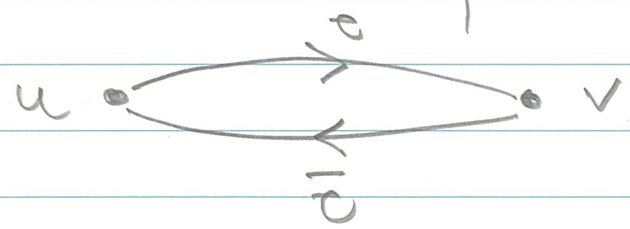
Expander graphs

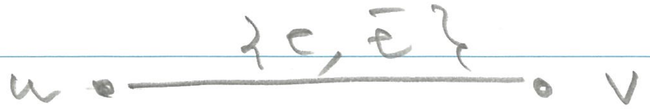
Last time, we define a graph to be a triple

$$\Gamma = (V, E, s\colon s^*(E) \to V)$$

of a set (of vertices) V, a set (of edges) with free left action by $G = \{1, g\}$, and a (source) map s that to an edge $e \in E$ assigns its source $s(e) \in V$. We call $\bar{e} = g \cdot e$ the reverse edge of e and call $t(e) = s(\bar{e}) \in V$ the target of e. We depict



as



and recall that, since G acts freely on E, we have $\bar{e} \neq e$ for all $e \in E$. An orientation of Γ is a pair of a set E_+ and an isomorphism of G-sets

$$s_!(E_+) \xrightarrow{a} E.$$

<u>Ex</u> Given an integer $n \geq 0$, we write $[0, n]$ for the oriented graph

$$0 \xrightarrow{e_1} 1 \xrightarrow{e_2} 2 \xrightarrow{e_3} \cdots \xrightarrow{e_n} n.$$

Let $\Gamma = (V, E, s)$ and $\Gamma' = (V', E', s')$ be graphs. A map of graphs

$$\Gamma \xrightarrow{f} \Gamma'$$

is a pair $f = (f_0, f_1)$ of a map $f_0 : V \to V'$ and a G-equivariant map $f_1 : E \to E'$ s.t. the diagram

$$\begin{array}{ccc} s^*(E) & \xrightarrow{s} & V \\ \downarrow{\scriptstyle s^*(f_1)} & & \downarrow{\scriptstyle f_0} \\ s^*(E') & \xrightarrow{s'} & V' \end{array}$$

commutes.

<u>Def</u> A path of length $n \geq 0$ in a graph Γ is a map of graphs

$$[0, n] \xrightarrow{\gamma} \Gamma.$$

A path $\gamma : [0, n] \to \Gamma$ determines n-tuple $(\gamma_1(e_i))_{1 \leq i \leq n}$ in E, and if $n \geq 1$, then this n-tuple, in turn, determines γ.

If $n \geq 2$ and if

$$\gamma_1(e_{i+1}) = \overline{\gamma_1(e_i)}$$

for some $1 \leq i \leq n-1$ then we say that γ is a path with back-tracking. In this case,

$$(\gamma_1(e_1), \dots, \gamma_1(e_{i-1}), \gamma_1(e_{i+2}), \dots, \gamma(e_n))$$

is a path of length $n-2$ from $\gamma(0)$ to $\gamma(n)$. One shows by induction on $n \geq 0$ that if a path from u to v exists in Γ, then one without backtracking also exists.

Def A graph Γ is connected if for every pair of vertices (u, v) there exists a path $\gamma : [0, n] \to \Gamma$ with $\gamma_0(0) = u$ and $\gamma_0(n) = v$.

Rmk A connected component of a graph Γ is a maximal connected subgraph.

Def A geodesic in a graph Γ is a path $\gamma: [0,n] \to \Gamma$ without backtracking.

Exercise Let Γ be a connected graph. Show that the map

$$V \times V \xrightarrow{d} [0, \infty)$$

that to (u,v) assigns the length $d(u,v) = n$ of a shortest geodesic

$$[0,n] \xrightarrow{\gamma} \Gamma$$

from u to v is a metric. This means that (1) for all $u, v \in V$, $d(u,v) = d(v,u)$ and $d(u,v) = 0$ if and only if $u = v$; and (2) for all $u, v, w \in V$,

$$d(u,w) \leq d(u,v) + d(v,w).$$

<u>Def</u> The diameter of a non-empty connected graph Γ is

$$\operatorname{diam}(\Gamma) = \operatorname{diam}(V, d).$$ //

Let Γ be a connected graph, and let $A \subset V$ be a subset. Given $v \in V$, we define

$$d(v, A) = \min_{a \in A} d(v, A).$$

We define the boundary of $A \subset V$ to the subset

$$\partial A = \{ v \in V \mid d(v, A) = 1 \} \subset V.$$

It consists of the vertices $v \in V$ that are adjacent to a vertex in A and are not in A.

<u>Def</u> A connected k-regular graph is a c-expander if

$$|\partial A| \geq c \cdot |A|$$

for all $A \subset V$ with $|A| \leq \frac{1}{2} |V|$. //

<u>Rmk</u> Best possible is $c = 1$.

What we really wish to find is a family $(X_n)_{n \geq N}$ of connected k-regular graphs s.t. X_n has n vertices and is a c-expander for some c that only depends on k. Obviously, we would like $c = c(k)$ to be as close to 1 as possible. We now show that families of Ramanujan graphs serve this purpose.

Let Γ be a finite, connected, k-regular graph, and let $a : S\Phi(E_+) \to E$ be a choice of orientation of Γ. We defined

$$\Omega^0(\Gamma) = \mathrm{Map}(V, \mathbb{R})$$

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and $d : \Omega^0(\Gamma) \to \Omega^1(\Gamma)$ by

$$d(f)(e) = f(t(e)) - f(s(e)).$$

We give $\Omega^0(V)$ the inner prod.

$$\langle f, g \rangle = \Sigma_{v \in V} f(v) g(v)$$

and similarly for $\Omega^1(\Gamma)$. The family $(\delta_v)_{v\in V}$, where

$$\delta_v(u) = \begin{cases} 1 & \text{if } u = v \\ 0 & \text{if } u \neq v \end{cases}$$

is a basis of $\Omega^0(V)$ and it is orthonormal w.r.t. $\langle -,- \rangle$. We let

$$\Omega^1(\Gamma) \xrightarrow{d^*} \Omega^0(\Gamma)$$

be the adjoint of d w.r.t. these inner products and define the Laplacian of Γ to be the composite map

$$\begin{tikzcd} \Omega^0(\Gamma) \arrow[rr, "\Delta"] \arrow[dr, "d"'] & & \Omega^0(\Gamma) \\ & \Omega^1(\Gamma) \arrow[ur, "d^*"'] & \end{tikzcd}$$

It is a self-adjoint positive definite operator, so by the spectral theorem, there exists a basis of $\Omega^0(V)$ that consists of eigenvectors of Δ and that is orthonormal w.r.t. $\langle -,- \rangle$.

Moreover the eigenvalues of Δ are real and non-negative.

A consequence of the spectral theorem is that for all $f \neq 0$,

$$\lambda_{min} \leq \frac{\langle \Delta(f), f \rangle}{\langle f, f \rangle} \leq \lambda_{max},$$

where λ_{max} (resp. λ_{min}) is the maximal (resp. the minimal) eigenvalue. We will use these inequalities with clever choices of $f \in \Omega^0(\Gamma)$ to connect information about the spectrum (= set of eigenvalues) of Δ and expander constants c for Γ.

Recall from last time that

$$\Delta = k \cdot \mathrm{Id} - \partial,$$

where $\partial : \Omega^0(\Gamma) \to \Omega^0(\Gamma)$ is the adjacency operator defined by

$$\partial(\delta_u)(v) = \# \{ e \in E \mid s(e) = v \text{ and } t(e) = u \}$$

This, in particular, proves that Δ does not depend on the choice of orientation of Γ, since ∂ visibly does not. Since ∂ is symmetric, its eigenvalues are real, and we proved last time that for every eigenvalue λ of ∂, we have

$$|\lambda| \leq k.$$

The constant fct. $1 \in \Omega^0(\Gamma)$ is an eigenvector of ∂ with eigenvalue k, and since we now assume Γ to be connected, it spans the eigenspace attached to $\lambda = k$. So we can order the eigenvalues

$$k = \lambda_0(\Gamma) > \lambda_1(\Gamma) \geq \lambda_2(\Gamma) \geq \cdots$$

$$\cdots \geq \lambda_{n-1}(\Gamma) \geq -k.$$

If Γ is bipartite, then we have $\lambda_{n-1}(\Gamma) = -k$, and one can show (easily?) that the opposite is true, too. So the smallest non-zero eigenvalue of the Laplacian Δ is $k-\lambda_1(\Gamma)$. We defined Γ to be Ramanujan if

$$\max_{\lambda \neq \pm k} |\lambda| \leq 2\sqrt{k-1},$$

and we will see today that, asymptotically as $|V| \longrightarrow \infty$, this is the best we can do.

Since $1 \in \Omega^0(\Gamma)$ spans the eigenspace attached to the eigenvalue $\lambda = 0$ of Δ, we conclude from the spectral theorem that

$$k - \lambda_1(\Gamma) \leq \frac{\langle \Delta(f), f \rangle}{\langle f, f \rangle} \qquad (*)$$

for all $0 \neq f \in \Omega^0(\Gamma)$ with $\langle f, 1 \rangle = 0$. We now fix $A \subset V$ and define $f \in \Omega^0(\Gamma)$ by

$$f(v) = \begin{cases} |V \setminus A| & \text{if } v \in A \\ -|A| & \text{if } v \notin A. \end{cases}$$

We have

$$\langle f, 1\rangle = \Sigma_{v\in V} f(v)\cdot 1$$

$$= \Sigma_{v\in A} |V\setminus A| + \Sigma_{v\in V\setminus A} (-|A|)$$

$$= |A||V\setminus A| - |V\setminus A||A| = 0,$$

so we apply $(*)$ to f. Next,

$$\langle f, f\rangle = \Sigma_{v\in V} f(v)^2$$

$$= \Sigma_{v\in A} |V\setminus A|^2 + \Sigma_{v\in V\setminus A} (-|A|)^2$$

$$= |A||V\setminus A|^2 + |V\setminus A||A|^2$$

$$= |A||V\setminus A|(|V\setminus A| + |A|)$$

$$= |A||V\setminus A||V|,$$

and finally,

$$\langle \Delta(f), f\rangle = \langle df, df\rangle$$

$$= \Sigma_{e\in E_+} (f(t(e)) - f(s(e)))^2$$

$$= \sum_{\substack{e\in E_+ \\ s(e)\in A \\ t(e)\notin A}} (-|A| - |V\setminus A|)^2 + \sum_{\substack{e\in E_+ \\ s(e)\notin A \\ t(e)\in A}} (|V\setminus A| + |A|)^2$$

$$= |V|^2 \cdot \# \{ e \in E \mid s(e) \in A, t(e) \notin A \}$$

$$\leq |V|^2 \cdot |\partial A| \cdot k.$$

Thus, we conclude that

$$k - \lambda_1(\Gamma) \leq \frac{|V|^2 |\partial A| \cdot k}{|V||A||V \setminus A|}$$

$$= \frac{|\partial A|}{|A|} \cdot \frac{|V|}{|V \setminus A|} \cdot k.$$

So if $|A| \leq \frac{1}{2}|V|$, we find

$$\frac{|\partial A|}{|A|} \geq \frac{1}{2}\left(1 - \frac{\lambda_1(\Gamma)}{k}\right)$$

Hence, if $(X_n)_{n \gg 0}$ is a family of connected, k-regular Ramanujan graphs s.t. X_n has n vertices, then this is a family of c-expanders with

$$c = \frac{1}{2}\left(1 - \frac{\lambda_1(\Gamma)}{k}\right).$$

It is possible to show that, in this situation, the constant c can be improved a bit by considering not only $\lambda_1(\Gamma)$ but the other eigenvalues of ∂ as well. The paper "Better expansion for Ramanujan graphs" by N. Kahale from 1991 apparently is the best known result. At any rate, an argument similar to the one above with a different (and more complicated) choice of $f \in \Omega^0(\Gamma)$ shows:

Thm (Alon, 1991) Let Γ be a non-empty connected k-regular graph such that

$$\operatorname{diam}(\Gamma) \geq 2(b+1) \geq 4.$$

In this situation,

$$\lambda_1(\Gamma) \geq 2\sqrt{k-1} - \frac{2\sqrt{k-1}-1}{b}$$

Thus, if $(X_n)_{n \gg 0}$ is any family of connected k-regular graphs (not necessarily Ramanujan) s.t. X_n has n vertices, then, since $\operatorname{diam}(X_n) \to \infty$ as $n \to \infty$, we have

$$\limsup_{n\to\infty} \frac{1}{2}\left(1 - \frac{\lambda_1(\Gamma)}{k}\right) \leq \frac{1}{2}\left(1 - \frac{2\sqrt{k-1}}{k}\right).$$

So, asymptotically Ramanujan graphs is the best that we can do as far as this number is concerned.