**Geometry III/Introduction to Geometry III: Riemann Surfaces**

**Problem 1.** Show that every automorphism $g\colon \mathbb{C} \to \mathbb{C}$ of the Riemann surface $\mathbb{C}$ is given by $f(z) = az + b$ with $a, b \in \mathbb{C}$ and $a \neq 0$.

[Hint: First, show that every homeomorphism $g\colon \mathbb{C} \to \mathbb{C}$ is proper, and hence, extends to a homeomorphism $\tilde{g}\colon \mathbb{P}^1 \to \mathbb{P}^1$. Next, use the removable singularities theorem from Lecture 1 to conclude that if $g$ is holomorphic, then so is $\tilde{g}$.]

**Problem 2.** We the stereographic projection from the north pole

$$\pi\colon S^2 \smallsetminus \{N\} \to \mathbb{C},$$

which we defined in Lecture 8. It restricts to a bijection of the lower hemisphere

$$U = \{(x, y, z) \in S^2 \mid z < 0\} \subset S^2$$

onto the open unit disc

$$D = \{a + ib \in \mathbb{C} \mid a^2 + b^2 < 1\} \subset \mathbb{C}.$$

It also restricts to a bijection of the back hemisphere

$$V = \{(x, y, z) \in S^2 \mid y < 0\} \subset S^2$$

onto the complex upper halfplane

$$\mathfrak{H} = \{a + ib \in \mathbb{C} \mid b > 0\} \subset \mathbb{C}.$$

Please answer the following questions.

(a) Find a rotation $k\colon S^2 \to S^2$ that restricts to a bijection

$$V \xrightarrow{g|_V} U$$

of the back hemisphere onto the lower hemisphere.

(b) Find an explicit formula for the unique bijection $h$ that makes the diagram

$$\begin{tikzcd} V \arrow[r, "g|_V"] \arrow[d, "\pi|_V"] & U \arrow[d, "\pi|_U"] \\ \mathfrak{H} \arrow[r, "h"] & D. \end{tikzcd}$$

commute.