Geometry III/Introduction to Geometry III: Riemann Surfaces

Problem 1. Let $f\colon \mathbb{C} \to \mathbb{C}$ be an $\mathbb{R}$-linear map. (Recall that this means that $f(z+w) = f(z) + f(w)$ for $z, w \in \mathbb{C}$ and that $f(\lambda z) = \lambda f(z)$ for all $\lambda \in \mathbb{R}$ and $z \in \mathbb{C}$.) Show that the following conditions are equivalent:

(1) The map $f\colon \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$-linear.
(2) The map $f\colon \mathbb{C} \to \mathbb{C}$ is given by multiplication by a complex number.
(3) Either $f\colon \mathbb{C} \to \mathbb{C}$ is the zero map, or else $f(i)$ is the 90-degree counterclockwise rotation of $f(1)$.
(4) Either f is the zero map, or else $f\colon \mathbb{C} \to \mathbb{C}$ preserves oriented angles.

Problem 2. Write down (for example as a power series) a holomorphic map

$$D(1,1) \xrightarrow{f} \mathbb{C}$$

with the property that $f(z)^5 = z$ for all $z \in D(1,1)$ and $f(1) = 1$. What is the result of analytically continuing f along a path, which starts at 1 and travels once counterclockwise around the origin, returning to the starting point 1? What about if you go N times counterclockwise around the origin, where N is an integer?