The uniformization theorem states that every non-empty connected Riemann surface is of the form $X \cong \Gamma \backslash Y$, where $Y \cong \mathbb{P}^1, \mathbb{C}$, or $\mathbb{D}$, and where

$$\Gamma \subset \mathrm{Aut}(Y)$$

acts properly discontinuously on Y. In particular, the action by Γ on Y must be free, so this forces

$$\Gamma \subset \mathrm{Isom}^+(Y) \subset \mathrm{Aut}(Y),$$

since every automorphism of Y that is not an isometry has a fixed point. (This occurs only for $Y = \mathbb{P}^1$ and $Y = \mathbb{C}$.) Thus, the requirement that Γ act properly discontinuously is equivalent to the requirement for every $y \in Y$,

$$\inf \{ d(y, \gamma(y)) \mid \gamma \in \Gamma \smallsetminus \{1\} \}$$

is attained for some $\gamma \in \Gamma \smallsetminus \{1\}$.

In the case $Y = \mathbb{P}^1$, the only possibility is $\Gamma = \{1\}$.

In the case $Y = \mathbb{C}$, we have

$$\Gamma \subset \mathrm{Isom}^+(\mathbb{C}) = \mathbb{C}$$

and the requirement is that Γ be a discrete subgroup. (The additive group $\mathbb{C}$ acts by translations.) To classify the possible such $\Gamma \subset \mathbb{C}$, we prove a more general result.

Prop If $\Gamma \subset \mathbb{R}^n$ is a discrete subgroup, then there exists a linearly independent family of vectors $(v_1, \ldots, v_r)$ in $\mathbb{R}^n$ s.t.

$$\Gamma = \mathrm{span}_{\mathbb{Z}}(v_1, \ldots, v_r) \subset \mathbb{R}^n$$

Pf We proceed by induction on $n \geq 0$, the case $n = 0$ being trivial. So we let $d \geq 1$ and assume that the statement has been proved for $n < d$ and prove it for $n = d$. If $\Gamma = \{0\}$, then $\Gamma = \mathrm{span}_{\mathbb{Z}}(\emptyset)$, so we are done. If $\Gamma \neq \{0\}$, then we can choose $0 \neq \gamma \in \Gamma$, and since $\Gamma \subset \mathbb{R}^d$ is discrete, we can assume that γ is chosen such that $d(0,\gamma) > 0$ is minimal. This choice implies that the map

$$\Gamma / \mathbb{Z}\cdot\gamma \longrightarrow \mathbb{R}^d / \mathbb{R}\cdot\gamma$$

is injective and that its image is a discrete subgroup of the quotient vector space, which is of dimension $d-1$. By the inductive hypothesis, we have

$$\Gamma / \mathbb{Z}\cdot\gamma = \mathrm{span}_{\mathbb{Z}}(\bar{v}_1, \dots, \bar{v}_r) \subset \mathbb{R}^d / \mathbb{R}\cdot\gamma$$

with $(\bar{v}_1, \dots, \bar{v}_r) \subset \mathbb{R}^d / \mathbb{R}\cdot\gamma$ linearly independent. We now choose $x_i \in \mathbb{R}^d$ s.t. $\bar{v}_i = v_i + \mathbb{R}\cdot\gamma$ for all $1 \le i \le r$. Then

$$(v_1, \dots, v_r, \gamma) \subset \mathbb{R}^d$$

is linearly independent and

$$\Gamma = \mathrm{span}_{\mathbb{Z}}(v_1, \dots, v_r, \gamma) \subset \mathbb{R}^d,$$

which proves the induction step. //

So the discrete subgroups $\Gamma \subset \mathbb{C}$ are free of rank $r = 0, 1$, or 2. If $r = 0$, then $\Gamma = \{0\}$, so $\Gamma \backslash \mathbb{C}$ is $\mathbb{C}$. If $r = 1$, then $\Gamma = \mathbb{Z} \cdot v$ for some $0 \neq v \in \mathbb{C}$. If $v, v' \in \mathbb{C}$ are nonzero, then so is $\lambda = v/v'$ and multiplication by λ gives an isomorphism

$$\mathbb{C}/\mathbb{Z}\cdot v' \xrightarrow{\sim} \mathbb{C}/\mathbb{Z}\cdot v\,,$$

so these Riemann surfaces are all isomorphic. Moreover, for $v = 2\pi i$, we have

$$\mathbb{C}/2\pi i\,\mathbb{Z} \xrightarrow[\sim]{\exp} \mathbb{C} \setminus \{0\}.$$

Finally, for $r = 2$, we have

$$\Gamma = \mathbb{Z}\cdot(v_1, v_2) \subset \mathbb{C}$$

with (v_1, v_2) an $\mathbb{R}$-linearly independent pair. Let

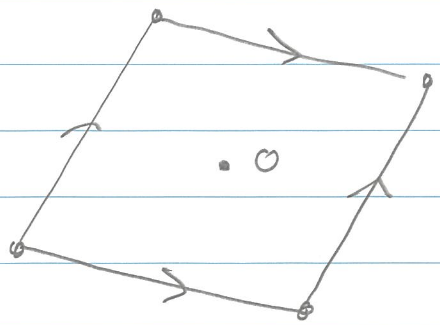
$$\bar{P} = \{\lambda_1 v_1 + \lambda_2 v_2 \in \mathbb{C} \mid -\tfrac{1}{2} \leq \lambda_1, \lambda_2 \leq \tfrac{1}{2}\}$$

be a fundamental domain for

the action by Γ on $\mathbb{C}$, centered at $0 \in \mathbb{C}$. The composition

$$\bar{P} \hookrightarrow \mathbb{C} \longrightarrow \mathbb{C}/\Gamma$$

of the canonical inclusion and the canonical projection is surjective, and if $z, w \in \bar{P}$ map to the same point in $\mathbb{C}/\Gamma$, then z and w lie on opposing sides in the parallelogram $\bar{P}$ and differ by a translation by v_1 or v_2. So the underlying smooth manifold of $\mathbb{C}/\Gamma$ is a torus:



In particular, $\mathbb{C}/\Gamma$ is compact, since it is the continuous image of the compact space $\bar{P} \subset \mathbb{C}$. By the algebraization theorem, $\mathbb{C}/\Gamma$ is determined by its field $\mathcal{M}(\mathbb{C}/\Gamma)$ of meromorphic functions. We proceed to describe this field.

Prop Let $\Gamma \subset \mathbb{C}$ be a discrete subgrp. of rank 2. There exists a unique Γ-invariant meromorphic function $\wp : \mathbb{C} \to \mathbb{P}^1$ s.t.

1) The set of poles of $\wp$ is exactly $\Gamma \subset \mathbb{C}$.

2) The Laurent series expansion of $\wp$ at $0 \in \mathbb{C}$ has the form

$$\wp(z) = \frac{1}{z^2} + (\text{terms of deg.} \geq 1).$$

(Hence, as a meromorphic fct.

$$\mathbb{C}/\Gamma \xrightarrow{\wp} \mathbb{C},$$

the only pole is $0 \in \mathbb{C}/\Gamma$ and it has depth 2.)

Pf Uniqueness: Suppose $\wp$ and p are two Γ-inv. merom. fct. s.t. 1) and 2) hold. Then $\wp - p$ gives a meromorphic fct.

$$\mathbb{C}/\Gamma \xrightarrow{\wp - p} \mathbb{P}^1$$

with no poles, since the $\frac{1}{z^2}$ terms cancel. So $\wp - p$ is a holomorphic map between connected Riemann surfaces, and its fiber over ∞ is empty. If $\wp - p$ is non-constant, then every fiber must be empty, but this is impossible. So $\wp - p$ is constant and $(\wp - p)(0) = 0$ since $\wp$ and p have constant term 0 at 0. This proves that $\wp = p$.

Existence: First guess is

$$\wp(z) = \sum_{\gamma \in \Gamma} \frac{1}{(z-\gamma)^2},$$

but this does not work, since the series does not converge. However, it does work for the would-be-derivative

$$\wp'(z) = \sum_{\gamma \in \Gamma} \frac{-2}{(z-\gamma)^3}.$$

Indeed, this sum converges uniformly on $\bar{D}(0,r)$ for all $r > 0$ and extends to a meromorphic function on $\mathbb{C}$, because

$$\int_{\mathbb{C}} \frac{1}{|z|^3} \, |dz| < \infty.$$

The set of poles of $\wp'$ is exactly $\Gamma \subset \mathbb{C}$, and its Laurent series expansion at $0 \in \mathbb{C}$ takes the form

$$\wp'(z) = \frac{-2}{z^3} + (\text{terms of deg.} \geq 1).$$

Moreover, the terms in deg. ≥ 1 only occur in odd degrees, so $\wp'$ is a odd function in the sense that $\wp'(-z) = -\wp'(z)$. Now, in the open parallelogram

$$Q = \{\lambda_1 v_1 + \lambda_2 v_2 \mid -1 < \lambda_1, \lambda_2 < 1\} \subset \mathbb{C},$$

the only pole of $\wp'$ is $z = 0$. So there is a unique anti-derivative $\wp$ of $\wp'$ defined on Q with constant term 0. This $\wp$ is given by integrating the Laurent series expansion of $\wp'$ term-wise, so it satisfies 2). Hence, it remains only to show that $\wp$ extends to a Γ-invariant meromorphic function on $\mathbb{C}$. It suffices to show that if both $z, z+v_1 \in Q$, then

$$\wp(z+v_1) = \wp(z)$$

and similarly with v_2. Indeed, granting this, we can inductively extend $\wp$ to all of $\mathbb{C}$ by using translation to move to neighboring parallelograms. So we consider the function $\wp(z+v_1)-\wp(z)$. It is constant, equal to, say, C, since it is defined on a connected region and its derivative is 0. But it is also even, because $\wp'$ is odd. So

$$C = \wp(z+v_1)-\wp(z)$$

$$= \wp(-z-v_1)-\wp(-z)$$

$$= \wp(w)-\wp(w+v_1) = -C$$

with $w=-z-v_1$. So $C=0$. //

Rmk The function $\wp$ that we constructed is even. But this is also a consequence of the uniqueness part. Indeed, if $\wp(z)$ is Γ-invariant and satisfies 1) and 2), then so does $\wp(-z)$. //

We view $\wp$ as a hol. map

$$\mathbb{C}/\Gamma \xrightarrow{\wp} \mathbb{P}^1.$$

Only $0 \in \mathbb{C}/\Gamma$ is mapped to $\infty \in \mathbb{P}^1$, and $\wp$ has mult. 2 at 0. So counted with multiplicity, every fiber of $\wp$ has 2 points. In particular, $\wp$ is surjective.

Lemma For all $z, w \in \mathbb{C}/\Gamma$, $\wp(z) = \wp(w)$ if and only if $z = \pm w$.

Pf If $z = \pm w$, then $\wp(z) = \wp(w)$, because $\wp$ is even. Conversely, if $\wp(z) = \wp(w)$, then $\{z, -z, w\} \subset \mathbb{C}/\Gamma$ is mapped to single pt. by $\wp$. So either $z = w$, $-z = w$, or $z = -z$. In the first two cases, we are done, so suppose $z = -z$. Given a (small) $\varepsilon \in \mathbb{C}$, let $z_0 = z + \varepsilon$. Since $z_0 = z + \varepsilon$ and $-z_0 = z - \varepsilon$ are mapped to the same pt. by $\wp$, we conclude that $\wp$ is not a local isomorphism at z. So it has mult. > 1 at z. But then this mult. must be 2, and hence, the fiber over $\wp(z)$ only has one pt. (with mult. 2), So $z = w$. //

<u>Rmk</u> The proof of the lemma shows that the points in $\mathbb{C}/\Gamma$ of mult. 2 is equal to the subset

$$\mathbb{C}/\Gamma[2] \subset \mathbb{C}/\Gamma$$

of 2-torsion pts. This is

$$\mathbb{C}/\Gamma[2] = \left\{0, \frac{v_1}{2}, \frac{v_2}{2}, \frac{v_1+v_2}{2}\right\}. \quad //$$

The pt. $0 \in \mathbb{C}/\Gamma[2]$ maps to $\infty \in \mathbb{P}^1$, so the remaining three pts. map to distinct pts. in $\mathbb{C}$. We write

$$S = \wp(\mathbb{C}/\Gamma[2] \setminus \{0\})$$

$$= \left\{\wp\left(\frac{v_1}{2}\right), \wp\left(\frac{v_2}{2}\right), \wp\left(\frac{v_1+v_2}{2}\right)\right\}$$

for this subset of $\mathbb{C}$. (We will see later that $\sum_{e \in S} e = 0$.)

We can now give a complete desciption of the field of meromorphic functions on $\mathbb{C}/\Gamma$.

<u>Thm</u> If $\Gamma \subset \mathbb{C}$ is a rank 2 lattice, then the map

$$\mathbb{C}(z)\left(\sqrt{4\cdot\prod_{e\in S}(z-e)}\right)$$

$$\longrightarrow \mathcal{M}(\mathbb{C}/\Gamma)$$

that maps

$$z \longmapsto \wp$$

$$\sqrt{4\cdot\prod_{e\in S}(z-e)} \longmapsto \wp'$$

is well-defined and an isom./

We must show that given $f \in \mathcal{M}(\mathbb{C}/\Gamma)$, there are unique $A, B \in \mathbb{C}(z)$ s.t.

$$f = A(\wp) + B(\wp)\cdot\wp',$$

and using that mult. by $\wp'$ gives an isomorphism

$$\mathcal{M}(\mathbb{C}/\Gamma)^{ev} \xrightarrow{\sim} \mathcal{M}(\mathbb{C}/\Gamma)^{od},$$

it suffices to show that for all $f \in M(\mathbb{C}/\Gamma)^{ev}$, there exists a unique $A \in \mathbb{C}(z)$ s.t. $f = A(\wp)$.
Given this, we must show that

$$(\wp')^2 = 4 \cdot \prod_{e \in S} (\wp - e).$$

I will ask you to read the proof in Clausen's Lecture 12.
In fact, one finds that

$$(\wp')^2 = 4\wp^3 - g_2 \wp - g_3$$

with

$$g_2 = 60 \cdot \sum_{\gamma \in \Gamma \setminus \{0\}} \frac{1}{\gamma^4}$$

$$g_3 = 140 \cdot \sum_{\gamma \in \Gamma \setminus \{0\}} \frac{1}{\gamma^6}.$$

The fact that there is no quadratic term is the statement

$$\sum_{e \in S} e = 0.$$