Given a Riemann surface X and a point $x \in X$, we will construct a map of Riemann surfaces

$$\tilde{X} \xrightarrow{p} X$$

that we call the universal cover of X at x. We will do so in two ways, the classical way, using paths in X, and the modern way using stacks. Both constructions begin with the definition of the fundamental groupoid $\pi_{\leq 1}(X)$ of X. The main difference is that, in stacks, X and $\pi_{\leq 1}(X)$ can interact with each other in the sense that there is a map of stacks $f : X \to \pi_{\leq 1}(X)$ for which the desired map is the fiber at x. So

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[d, "p"] & 1 \arrow[d, "x"] \\ X \arrow[r, "f"] & \pi_{\leq 1}(X) \end{tikzcd}$$

is a cartesian diagram of stacks, making the dependence on x clear.

Def Let X be a Riemann surface, and let $x, y \in X$. A path from y to x is a piecewise continuously differentiable map

$$[0,1] \xrightarrow{\gamma} X$$

s.t. $\gamma(0) = y$ and $\gamma(1) = x$. If both γ and γ' are paths from y to x, then a homotopy from γ' to γ is a piecewise continuously differentiable map

$$[0,1] \times [0,1] \xrightarrow{h} X$$

s.t. $h(0,-) = \gamma'$, $h(1,-) = \gamma$, $h(-,0) = y$, and $h(-,1) = x$. ✓

Homotopy is an equivalence relation on the set of paths from y to x in X, and we write

$$\pi_X(y,x)$$

for the set of homotopy classes of paths from y to x in X. It turns out that, while the set of from y to x is a "big" uncountable set, its quotient $\pi_X(y,x)$ by the equivalence relation of homotopy is typically countable. This is a hint that the mathematical nature of X and $\pi_X(y,x)$ is rather different.

Ex 1) If X is a convex subset of $\mathbb{C}$, e.g. $X = D$, then for all $x, y \in X$, $\pi_X(y,x)$ has a single element given by the homotopy class of the line segment

$$\gamma(t) = (1-t)x + ty.$$

connecting x and y. Indeed, if γ and γ' are any two paths from y to x, then

$$h(s,t) = (1-s)\gamma'(t) + s\gamma(t)$$

is a homotopy from γ' to γ.

2) If $X = \mathbb{P}^1$, then again $\pi_X(y,x)$ has a single element for all $x, y \in X$. Indeed, a path $\gamma : [0,1] \to \mathbb{P}^1$ must miss a point $z \in \mathbb{P}^1$, and $\mathbb{P}^1 \setminus \{z\} \simeq \mathbb{C}$ is convex.

3) If $X = \mathbb{C} \setminus \{0\}$, then $\pi_X(y,x)$ can be identified, non-canonically, with $\mathbb{Z}$. Informally, $n \in \mathbb{Z}$ corresp. to a path from y to x that wraps around $0 \in \mathbb{C}$ n times.

If $\pi_X(y,x)$ consists of a single element for all $x, y \in X$, then we say that X is simply connected (or 2-connective). Informally, X is simply connected if there are no holes in X. So $\mathbb{C}$ and $\mathbb{P}^1$ are simply connected, by $\mathbb{C} \setminus \{0\}$ is not.

Given $x, y, z \in X$, we have a map

$$\pi_X(y,x) \times \pi_X(z,y) \xrightarrow{\circ} \pi_X(z,x)$$

that to the pair of the homotopy class of the path $\gamma : [0,1] \to X$ from y to x and the homotopy class of the path $\mu : [0,1] \to X$ from z to y assigns the homotopy class of the path $\gamma \cdot \mu : [0,1] \to X$ from z to x defined by

$$(\gamma\cdot\mu)(t) = \begin{cases} \mu(2t) & 0 \le t \le 1/2 \\ \gamma(2t-1) & 1/2 \le t \le 1. \end{cases}$$

This composition law satisfies:

1) (Associativity) Given $f \in \pi_X(y,x)$, $g \in \pi_X(z,y)$, and $h \in \pi_X(w,z)$,

$$f \circ (g \circ h) = (f \circ g) \circ h$$

in $\pi_X(w,x)$.

2) (Unitality) For all $x \in X$, there exists $id_x \in \pi_X(x,x)$ s.t. for all $f \in \pi_X(y,x)$ and $g \in \pi_X(x,y)$,

$$id_x \circ g = g \quad \text{and}$$

$$f \circ id_x = f.$$

3) (Inverse) Given $f \in \pi_X(y,x)$, there exists $g \in \pi_X(x,y)$ s.t.

$$f \circ g = id_x \quad \text{and}$$

$$g \circ f = id_y.$$

Exercise: Write down the homotopies

that verify 1) – 3).

The element $\mathrm{id}_x \in \pi_X(x,x)$ in 2) is unique with property, and so is the element $g \in \pi_X(x,y)$ in 3). So we also write $f^{-1} = g$.

Now, our first definition of the fundamental groupoid of X is that it is the groupoid

$$\pi_{\leq 1}(X)$$

with objects the points $x \in X$, with maps from y to x given by the set $\pi_X(y,x)$, and with composition

$$\pi_X(y,x) \times \pi_X(z,y) \xrightarrow{\circ} \pi_X(z,x).$$

The properties 1) – 3) show that this is indeed a groupoid. In particular, if $x \in X$, then

$$\Gamma_x = \pi_X(x,x)$$

is a group under composition. We call $\pi_X(x,x)$ the fundamental group of X at x. Moreover, as you will on the problem set, if the set $\pi_X(y,x)$ is non-empty, then it is simultaneously a torsor for left action by Γ_x and for right action by Γ_y.

We now give the classical definition of the universal cover of X at $x \in X$.

<u>Def</u> Let X be a Riemann surface, and let $x \in X$ be a point. Let $\tilde{X}$ be the set of homotopy classes of paths $\gamma : [0,1] \longrightarrow X$ such that $\gamma(1) = x$ and such that homotopies between paths fix the end-points, and let

$$\tilde{X} \xrightarrow{p} X$$

be the map that to the homotopy class of γ assigns $\gamma(0) \in X$.

<u>Rmk</u> For all $y \in X$, we have

$$p^{-1}(y) = \pi_X(y,x).$$

Let $U \subset X$ be simply connected. Given $f_1, f_2 \in p^{-1}(U)$ with $p(f_i) = y_i$, we say that f_1 and f_2 are horizontally situated if

$$f_1^{-1} \circ f_2 \in \pi_X(y_2, y_1)$$

can be represented by a path in U. This is an equivalence relation on $p^{-1}(U)$, and we let

$$p^{-1}(U) \xrightarrow{q_U} S_U$$

be the quotient.

<u>Lemma</u> In this situation, the map

$$p^{-1}(U) \xrightarrow{(q_U, p)} S_U \times U$$

is a bijection.

<u>Pf</u> Given $y \in U$ and $f_1 \in p^{-1}(U)$, we must show that there is a unique $f_2 \in p^{-1}(y)$ s.t. f_1 and f_2 are horizontally situated. Let $y_1 = p(f_1)$. Since U is simply connected (and non-empty — if $U = \emptyset$, there is no —

thing to show), there is a unique

$$g \in \pi_U(y, y_1) \subset \pi_X(y, y_1)$$

so $f_2 = f_1 \circ g \in \pi_X(y, x)$ and $f_1 \in \pi_X(x, y_1)$ are horizontally situated and f_2 is unique with this property. ∎

If both $\emptyset \neq V \subset U \subset X$ are simply connected, then the can. map

$$S_V \longrightarrow S_U$$

is a bijection, and the diagr.

$$\begin{array}{ccc} p^{-1}(V) & \hookrightarrow & p^{-1}(U) \\ \downarrow{\scriptstyle (q_V, p)} & & \downarrow{\scriptstyle (q_U, p)} \\ S_V \times V & \hookrightarrow & S_U \times U \end{array}$$

commutes. So we can use these bijections (and the obvious str. of Riemann surface on $S_U \times U$) as charts for a structure of Riemann surface on $\tilde{X}$.

Prop The charts give $\tilde{X}$ the structure of a simply connected Riemann surface. Moreover, the action by Γ_x on $\tilde{X}$ defined by

$$\begin{aligned} \Gamma_x \times \tilde{X} &\longrightarrow \tilde{X} \\ (g, f) &\longmapsto g \circ f \end{aligned}$$

is properly discontinuous, and

$$\tilde{X} \xrightarrow{p} X$$

exhibits the connected component of $x \in X$ as quotient of $\tilde{X}$ by this left Γ_x action.

Pf If U is a non-empty simply connected subset of the connected component of X that contains x, then, under the bijection

$$p^{-1}(U) \xrightarrow{(q_U, p)} S_U \times U,$$

the left action by Γ_x corresponds to a free and transitive action on S_U and trivial action on U. So the statement concer-

In this case, it is clear that $\tilde{X}$ is paracompact, because the group Π_X is countable (and known). It is a theorem of Radó that, in fact, every connected Riemann surface is paracompact, but this is a non-trivial thm., the analogue of which fails for complex mfds. of dim. ≥ 2.

Let me talk about stacks. As Grothendieck told us, it is better to have a good category with bad objects that a bad category with good objects. The category of complex manifolds and holomorphic maps is a bad category with good objects. For example, it does not have colimits, except for ones corresponding to gluing together coverings by open submanifolds. In general, if C is a small category, then the Yoneda emb.

$$C \xrightarrow{h} \mathrm{Fun}(C^{op}, \mathrm{Set})$$

$$X \longmapsto \mathrm{Map}(-, X)$$

is initial among functors

$$C \xrightarrow{f} D$$

to categories that admit all small colimits, but it does not preserve any colimits that may already exist in C. But if $\mathcal{K}$ is a family of colimits in C that we wish to preserve, then we let

$$\mathrm{Fun}^{\mathcal{K}}(C^{op}, \mathrm{Set})$$

$$\subset \mathrm{Fun}(C^{op}, \mathrm{Set})$$

be the full subcategory spanned by the functors F s.t.

$$F(\mathrm{colim}_{i \in I} X_i)$$

$$\longrightarrow \lim_{i \in I} F(X_i)$$

is a bijection for all diagr. $i \mapsto X_i$ in $\mathcal{K}$. Since the colimits of these diagrams exist in C, the Yoneda emb. induces

$$\begin{aligned} C &\xrightarrow{\ h\ } \mathrm{Fun}^{\times}(C^{op}, \mathrm{Set}) \\ X &\longmapsto \mathrm{Map}(-, X) \end{aligned}$$

and this is the initial functor to a category that admits all small colimits that preserves the colimits in X.

Now, this is the situation for 1-categorical colimits. But we really want ∞-categorical colimits. The situation is complete parallel, provided that we replace the 1-category Set of sets by the ∞-category

$$\mathrm{An}$$

of anima (aka animated sets, ∞-groupoids, homotopy types, ...) This gives a "richer" kind of colimits, and it is this added "richness" that produces anything "derived."[4] So we take C to be the category of complex manifolds and holomorphic maps, and consider

$$C \xrightarrow{h} \mathrm{Fun}^{\mathcal{X}}(C^{op}, \mathrm{An}),$$

where $\mathcal{X}$ is the family of diagrams corresponding to coverings by open submanifolds. The target ∞-category, by definition, is the ~~∞~~ ∞-category of stacks on complex mfds.

The unique functor $p : C \to 1$ gives rise to adjoint functors

$$\begin{tikzcd} \mathrm{Fun}^{\mathcal{X}}(C^{op}, \mathrm{An}) \arrow[d, shift left, "p_*"] \\ \mathrm{An} \arrow[u, shift left, "p^*"] \end{tikzcd}$$

We call p_* the (derived) global sections functor and we call p^* the constant sheaf functor. The latter is fully faithful, so we can view An as a full subcategory of the ∞-category of stacks. Similarly, we can view C as a full subcategory of the ∞-category of stacks. Now, given $X \in C$, there exists an initial map of stacks

$$X \xrightarrow{f} \pi_{\leq \infty}(X)$$

to an anima. We say that its target is the fundamental ∞-groupoid of X. If we write X as a colimit in C

$$X \simeq \operatorname{colim}_{D \subset X} D$$

of discs, then

$$\pi_{\leq \infty}(X) \simeq \operatorname{colim}_{D \subset X} 1$$

is the corresponding colimit in anima of the diagram $D \mapsto 1$, and the map f is the map of colimits induced by the unique map of stacks

$$D \longrightarrow 1.$$

Now, the fundamental groupoid of X is the 1-truncation

$$\pi_{\leq 1}(X) \simeq \tau_{\leq 1}\pi_{\leq \infty}(X)$$

of the fundamental ∞-groupoid, and the universal cover of X at $x \in X$ is the pullback

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[d] & 1 \arrow[d, "x"] \\ X \arrow[r] & \pi_{\leq 1}(X) \end{tikzcd}$$

in the ∞-category of stacks. It is then a theorem that this pullback is represented by a complex manifold.

So the morale is that the paths that appear in the classical construction of the universal cover are an artifact of the model of ∞-categories as the coherent nerve of the self-enriched category of top. spaces.