Geometry III/Introduction to Geometry III: Riemann Surfaces

Problem 1. Let $f\colon Y \to X$ be a non-constant proper holomorphic map between connected Riemann surfaces. Let $S \subset X$ be the set of point $x \in X$ for which there exists $y \in Y$ with $f(y) = x$ such that $m_y(f) > 1$. (We say that S is the set of branch points of $f\colon Y \to X$.)

(1) Show that the subspace $S \subset X$ is discrete.
(2) Let $T = f^{-1}(S) \subset Y$. Show that the restricted map

$$Y \setminus T \xrightarrow{f|_{Y \setminus T}} X \setminus S$$

is a finite covering map. This means that for every $x \in X \setminus S$, $f^{-1}(x)$ is finite and there exists $x \in U \subset X \setminus S$ open and a commutative diagram

$$\begin{array}{ccc} f^{-1}(x) \times U & \xrightarrow{\quad\varphi\quad} & f^{-1}(U) \\ & {\scriptstyle p} \searrow \qquad \swarrow {\scriptstyle f|_{f^{-1}(U)}} & \\ & U & \end{array}$$

in which the top horizontal map φ is a homeomorphism.