Geometry III/Introduction to Geometry III: Riemann Surfaces

Problem 1. This problem will prove some general facts about groupoids. They apply, in particular, to the fundmental groupoid of a Riemann surface. So let $\mathcal{G}$ be a groupoid. Given objects x and y in $\mathcal{G}$, we write $\operatorname{Map}(y,x)$ for the set of maps from y to x in $\mathcal{G}$, and given objects x, y, and z in $\mathcal{G}$, we write

$$\operatorname{Map}(y,x) \times \operatorname{Map}(z,y) \xrightarrow{\circ} \operatorname{Map}(z,x)$$

for the composition. In particular, given an object x in $\mathcal{G}$, the composition defines a group structure on $\operatorname{Map}(x,x)$. We write $\operatorname{Aut}(x)$ for this group. Moreover, given objects x and y in $\mathcal{G}$, the composition

$$\operatorname{Map}(x,x) \times \operatorname{Map}(y,x) \xrightarrow{\circ} \operatorname{Map}(y,x)$$

is a left action by the group $\operatorname{Aut}(x)$ on the set $\operatorname{Map}(y,x)$.

(1) Let x and y be objects in $\mathcal{G}$. Show that if $\operatorname{Map}(y,x)$ is non-empty, then the left action by the group $\operatorname{Aut}(x)$ on this set is free and transitive.

Of course, the right action by the group $\operatorname{Aut}(y)$ on $\operatorname{Map}(y,x)$ is also free and transitive, provided that $\operatorname{Map}(y,x)$ is non-empty. Next, we let

$$B\operatorname{Aut}(x) \subset \mathcal{G}$$

be the full subcategory spanned by the object x of $\mathcal{G}$. The category $B\operatorname{Aut}(x)$ is a groupoid with a single object x and set of maps from x to x in $B\operatorname{Aut}(x)$ is the same as the set of maps from x to x in $\mathcal{G}$.

(2) Show that if $\operatorname{Map}(y,x)$ is non-empty for all y in $\mathcal{G}$, then the inclusion

$$B\operatorname{Aut}(x) \xrightarrow{f} \mathcal{G}$$

is an equivalence of categories. [Hint: To prove this, you must produce a functor $g\colon \mathcal{G} \to B\operatorname{Aut}(x)$ and natural transformations $\epsilon\colon g \circ f \to \mathrm{id}$ and $\eta\colon \mathrm{id} \to f \circ g$. The latter will automatically be natural isomorphisms, because $\mathcal{G}$ and $B\operatorname{Aut}(x)$ are groupoids.]

As your proof of (2) will make clear, the inverse g of f is not unique. However, this is because we only consider 1-categories. In the world of ∞-categories, the inverse of an equivalence of ∞-categories is unique, up to contractible ambiguity, which is as unique as it can be.