In the first lecture, we defined Ramanujan graphs, and in the second lecture, we saw that, asymptotically, they provide the best possible expander graphs. Today, we begin work to construct infinite families of k-regular Ramanujan graphs. The construction requires that $k = p+1$ with p a prime number, or at least $k = q+1$ with $q = p^r$ a power of a prime number.

The idea behind the construction, due to Lubotzky–Phillips–Sarnak, is that to consider graphs that are p-adic analogues of (hypebolic) compact Riemann surfaces. Let

$$\mathcal{H} = \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \} \subset \mathbb{C}$$

be complex upper halfplane. It carries a left action by $SL_2(\mathbb{R})$ via Möbius transformations,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

If X is a compact Riemann

surface of genus $g \geq 2$, then there exists a subgroup

$$\Gamma \subset SL_2(\mathbb{R})$$

that acts properly cocompactly on $\mathfrak{h}$ and an isomorphism of Riemann surfaces

$$\Gamma \backslash \mathfrak{h} \xrightarrow{\sim} X.$$

We can express $\mathfrak{h}$ in purely group-theoretic terms as follows: The action by $SL_2(\mathbb{R})$ on $\mathfrak{h}$ is transitive. Indeed, given

$$z = a + ib \in \mathfrak{h},$$

we have

$$\begin{pmatrix} b^{1/2} & ab^{-1/2} \\ 0 & b^{-1/2} \end{pmatrix} \cdot i$$

$$= \frac{b^{1/2} i + ab^{-1/2}}{b^{-1/2}} = a + ib = z.$$

But it is not a free action.

We calculate:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot i = i \iff$$

$$ai + b = -c + di \iff$$

$$a - d = 0 = b + c \iff$$

$$\left\langle \begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix} \right\rangle = ab + cd = 0,$$

$$\left\langle \begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} a \\ c \end{pmatrix} \right\rangle = a^2 + c^2 = ad - bc = 1$$

$$\left\langle \begin{pmatrix} b \\ d \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix} \right\rangle = b^2 + d^2 = -bc + ad = 1$$

This shows that the stabilizer of $i \in \mathfrak{h}$ is the subgroup

$$SO(2) < SL_2(\mathbb{R}).$$

Hence, the map

$$SL_2(\mathbb{R})/SO(2) \longrightarrow \mathfrak{h}$$

$$g\,SO(2) \longmapsto g \cdot i$$

is a bijection. In fact, it is

diffeomorphism with respect to the standard structures of smooth manifolds on the source and target. But more is true: Up to scaling, there is a unique left invariant (Riemannian) metric on $SL_2(\mathbb{R})/SO(2)$, which, under this diffeomorphism, defines the hyperbolic metric

$$ds^2 = \operatorname{Im}(z)^{-2}\,|dz|^2$$

on $\mathfrak{h}$. Moreover, the subgroup

$$SO(2) < SL_2(\mathbb{R})$$

is a maximal compact subgroup, and any two maximal compact subgroups are conjugate. So we may restate the uniformization theorem in group-theoretic terms as saying that given subgroups

$$\Gamma,\ K < SL_2(\mathbb{R})$$

with K maximal compact and

with Γ acting properly cocompactly on $SL_2(\mathbb{R})/K$, then

$$X \simeq \Gamma \backslash SL_2(\mathbb{R})/K$$

promotes to a compact Riemann surface of genus $g \geq 2$, and conversely, every compact Riemann surface of genus $g \geq 2$ arises in this way, albeit not in a unique way.

Rmk Alternative, one may use

$$\mathfrak{h}^{\pm} = \mathbb{C} \setminus \mathbb{R} = \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R}),$$

in which case, $SL_2(\mathbb{R})$ is replaced by $PGL_2(\mathbb{R})$. //

This formulation of what a compact hyperbolic Riemann surface is admits an obvious p-adic analogue. But first, I must explain what "p-adic" means. Recall that we define $\mathbb{R}$ to be the completion of $\mathbb{Q}$ with respect to the usual absolute value

$$|a| = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{if } a \leq 0. \end{cases}$$

So this absolute value extends uniquely to $\mathbb{R}$ and sequences in $\mathbb{R}$ that are Cauchy with respect to the extended absolute value converge, and

$$(\mathbb{Q}, |-|) \longrightarrow (\mathbb{R}, |-|)$$

is initial among extensions of absolute valued fields with this property.

Let us write $|-|_\infty$ for $|-|$ and call it the archimedean absolute value on $\mathbb{Q}$. Is it the only one? No, for every prime number p, there is an absolute value

$$\mathbb{Q} \xrightarrow{|-|_p} \mathbb{R}$$

called the p-adic absolute value

that is defined as follows: We can write every nonzero $a \in \mathbb{Q}$ as

$$a = p^r \cdot \frac{m}{n}$$

with $r \in \mathbb{Z}$ and $m, n \in \mathbb{Z} \setminus p\mathbb{Z}$, and the integer r is uniquely determined by a. We write

$$v_p(a) = r$$

and call it the p-adic valuation of a. Now, the p-adic absolute value on $\mathbb{Q}$ is defined by

$$|a|_p = \begin{cases} p^{-v_p(a)} & \text{if } a \neq 0 \\ 0 & \text{if } a = 0. \end{cases}$$

It is a non-archimedean (or ultrametric) absolute value in the sense that it satisfies:

1) $|a|_p \geq 0$ and $|a|_p = 0 \Leftrightarrow a = 0$.
2) $|a \cdot b|_p = |a|_p \cdot |b|_p$
3) $|a+b|_p \leq \max\{|a|_p, |b|_p\}$.

Note that 3) implies, but is

stronger than, the triangle-ineq.

$$|a+b|_p \leq |a|_p + |b|_p.$$

The (valued) field of p-adic number is defined to be the completion

$$(\mathbb{Q}, |-|_p) \longrightarrow (\mathbb{Q}_p, |-|_p)$$

of $\mathbb{Q}$ with respect to the p-adic absolute value. Every $a \in \mathbb{Q}_p$ admits a unique expansion

$$a = \Sigma_{i \in \mathbb{Z}} a_i p^i$$

with $0 \leq a_i < p$ integers such that $a_i = 0$ for $i \ll 0$, and sum and multiplication of such expansions are given by the usual formulas with carrys. For example,

$$-1 = \Sigma_{i \geq 0} (p-1) p^i.$$

A consequence of 3) is that the closed unit disc

$$\mathbb{Z}_p = \{ a \in \mathbb{Q}_p \mid |a|_p \leq 1 \} \subset \mathbb{Q}_p$$

is a subring. The open unit disc

$$p\mathbb{Z}_p = \{ a \in \mathbb{Q}_p \mid |a|_p < 1 \} \subset \mathbb{Z}_p$$

is a maximal ideal. It is the unique maximal ideal in $\mathbb{Z}_p$ and it is nonzero. Furthermore, every nonzero ideal $I \subset \mathbb{Z}_p$ is equal to $p^r \mathbb{Z}_p$ for a unique $r \geq 0$, so $\mathbb{Z}_p$ is a principal ideal domain. So $\mathbb{Z}_p$ is a complete discrete valuation ring and $\mathbb{Q}_p = \mathbb{Z}_p[1/p]$ is a complete discrete valuation field. The residue field of $\mathbb{Z}_p$ is

$$\mathbb{Z}_p / p\mathbb{Z}_p \simeq \mathbb{F}_p .$$

The metric topology on $\mathbb{Q}_p$ makes it a locally compact topological space. The subspace $\mathbb{Z}_p \subset \mathbb{Q}_p$ is compact. The induced topology on $SL_2(\mathbb{Q}_p)$ makes it a locally compact topological group, and

$$K = SL_2(\mathbb{Z}_p) \subset SL_2(\mathbb{Q}_p)$$

is a maximal compact subgroup. Thus, if $\Gamma < SL_2(\mathbb{Q}_p)$ is a subgrp. that acts properly compactly on $SL_2(\mathbb{Q}_p)/K$, then

$$\Gamma \backslash SL_2(\mathbb{Q}_p) / K$$

would be our candidate for a p-adic analogue of

$$\Gamma \backslash SL_2(\mathbb{R}) / K.$$

However, whereas $SL_2(\mathbb{R})/K$ promotes to a hyperbolic Riemannian manifold, $SL_2(\mathbb{Q}_p)/K$ instead promotes to a tree, called the Bruhat-Tits building of SL_2. Being a $(p+1)$-regular tree, it shares the feature of the hyperbolic metric that there is "lots of room" at infinity. We will define this tree next time.

But let us define trees. A graph is a tree if it connected and if it has no circuits. Any graph $\Gamma = (V, E, s : s^*(E) \to V)$ has an underlying anima $|\Gamma|$ defined to be the pushout

$$\begin{tikzcd} p_!s_!s^*(E) \simeq s^*(E) \arrow[r, "s"] \arrow[d, "p_!\eta"] & V \arrow[d] \\ p_!(E) \arrow[r] & |\Gamma| \end{tikzcd}$$

in the ∞-category of anima. (Lurie says "spaces" instead of anima.) Now, a graph Γ is a tree if and only if its underlying anima $|\Gamma|$ is contractible,

$$|\Gamma| \simeq 1.$$

By Zorn's lemma, every non-empty graph Γ contains a maximal subtree $\Lambda \subset \Gamma$. Moreover, if a graph Γ is non-empty and connected, then every maximal tree $\Lambda \subset \Gamma$ contains all vertices. Thus, an orientation of Γ determines an equivalence

$$|\Gamma| \xrightarrow{\sim} |\Gamma/\Lambda| \xleftarrow{\sim} \bigvee_{\substack{e \in E_+ \\ e \notin \Lambda}} |\Delta^1/\partial\Delta^1|.$$