Ramanujan graphs

$$\begin{tikzcd} BG \arrow[d, shift right, "p"'] \\ 1 \arrow[u, shift right, "s"'] \end{tikzcd} \qquad \begin{tikzcd} Set^{BG} \arrow[d, bend right, "p_!"'] \arrow[d, bend left, "p_*"] \\ Set \arrow[u, "p^*"] \end{tikzcd} \qquad \begin{tikzcd} Set^{BG} \arrow[d, "s^*"] \\ Set \arrow[u, bend left, "s_!"] \arrow[u, bend right, "s_*"'] \end{tikzcd}$$

$p^*(X) = X$ with trivial G-action

$p_*(Y) = Y^G = \{ y \in Y \mid gy = y \text{ for all } g \in G \}$

$p_!(Y) = Y/G = Y/_{gy \sim y}$

$s^*(Y) =$ underlying set (forget G-action)

$s_*(X) = \mathrm{Map}(G, X)$

$$(g \cdot f)(h) = f(hg^{-1})$$

$s_!(X) = G \times X$

$$g \cdot (h, x) = (gh, x)$$

Say that $Y \in Set^{BG}$ is a free G-set, if $Y \simeq s_!(X)$ for some X. Equivalently, Y is a free G-set, if for all $y \in Y$ and $e \neq g \in G$, $gy \neq y$.

Def A graph is a triple

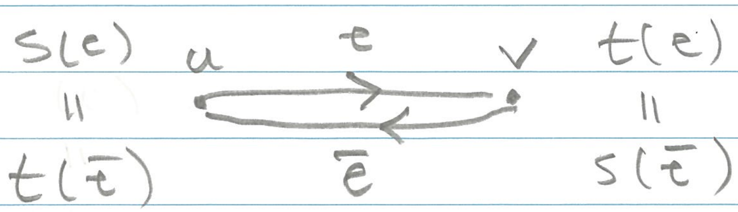
$$\Gamma = (V, E, s : s^*(E) \longrightarrow V)$$

with V a set, E a free G-set, and $s : s^*(E) \to V$ a map of sets.

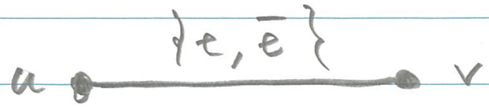
We say that $v \in V$ is a vertex of Γ, that $e \in E$ is an edge of Γ, and that $s(e) \in V$ is the source of e. If $e \in E$ and $1 \neq g \in G$, then we also write $\bar{e} = g \cdot e$ and call it the reverse edge of e. For $e \in E$, we define

$$t(e) = s(\bar{e}) \in V$$

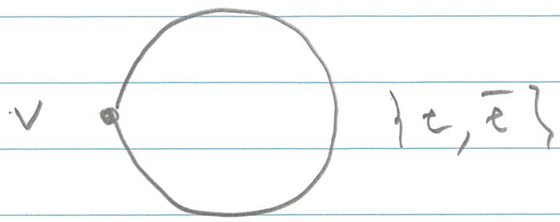
and call it the target of V. We picture



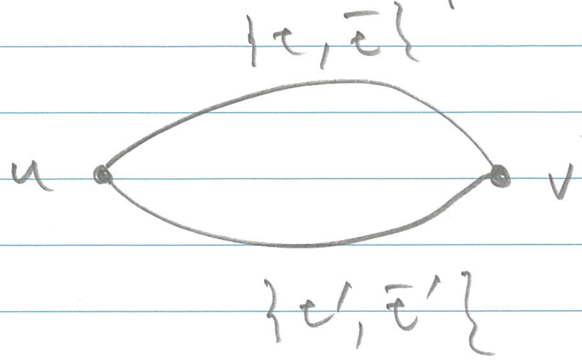
as a single unoriented edge



between the vertices u and v. We allow $u = v$, which gives a loop



and we allow parallel edges

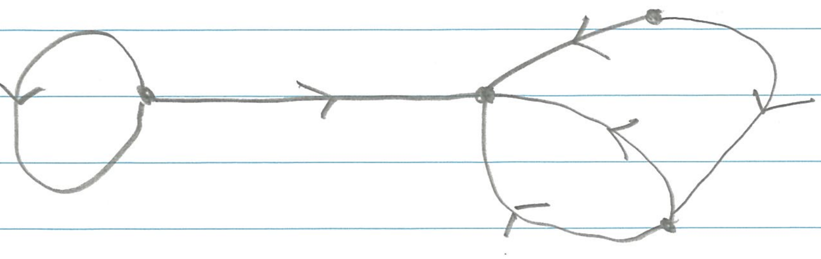


Def An orientation of a graph $\Gamma = (V, E, s : s^*(E) \to V)$ is a pair of a set E_+ and an isomorphism of G-sets $a : s_!(E_+) \to E$.

Given an orientation of a graph Γ, the mate of $a : s_!(E_+) \xrightarrow{\sim} E$ is a map

$$E_+ \longrightarrow s^*(E)$$

that for every $\{e, \bar{e}\} \subset E$ chooses one of e and $\bar{e}$. We depict an oriented graph as



An oriented graph determines and is determined, up to unique isomorphism, by the triple

$$(V, E_+, E_+ \xrightarrow{(s,t)} V \times V),$$

where $V, E_+ \in S$, and where the map (s,t) is double mate of

$$s^* s_!(E_+) \underset{\sim}{\xrightarrow{s^* a}} s^*(E) \xrightarrow{s} V,$$

that is,

$$E_+ \longrightarrow s^* s_*(V) \simeq V \times V.$$

Def Let $\Gamma = (V, E, s: s^*(E) \to V)$ be a graph. The degree of $v \in V$ is the number

$$\deg(v) = \# s^{-1}(v)$$

of edges, whose source is v.

Rmk If $\deg(v) < \infty$ for all $v \in V$, then we say that Γ is locally finite. If Γ is locally finite and V is finite, then we say that Γ is finite. Equivalently, Γ is finite if both V and E are finite. /

We would like to understand how information can spread through a graph. So we think of a graph $\Gamma = (V, E, s : s^*(E) \to V)$ as a discrete version of a Riemannian manifold (M, g) and of information as heat. On a Riemannian manifold, we have the Laplace operator

$$\begin{tikzcd} \Omega^0(M) \arrow[rr, "\Delta"] \arrow[dr, "d"'] & & \Omega^0(M) \\ & \Omega^1(M) \arrow[ur, "d^*"'] & \end{tikzcd}$$

and the heat equation

$$\frac{\partial f}{\partial t} = \Delta(f).$$

Let Γ be an oriented graph, which we will assume to be finite. By analogy, we define

$$\Omega^0(\Gamma) = L^2(V)$$

$$\Omega^1(\Gamma) = L^2(E_+)$$

and $d : \Omega^0(\Gamma) \to \Omega^1(\Gamma)$ by

$$df(e) = f(t(e)) - f(s(e)).$$

So d is a difference operator rather than a differential operator. We give $L^2(V)$ the "standard" inner product

$$\langle f, g \rangle = \Sigma_{v \in V} f(v) g(v)$$

and similarly for $L^2(E_+)$. We define $d^* : \Omega^1(\Gamma) \to \Omega^0(\Gamma)$ to be the adjoint of d w.r.t. these inner products.

<u>Def</u> The Laplacian of a finite graph Γ is the composite map

$$\begin{tikzcd} \Omega^0(\Gamma) \arrow[rr, "\Delta"] \arrow[dr, "d"'] & & \Omega^0(\Gamma) \\ & \Omega^1(\Gamma) \arrow[ur, "d^*"'] & \end{tikzcd}$$

Exercise Show that the Laplacian of a finite graph Γ is independent on the choice of orientation. ✓

Prop The Laplacian of a finite graph is a self-adjoint and positive semi-definite operator.

Pf It self-adjoint, because

$$\Delta^* = (d^*d)^* = d^*d^{**}$$

$$= d^*d = \Delta,$$

and it is positive semi-definit, because

$$\langle f, \Delta(f)\rangle = \langle f, d^*d(f)\rangle$$

$$= \langle df, df\rangle \geq 0. \quad //$$

Cor The eigenvalues of Δ are real and non-negative. /

Lemma The dimension of the eigenspace attached to the eigenvalue $\lambda = 0$ is equal to the number of connected components of Γ.

Pf Let us give a physics proof: We think of $f \in \Omega^0(\Gamma)$ as a heat distribution. That $\Delta(f) = 0$ means that the distribution is in an equilibrium. But then it must be constant on each connected component of Γ. //

We would like information to spread quickly around Γ. So we would like the nonzero eigenvalues of Γ to be large. How do find Δ?

Def The adjacency operator of a finite graph Γ is the linear map

$$\Omega^0(\Gamma) \xrightarrow{\partial_\Gamma} \Omega^0(\Gamma)$$

defined by

$$\partial_\Gamma(\delta_u)(\delta_v) = \#\{e \in E \mid s(e) = v \text{ and } t(e) = u\},$$

where $(\delta_u)_{u\in V}$ is the basis of $\Omega^0(\Gamma) = \mathrm{Map}(V, \mathbb{R})$ given by

$$\delta_u(v) = \begin{cases} 1 & \text{if } u = v \\ 0 & \text{if } u \neq v. \end{cases}$$ //

We remark that ∂_Γ is self-adjoint and that $\partial_\Gamma(\delta_u)(\delta_u)$ is even. We also have the linear map

$$\Omega^0(\Gamma) \xrightarrow{\deg\cdot\mathrm{id}} \Omega^0(\Gamma)$$

$$\delta_u \longmapsto \deg(u)\cdot\delta_u$$

Prop The Laplacian of a finite graph Γ is given by

$$\Delta_\Gamma = \deg\cdot\mathrm{id} - \partial_\Gamma.$$

Pf We choose an orientation of Γ so that for all $f, g \in \Omega^0(\Gamma)$,

$$\langle f, \Delta(g)\rangle = \langle df, dg\rangle.$$

We wish to show that this (real) number is equal to

$$\langle f, \deg(g)\cdot g - \partial_\Gamma(g)\rangle.$$

We write the latter as

$$\sum_{u\in V} f(u)\left(\deg(u)g(u) - \sum_{v\in V} \partial_p(\delta_u)(\delta_v)g(v)\right)$$

$$= \sum_{u\in V} \deg(u) f(u) g(u)$$

$$- \sum_{u,v\in V} \partial_p(\delta_u)(\delta_v) f(u) g(v),$$

and we write the former as

$$\sum_{e\in E_+} df(e)\cdot dg(e)$$

$$= \sum_{e\in E_+} \left(f(t(e)) - f(s(e))\right)\left(g(t(e)) - g(s(e))\right)$$

$$= \sum_{e\in E_+} \left(f(t(e))g(t(e)) + f(s(e))g(s(e))\right)$$

$$- \sum_{e\in E_+} \left(f(t(e))g(s(e)) + f(s(e))g(t(e))\right)$$

$$= \sum_{u\in V} \deg(u) f(u) g(u)$$

$$- \sum_{u,v\in V} \partial_p(\delta_u)(\delta_v) f(u) g(v). \quad //$$

Def A graph Γ is k-regular with $k\geq 0$ an integer if

$$\deg(v) = k$$

for all $v\in V$. //

Cor If Γ is a finite k-regular graph, then

$$\Delta_\Gamma = k \cdot \mathrm{id} - \partial_\Gamma .$$

Hence, if Γ is a finite k-regular graph, then the eigenvalues of ∂_Γ are real and $\leq k$. In fact:

Prop Let Γ be a finite graph, and let $\Delta(\Gamma)$ be the maximal degree of any $v \in V$. If λ is an eigenvalue of ∂_Γ, then

$$|\lambda| \leq \Delta(\Gamma).$$

Pf Let (a_{uv}) be the matrix of ∂_Γ with respect to the basis $(\delta_u)_{u \in V}$ of $\Omega^0(\Gamma)$. Let

$$x = \Sigma_{v \in V} \delta_v \cdot x_v$$

be an eigenvector for ∂_Γ with eigenvalue λ. We choose $u \in V$ s.t. $|x_u| \geq |x_v|$ for all $v \in V$. Now, the u'th coordinate of

$$\partial_\Gamma(x) = \lambda \cdot x$$

is given by

$$\sum_{v \in V} a_{uv} x_v = \lambda \cdot x_u .$$

Hence, we have

$$|\lambda| \cdot |x_u| = \left| \sum_{v \in V} a_{uv} x_v \right|$$

$$\leq \sum_{v \in V} |a_{uv}| \cdot |x_v|$$

$$\leq \left(\sum_{v \in V} |a_{uv}| \right) \cdot |x_u|$$

$$= \deg(u) \cdot |x_u| .$$

Since x is an eigenvector, it is nonzero, so $|x_u| \neq 0$, and hence, $|\lambda| \leq \deg(u)$, and $\deg(u) \leq \Delta(X)$ by definition. //

Cor If Γ is a k-regular finite graph and if λ is an eigenvalue of ∂_P, then

$$|\lambda| \leq k . \quad //$$

Rmk If Γ is a finite k-regular graph, then k is an eigenvalue of ∂_P with multiplicity the number of connected components of Γ. Indeed, 0 is an eigenvalue of Δ_P with this mult.

If Γ is a finite k-regular graph, then we define

$$\lambda(\Gamma) < k$$

to be the maximum of $|\lambda|$ as λ ranges over the eigenvalues of ∂_P not equal to $\pm k$.

<u>Def</u> A finite k-regular graph Γ is Ramanujan if

$$\lambda(\Gamma) \leq 2\sqrt{k-1}.$$

So for a Ramanujan graph, the nonzero eigenvalues of its Laplacian satisfy

$$\lambda \geq k - 2\sqrt{k-1}.$$

We will see that for k fixed, this is asymptotically the best possible as $\# V \to \infty$.