**Perspectives in Mathematical Sciences I/III — Part 2**

**Problem 1.** We recall that, in general, a graph is a triple

$$\Gamma = (V, E, s\colon s^*(E) \to V)$$

of a set (of vertices) $V$, a set (of edges) $E$ with free action by the group

$$G = \{1, g\},$$

and a map $s$ from the underlying set of $E$ to $V$ that to an edge $e \in E$ assigns its source $s(e) \in V$. We write $\bar{e} = g \cdot e \in E$ and call it the reverse edge of $e \in E$, and we define the target of $e \in E$ to be $t(e) = s(\bar{e}) \in V$.

Now, let $V$ be a group. (I will abuse notation and write $V$ both for the group and for its underlying set.) We let $G$ act on $V$ by $g \cdot v = v^{-1}$. (We say that $G$ acts on $V$ by inversion.) Let $K$ be a set with free action by $G$, and let $f\colon K \to V$ be a $G$-equivariant map. (This means that $f(g \cdot k) = g \cdot f(k) = f(k)^{-1}$ for all $k \in K$.) Given this data, we define the Cayley graph

$$\Gamma = (V, E, s\colon s^*(E) \to V)$$

to have vertex set $V$ the underlying set of the group $V$, to have $G$-set of edges $E$ the set $K \times V$ with the (free) $G$-action defined by $g \cdot (k, v) = (g \cdot k, f(k) \cdot v)$, and to have source map $s$ the map defined by $s(k, v) = v$.

(1) Show that the Cayley graph $\Gamma$ contains a loop if and only if there exists $k \in K$ such that $f(k) = 1 \in V$ is the identity element of the group $V$.

(2) Show that the Cayley graph $\Gamma$ is connected if and only if for every $v \in V$, there exists an $n$-tuple $(k_1, \dots, k_n)$ of elements of $K$ with $n \geq 0$, such that

$$v = f(k_n) \cdots f(k_1).$$

(3) Show that the Cayley graph $\Gamma$ is a tree if and only if for every $v \in V$, there exists a unique $n$-tuple $(k_1, \dots, k_n)$ of elements in $K$ with $n \geq 0$, such that

$$v = f(k_n) \cdots f(k_1).$$