**Perspectives in Mathematical Sciences I/III — Part 2**

**Problem 1.** Let $K_n$ be the complete graph with vertex set $V = \{1, 2, \dots, n\}$.[1]

(1) Find the adjacency matrix $\partial_{K_n}$.
(2) Find the characteristic polynomial of $\partial_{K_n}$
[Hint: Try some small $n$; guess the answer; prove that guess is correct.]
(3) Conclude that $K_n$ is an $(n-1)$-regular Ramanujan graph.

**Problem 2.** Let $\Gamma = (V, E, s\colon s^*(E) \to V)$ be a connected graph, and let

$$V \times V \xrightarrow{d} \mathbb{R}$$

be the map that to a pair of vertices $(u, v)$ assigns the smallest non-negative integer $m$ for which there exists a geodesic $\gamma\colon [0, m] \to \Gamma$ with $\gamma(0) = u$ and $\gamma(m) = v$.[2] Prove the following statement:

(1) The map $d$ is a metric. This means:
  (a) For all $u, v \in V$, $d(u, v) \geq 0$ and $d(u, v) = 0$ if and only if $u = v$.
  (b) For all $u, v \in V$, $d(u, v) = d(v, u)$.
  (c) For all $u, v, w \in V$, $d(u, w) \leq d(u, v) + d(v, w)$.

We define the diameter of $\Gamma$ to be the diameter of the metric space $(V, d)$. It is the maximum value $\operatorname{diam}(\Gamma)$ of the function $d\colon V \times V \to \mathbb{R}$, if such a maximum exists, and it is $\infty$, otherwise. Obviously, if $V$ is finite, then $\operatorname{diam}(\Gamma) < \infty$.

(2) Let $(X_n)_{n \geq N}$ be a family of connected $k$-regular graphs such that $X_n$ has $n$ vertices. Show that $\operatorname{diam}(X_n) \to \infty$ as $n \to \infty$.

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[1] So for all $1 \leq i, j \leq n$, there is a single edge $e$ with $s(e) = j$ and $t(e) = i$, if $i \neq j$, and there is no edge $e$ with $s(e) = j$ and $t(e) = i$, if $i = j$.

[2] We view $d$ as a real-valued function, although it takes non-negative integer values.