

Prop A map $f: \mathbb{R}^3 \to \mathbb{R}^3$ is a rotation around a line in $\mathbb{R}^3$ through the origin if and only if it is a linear isometry and its determinant is positive.

Pf A rotation around a line is a linear isometry and it is orientation-preserving, so its determinant is positive. Conversely, let $f: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear isometry with $\det(f) > 0$, and let

$$p(t) = \det(\mathrm{id} - t \cdot f)$$

be its characteristic polynomial. Since $p(t)$ is a real polynomial of degree 3, it has 3 complex roots, counted with multiplicity, and the subset of $\mathbb{C}$ consisting of the roots of p is stable under conjugation. Thus, there are two possibilities:

1) All three roots are real.
2) One root is real, and the other two form a pair of conjugate non-real complex numbers.

In either case, $p(t)$ has a positive real root, since $\det(f) > 0$ is the product of the three roots. So let $0 \neq v_0 \in \mathbb{R}^3$ be an eigenvector for f with positive real eigen value $\lambda > 0$. So $f(v_0) = \lambda v_0$, and since f is a linear isometry, we see that $\lambda = 1$. Hence, f fixes the line $L \subset \mathbb{R}^3$ spanned by v_0. It also preserves the decomposition

$$L \oplus L^\perp \xrightarrow{\sim} \mathbb{R}^3$$

and restricts to a linear isometry of the 2-dimensional subspace $L^\perp$. Finally, the matrix that represents $f|_{L^\perp}$ w.r.t. of a basis of $L^\perp$ that is orthonormal w.r.t. the inner product necessarily is of the form

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for some θ. So f is a rotation around L. //

Since $S^2 \subset \mathbb{R}^3$ is the subset of vectors of length 1, a linear isometry of $\mathbb{R}^3$ restricts to an isometry of S^2, and if its determinant is positive, then it is orientation-preserving. So we have a group homorphism

$$SO(3) \longrightarrow \mathrm{Isom}^+(S^2)$$
$$f \longmapsto f|_{S^2}$$

We will conclude later that it is an isomorphism as a consequence of the following theorem.

<u>Thm 2</u> Let $K, A, N \subset \mathrm{Aut}(\mathbb{P}^1)$ be the following subsets: K is the set of automorphisms of the form $\pi f \pi^{-1}$ with $f \in SO(3)$; A is the set of automorphisms of the form $z \mapsto \lambda \cdot z$ with $\lambda > 0$ real; and N is the set of automorphisms of the form $z \mapsto z + a$ with a a complex number. In this situation, the map

$$K \times A \times N \longrightarrow \mathrm{Aut}(\mathbb{P}^1)$$
$$(k, a, n) \longmapsto k \circ a \circ n$$

is a bijection.

Rmk This is the Iwasawa decomposition of the Lie group $\mathrm{Aut}(\mathbb{P}^1)$. //

Pf We have proved earlier that for every triple (p, q, r) of distinct points in $\mathbb{P}^1$, there exists a unique automorphism

$$\mathbb{P}^1 \xrightarrow{g} \mathbb{P}^1$$

s.t. $g(0) = p$, $g(1) = q$ and $g(\infty) = r$. So we must show that there exist unique $k \in K$, $a \in A$, and $n \in N$ s.t. $g = k \circ a \circ n$ has this property.

We first prove existence. We can find a rotation $f \in K$ s.t. $f(r) = \infty$. Let $z = f(p)$ and $w = f(q)$. We have $z, w \in \mathbb{C} = \mathbb{P}^1 \setminus \{\infty\}$, because f is a bijection, so there are unique $s \in \mathbb{C} \setminus \{0\}$ and $n \in N$ s.t.

$$s \cdot n(0) = z$$
$$s \cdot n(1) = w$$
$$s \cdot n(\infty) = \infty,$$

namely, $s = w - z$ and $n = \frac{z}{s} + \mathrm{id}$. We now let $\lambda = |s|$ and write

$$s = s/\lambda \cdot \lambda .$$

Since $s/\lambda \cdot \mathrm{id}$ is a rotation, the proposition shows that

$$k = f^{-1} \circ (s/\lambda \cdot \mathrm{id}) \in K,$$

and since $a = \lambda \cdot \mathrm{id} \in A$, the automorphism

$$g = k \circ a \circ n$$

maps $(0, 1, \infty)$ to (p, q, r) as we wanted. For uniqueness, one must show that a different choice of rotation $f \in K$ with $f(r) = \infty$ leads to the same (k, a, n). See Clausen's notes or prove it yourself. //

Pf of Thm 1 We can now prove that

$$SO(3) \longrightarrow \mathrm{Isom}^+(S^2)$$

is an isomorphism. The composition of this map and the inclusion

$$\mathrm{Isom}^+(S^2) \simeq \mathrm{Isom}^+(\mathbb{P}^1)$$

$$\hookrightarrow \mathrm{Aut}(\mathbb{P}^1)$$

has image the subgroup K. Hence, it suffices to prove that if

$$g = k \circ a \circ n$$

is an automorphism of $\mathbb{P}^1$, and if g is an isometry for the round metric, then $a \circ n = \mathrm{id}$. But k is an isometry, and hence, so is

$$a \circ n = k^{-1} \circ g.$$

But $a \circ n$ can only be an isometry if $a = \mathrm{id} = n$. //

In conclusion, we have identified

$$\mathrm{Isom}^+(\mathbb{P}^1) \hookrightarrow \mathrm{Aut}(\mathbb{P}^1)$$

with the inclusion

$$SO(3) \longrightarrow PGL_2(\mathbb{C}).$$

This is a strict inclusion with

$$SO(3) \backslash PGL_2(\mathbb{C}) \xrightarrow{\sim} AN.$$

$$\mathfrak{h} \longrightarrow D$$
$$z \longmapsto \frac{z-i}{z+i}$$

and its inverse

$$D \longrightarrow \mathfrak{h}$$
$$z \longmapsto (-i)\frac{z+i}{z-i}$$

We note that the Cayley transf. takes 0, i, and ∞ to -1, 0, and 1, respectively. On $\mathfrak{h}$, the hyperbolic metric becomes

$$\langle -,- \rangle_z^{hyp} = \frac{1}{\mathrm{Im}(z)^2} \langle -,- \rangle_z^{std}$$

It is convenient to use both D and $\mathfrak{h}$ as models. We now define three families of isometries of D or, equivalently, $\mathfrak{h}$.

1) (Parabolic) For every $a \in \mathbb{R}$, the map $z \longmapsto z+a$ is an isometry of $\mathfrak{h}$.

2) (Hyperbolic) For every $\lambda \in \mathbb{R}_{>0}$, the map $z \longmapsto \lambda \cdot z$ is an isometry of $\mathfrak{h}$.

Next, we define the hyperbolic (Riemannian) metric on

$$D = \{ z \in \mathbb{C} \mid |z| < 1 \}$$

and show that orientation-pres. isometries for this metric are automorphisms of D as a Riemann surface. This gives

$$\mathrm{Isom}^+(D) \longrightarrow \mathrm{Aut}(D).$$

We then show that this map is an isomorphism!

The hyperbolic metric on D is

$$\langle -,- \rangle_z^{hyp} = \left(\frac{2}{1-|z|^2} \right)^2 \langle -,- \rangle_z^{std},$$

so it differs from the round metric only in the sign $-|z|^2$ instead of $+|z|^2$. On the problem set for this week, you will show that the unit disc D and the upper half-plane $\mathfrak{h}$ are isomorphic as Riemann surfaces via the Cayley transform

3) (Elliptic) For every $u \in \mathbb{C}$ with $|u| = 1$, the map $z \longmapsto u \cdot z$ is an isometry of D.

It is clear that these are all isometries for the hyperbolic metric, and it is also clear that they are orientation-pres. It is also clear that we have a group homomorphism

$$\mathrm{Isom}^+(D) \longrightarrow \mathrm{Aut}(D).$$

Indeed, the hyperbolic metric is defined by scaling the euclidean metric, so every isometry will preserve angles, and orientation-pres. ones will preserve oriented angles; thus, these are holomorphic maps. Now, here are the basic theorems:

<u>Thm 3</u> If $K, A, N \subset \mathrm{Aut}(D)$ are the subgroups of elliptic, parabolic, and hyperbolic isometries, then

$$K \times A \times N \longrightarrow \mathrm{Aut}(D)$$
$$(k, a, n) \longmapsto k \circ a \circ n$$

is an isomorphism. /

In particular, the map

$$\mathrm{Isom}^+(D) \longrightarrow \mathrm{Aut}(D)$$

is an isomorphism.

<u>Thm</u> 4 Given $h \in \mathrm{Aut}(D)$, there exists $g \in \mathrm{Aut}(D)$ s.t. ghg^{-1} is in K, A, or N. /

Let $GL_2^+(\mathbb{R}) \subset GL_2(\mathbb{R})$ be the subgroup of invertible matrices with positive determinant, and

$$PGL_2^+(\mathbb{R}) = GL_2^+(\mathbb{R}) / \mathbb{R}_{>0} \cdot I$$

<u>Thm</u> 5 The map

$$PGL_2^+(\mathbb{R}) \longrightarrow \mathrm{Aut}(\mathfrak{h})$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbb{R}_{>0} \longmapsto \left(z \longmapsto \frac{az+b}{cz+d} \right)$$

is a group isomorphism. /

As a consequence of Thm. 4, every automorphism of $\mathbb{D}$ or, equivalently, $\mathcal{H}$ extends uniquely to an automorphism of $\mathbb{P}^1$. Indeed,

$$PGL_2^+(\mathbb{R}) \subset PGL_2(\mathbb{C}).$$

I will not prove these results, but you can find the proofs in Clausen's Lecture 10.