We wish to find discrete subgroups

$$\Gamma \subset PGL_2(\mathbb{Q}_p)$$

such that the double coset space

$$\Gamma \backslash PGL_2(\mathbb{Q}_p) / K$$

is compact. Here $K \subset PGL_2(\mathbb{Q}_p)$ is a maximal compact subgroup, which we take to be $K = PGL_2(\mathbb{Z}_p)$. Since K is compact, it is equivalent to ask that

$$\Gamma \backslash PGL_2(\mathbb{Q}_p)$$

be compact.

Let us consider the analogous problem of finding discrete

$$\Gamma \subset PGL_2(\mathbb{R})$$

such that

$$\Gamma \backslash PGL_2(\mathbb{R}) / K$$

is compact. We have identified

$PGL_2(\mathbb{R})/K$ with the complex upper halfplane $\mathfrak{h}$, and moreover, the unique, up to scaling, left-invariant metric on $PGL_2(\mathbb{R})/K$ gives rise to the hyperbolic metric on $\mathfrak{h}$. This suggest one way to find appropriate $\Gamma \subset PGL_2(\mathbb{R})$, namely, to find a tesselation of $\mathfrak{h}$ by polygons (This idea goes back to Poincaré.) That this is possible is more readily apparent in the Poincare disc model

$$\Delta = \{ z \in \mathbb{C} \mid |z| < 1 \}$$

with the hyperbolic metric

$$ds^2 = \left(\frac{2}{1-|z|^2} \right)^2 |dz|^2 .$$

We have isometric isomorphisms

$$\mathfrak{h} \longrightarrow \Delta$$

$$z \longmapsto \frac{z-i}{z+i}$$

$$(-i)\,\frac{z+i}{z-i} \longleftarrow\!\shortmid\; z$$

It is intuitively clear that Δ admits a tesselation by regular $2g$-gons for every $g \geq 1$, albeit it may not be so easy to write down the discrete group

$$\Gamma < PGL_2(\mathbb{R})$$

of symmetries of Δ that map tiles to tiles. It is also not clear that this helps with the problem of finding $\Gamma \subset PGL_2(\mathbb{Q}_p)$.

Instead, we will use arithmetic methods developed by Borel and Harish-Chandra, which work simultaneously in the archimean and non-archimedean case. (These methods also work in higher dimension, e.g. for $G = GL_n$, where the geometric methods are unavailable.) Let us consider the group

$$\Gamma = PGL_2(\mathbb{Z}),$$

which we consider simultaneously

as a subgroup $\Gamma \subset PGL_2(\mathbb{R})$ and $\Gamma \subset PGL_2(\mathbb{Q}_p)$ for all p. Every locally compact (topological) group such as $PGL_2(\mathbb{R})$ or $PGL_2(\mathbb{Q}_p)$ admits a unique, up to scaling, left-invariant measure called the Haar measure. It induces a measure on the quotient spaces

$$\Gamma \backslash PGL_2(\mathbb{R}), \quad \Gamma \backslash PGL_2(\mathbb{Q}_p),$$

so we can ask if their volume is finite. For $\Gamma = PGL_2(\mathbb{Z})$, the answer is yes. However, again for $\Gamma = PGL_2(\mathbb{Z})$, the quotient spaces are not compact.

To find different arithmetic subgroups of PGL_2, we will consider quaternion algebras over $\mathbb{Q}$. In general, if F is a field and $u, v \in F^\times$ are units in F, then the associated quaternion algebra

$$D(u, v)_F$$

is the (noncommutative) F-algebra given by the 4-dimensional F-vector space with basis

$$(1, i, j, ij)$$

and with the multiplication determined by the relations

$$i^2 + u = j^2 + v = ij + ji = 0.$$

So for example,

$$(ij)^2 = ijij = -i^2j^2$$
$$= -(-u)(-v) = -uv.$$

Ex The Hamiltonian quaternions is the $\mathbb{R}$-algebra $\mathbb{H} = D(1,1)_{\mathbb{R}}$. //

There is an anti-involution

$$D(u,v)_F^{op} \xrightarrow{\sim} D(u,v)_F$$

that to $q = a + ib + jc + ijd$ assigns its conjugate

$$\bar{q} = a - ib - jc - ijd.$$

We define the reduced norm and the reduced trace of $q \in D(a,b)_F$ by

$$\mathrm{Nrd}(q) = q\bar{q} = a^2 + ub^2 + vc^2 + uvd^2$$

$$\mathrm{Trd}(q) = q + \bar{q} = 2a.$$

We have $\mathrm{Nrd}(q), \mathrm{Trd}(q) \in F$ and

$$\mathrm{Nrd}(q \cdot q') = \mathrm{Nrd}(q) \cdot \mathrm{Nrd}(q')$$

$$\mathrm{Trd}(q + q') = \mathrm{Trd}(q) + \mathrm{Trd}(q').$$

The algebra $D(a,b)_F$ is either a division algebra in the sense that every $q \neq 0$ is invertible, or else it is isomorphic to the matrix algebra $M_2(F)$. The former case happens if and only if $\mathrm{Nrd}(q) = 0$ only for $q = 0$, in which case,

$$q^{-1} = \bar{q}/\mathrm{Nrd}(q).$$

Now, let $F = \mathbb{Q}$, and let

$$D = D(u,v)_{\mathbb{Q}}$$

be a quaternion algebra. We say that D is split at p if

$$D \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq D(u,v)_{\mathbb{Q}_p}$$

is isomorphic to $M_2(\mathbb{Q}_p)$. If not, then we say that D is non-split at p. Similarly, we say that D is split at ∞, if

$$D \otimes_{\mathbb{Q}} \mathbb{R} \simeq D(u,v)_{\mathbb{R}}$$

is isomorphic to $M_2(\mathbb{R})$, and that it is non-split at ∞, otherwise, in which case, $D \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{H}$.

<u>Rmk</u> The set of places S where $D = D(u,v)_{\mathbb{Q}}$ is non-split is finite and $\# S$ is even. For example, $D = D(-1,1)_{\mathbb{Q}}$ is non-split only for $v \in S = \{2, \infty\}$. Moreover, the set S determines D uniquely, up to isomorphism. //

<u>Ex</u> If $p-1$ is divisible by 4, then -1 has a square root in $\mathbb{Q}_p$, and in fact, in $\mathbb{Z}_p$.

In this case, an isomorphism

$$D(1,1)_{\mathbb{Q}_p} \xrightarrow{\sigma} M_2(\mathbb{Q}_p)$$

is given by

$$\sigma(q) = \begin{pmatrix} a+\varepsilon b & c+\varepsilon d \\ -c+\varepsilon d & a-\varepsilon b \end{pmatrix}$$

where $q = a + ib + jc + ijd$ and $\varepsilon^2 = -1$. //

We now fix $D = D(u,v)_{\mathbb{Q}}$ and for $F/\mathbb{Q}$ a field, define

$$G(F) = D_F^\times / Z(D_F)^\times$$

to be the group given by the group of units in $D_F = D(u,v)_F$ modulo its center. If D is split at p, then

$$G(\mathbb{Q}_p) \simeq PGL_2(\mathbb{Q}_p),$$

but if D is non-split at p, then this is a different group.

Ex If $D = D(1,1)_{\mathbb{Q}}$, then

$$G(\mathbb{R}) \simeq \mathbb{H}^\times / \mathbb{R}^\times$$

$$\simeq Sp(1)/\{\pm 1\} \simeq SO(3). \quad //$$

We now assume that $u, v \in \mathbb{Z}$ and that $D = D(u,v)_{\mathbb{Q}}$ is non-split at ∞. We choose a prime p s.t. D is split at p and define

$$\Gamma = G(\mathbb{Z}[\tfrac{1}{p}]) \subset G(\mathbb{Q}).$$

The following is a special case of a theorem of Borel and Harish-Chandra known as "Godement's criterion."

Thm The quotient space

$$\Gamma \backslash (G(\mathbb{Q}_p) \times G(\mathbb{R}))$$

is compact. //

It follows that $\Gamma \backslash G(\mathbb{Q}_p)$ and $\Gamma \backslash G(\mathbb{Q}_p)/K$ are compact. But is the vertex set in a $(p+1)$-regular Ramanujan graph.