The tree of $PGL_2(\mathbb{Q}_p)$

Last time, we discussed the idea of Lubotzky-Phillips-Sarnak to construct Ramanujan graphs as p-adic analogues of connected compact hyperbolic Riemann surfaces. The uniformization thm. for Riemann surfaces may be stated as saying that every such Riemann surface X arises as a double coset space

$$X \simeq \Gamma \backslash PGL_2(\mathbb{R}) / K$$

with $K, \Gamma \subset PGL_2(\mathbb{R})$ a maximal compact subgroup and a discrete subgroup that acts properly freely on $Y = PGL_2(\mathbb{R})/K$, respectively. We defined

$$(\mathbb{Q}, |-|_p) \longrightarrow (\mathbb{Q}_p, |-|_p)$$

to be the completion of $\mathbb{Q}$ w.r.t. the p-adic absolute value

$$|a|_p = p^{-v_p(a)},$$

and we saw that the fact that the p-adic absolute value is non-archimedean implies that the unit disc

$$\mathbb{Z}_p = \{ a \in \mathbb{Q}_p \mid |a|_p \leq 1 \} \subset \mathbb{Q}_p$$

is a subring, the ring of p-adic integers. As topological spaces, $\mathbb{Z}_p$ is compact, and $\mathbb{Q}_p$ is locally compact. The subgroup

$$K = PGL_2(\mathbb{Z}_p) \subset PGL_2(\mathbb{Q}_p)$$

is a maximal compact subgroup, so the coset space

$$V = PGL_2(\mathbb{Q}_p)/K$$

is our candidate for a p-adic analogue of the complex upper half-plane

$$\mathfrak{h} \simeq PGL_2(\mathbb{R})/K.$$

However, whereas $\mathfrak{h}$ has the hyperbolic metric, characterized as the unique, up to scaling, left-invariant metric, the space V does not even have a useful topology. Indeed, the quotient topology on V is discrete, because the subgroup

$$K \subset PGL_2(\mathbb{Q}_p)$$

is open, and hence, also closed. Instead, we define a graph

$$Y = (V, E, s \colon s^*(E) \to V)$$

with V as its set of vertices.

To do so, it is helpful to identify V with the set of orbits for the action by the group of units $\mathbb{Q}_p^* \subset \mathbb{Q}_p$ on the set of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^2$. So let

$$\mathbb{Q}_p^2 = M_{2,1}(\mathbb{Q}_p)$$

be the (right) $\mathbb{Q}_p$-vector space of 2-dimensional column vectors. We may view $\mathbb{Q}_p^2$ as a (right) $\mathbb{Z}_p$-module $j_*(\mathbb{Q}_p^2)$ by restriction of scalars along the inclusion $j \colon \mathbb{Z}_p \to \mathbb{Q}_p$. Now, a $\mathbb{Z}_p$-lattice in $\mathbb{Q}_p^2$ is a submodule

$$L \subset j_*(\mathbb{Q}_p^2)$$

that is free of rank 2. (It is equivalent to require that L be finitely generated as a $\mathbb{Z}_p$-module and that the map of $\mathbb{Q}_p$-vector spaces

$$j^*(L) = L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow \mathbb{Q}_p^2$$

given by the mate of the inclusion be an isomorphism.)

Ex If (e, f) is a basis of $\mathbb{Q}_p^2$ as a $\mathbb{Q}_p$-vector space, then

$$L = \operatorname{span}_{\mathbb{Z}_p}(e, f) \subset j_*(\mathbb{Q}_p^2)$$

is a $\mathbb{Z}_p$-lattice, and conversely, every $\mathbb{Z}_p$-lattice in $\mathbb{Q}_p^2$ arises in this way. Thus, if L_0 is any $\mathbb{Z}_p$-lattice in $\mathbb{Q}_p^2$, say,

$$L_0 = \mathbb{Z}_p^2 \subset j_*(\mathbb{Q}_p^2),$$

then the action by left multiplication by $GL_2(\mathbb{Q}_p)$ on the set $\tilde{V}$ of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^2$ defines a bijection

$$GL_2(\mathbb{Q}_p)/GL_2(\mathbb{Z}_p) \xrightarrow{\sim} \tilde{V}$$

$$g\,GL_2(\mathbb{Z}_p) \longmapsto g \cdot L_0$$

Prop Given two $\mathbb{Z}_p$-lattices

$$L, L' \subset j_*(\mathbb{Q}_p),$$

there exists a basis (e, f) of L and a pair of integers (m, n) s.t. $(p^m e, p^n f)$ is a basis of L'. Moreover, the set $\{m, n\}$ depends only on L and L' and not on the choice of basis (e, f).

Pf The proposition is equivalent to the statement: Given

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q}_p),$$

there exist $k_1, k_2 \in GL_2(\mathbb{Z}_p)$ s.t.

$$g = k_1 \begin{pmatrix} p^m & 0 \\ 0 & p^n \end{pmatrix} k_2 ,$$

and the diagonal matrix is unique, up to permuting the diagonal entries. To prove this, we use "p-adic integer row and column operations" corresponding to right and left multiplication by the following three types of "elementary matrices" in $GL_2(\mathbb{Z}_p)$:

1) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

2) $\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$, $u \in \mathbb{Z}_p^*$

3) $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$, $x \in \mathbb{Z}_p$

We can assume that $|a|_p \geq |b|_p$ and $|a|_p \geq |c|_p$. If not, then we use row and column operations of type 1) to accomplish this. We now use row and column operations of type 3), to get rid of the off-diagonal entries:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow \begin{pmatrix} a & b \\ 0 & d - bc/a \end{pmatrix} \rightsquigarrow$$

$$\begin{pmatrix} a & 0 \\ 0 & \Delta/a \end{pmatrix}, \quad \Delta = \det(g).$$

Finally, we use row operations of type 2) to change this to

$$\begin{pmatrix} p^m & 0 \\ 0 & p^n \end{pmatrix}$$

as desired. Actually, we could as well have arranged that $|a|_p$ be greater than or equal to all of $|b|_p$, $|c|_p$, and $|d|_p$. In that case, at the end, we will have

$$m = \min\{v_p(a), v_p(b), v_p(c), v_p(d)\}$$

$$m+n = v_p(\Delta).$$

I leave it as an exercise to show that these do not change under row and column operations. //

The group $\mathbb{Q}_p^*$ acts on $\tilde{V}$ from the left by $a \cdot L = aL$, and

$$V = PGL_2(\mathbb{Q}_p)/PGL_2(\mathbb{Z}_p)$$

$$\simeq \mathbb{Q}_p^* \backslash GL_2(\mathbb{Q}_p)/GL_2(\mathbb{Z}_p)$$

$$\simeq \mathbb{Q}_p^* \backslash \tilde{V}$$

is the set of orbits for the action. Let $\Lambda, \Lambda' \in V$, and let $L \in \Lambda$ and $L' \in \Lambda'$ be representative lattices. If we choose bases (e, f) of L and $(p^m e, p^n f)$ of L' as in the proposition, then

$$d(\Lambda, \Lambda') = |m-n|_\infty \in \mathbb{Z}.$$

does not depend on the choice of L and L'. Indeed, changing L to aL changes m and n to $m - v_p(a)$ and $n - v_p(a)$, so their difference is unchanged. We define the graph Y be declaring that there is a unique edge e from Λ to Λ' if and only if $d(\Lambda, \Lambda') = 1$.

Thm The graph Y is a tree.

Pf It suffices to show that there is a unique path from Λ_0, the class of $L_0 = \mathbb{Z}_p^2$, to any other vertex Λ. We can choose a representative $L \in \Lambda$ s.t. $L \subset L_0$ and s.t. L_0/L is a finite cyclic p-group. Now, the existence and uniqueness of the Jordan-Hölder sequence of a finite p-group gives the existence and uniqueness of a path from Λ to Λ_0. //

The group $PGL_2(\mathbb{Q}_p)$ acts transitively on V, preserving the relation of adjancency. Thus, Y is a k-regular graph for some k. The vertices adjacent to Λ_0 correspond to the sublattices of L_0 of index p, or equivalently, the subspaces of the $\mathbb{F}_p$-vector space

$$L_0/pL_0 \simeq \mathbb{F}_p^2$$

of dimension 1. These are the

$\mathbb{F}_p$-points on $\mathbb{P}^1$; there are $p+1$ of them. We can represent these by lattices as follows: Let

$$g_i = \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix}, \quad 0 \leq i < p,$$

$$g_\infty = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix},$$

and let $L_i = g_i L_0$, $0 \leq i < p$, $i = \infty$. Then the vertices Λ_i, $0 \leq i < p$ and $i = \infty$, are the vertices adjacent to Λ_0.

We have proved that Y is a $(p+1)$-regular tree equipped with a transitive left action by the group $PGL_2(\mathbb{Q}_p)$. Next time, we will construct subgroups

$$\overline{\Gamma}(N) \subset PGL_2(\mathbb{Q}_p)$$

s.t. $X(N) = \overline{\Gamma}(N) \backslash Y$ is a finite $(p+1)$-regular Ramanujan graph.

We conclude this time with some general remarks about actions by groups on graphs. So let

$$Y = (V, E, s : s^*(E) \to V)$$

be a graph and let

$$\Gamma \xrightarrow{\lambda} \mathrm{Aut}(Y)$$

be a (left) action by a grp. Γ on Y. We would like to say that $\Gamma \backslash V$ and $\Gamma \backslash E$ are the sets of vertices and edges in a graph $\Gamma \backslash Y$. But this is not always true. The issue is that the action by $G = \{1, g\}$ on $\Gamma \backslash E$ may no longer be free. Indeed, if $e \in E$ with $\bar{e} = g \cdot e$, and if there exists $\gamma \in \Gamma$ s.t. $\gamma \cdot e = \bar{e}$, then $\Gamma \cdot e \in \Gamma \backslash E$ is a fixed pt. for the action by G. If such $e \in E$ and $\gamma \in \Gamma$ exist, then we say that Γ acts on Y with inversion. If not, then we say that Γ acts on Y without inversion. If the latter case, the triple consisting of $\Gamma\backslash V$, $\Gamma\backslash E$, and

$$s^*(\Gamma\backslash E) \xrightarrow{\bar{s}} \Gamma\backslash V$$

is a graph $\Gamma\backslash \Upsilon$.

Ex Not every subgroup

$$\Gamma \subset PGL_2(\mathbb{Q}_p)$$

acts without inversion on the Bruhat–Tits building Υ. Indeed, let $\Lambda, \Lambda' \in V$ correspond to the lattices $L, L' \subset j_*(\mathbb{Q}_p^2)$ with bases (e_1, e_2) and (e_1, pe_2), and let $\gamma \in PGL_2(\mathbb{Q}_p)$ be the class of

$$\tilde{\gamma} = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \in GL_2(\mathbb{Q}_p).$$

We have $\tilde{\gamma} L = L'$ and $\tilde{\gamma} L' = pL$, so $\gamma \cdot \Lambda = \Lambda'$ and $\gamma \cdot \Lambda' \overset{!}{=} \Lambda$. So γ maps the unique edge e from Λ to Λ' to the unique edge $\bar{e}$ from Λ' to Λ. //