Perspectives in Mathematical Sciences I/III — Part 2

Problem 1. If $D = D(1,1)_{\mathbb{Q}}$ is the standard quaternion algebra, and if p and ℓ are distinct prime numbers such that both $p-1$ and $\ell-1$ are divisible by 4, then the $(p+1)$-regular Ramanujan graph with vertex set

$$\Gamma(2\ell)\backslash\operatorname{PGL}_2(\mathbb{Q}_p)/\operatorname{PGL}_2(\mathbb{Z}_p),$$

which we defined in Lecture 5, may be identified with the Cayley graph[1] associated with the pair $(V, f\colon K \to V)$ defined as follows. The group V is given by

$$V = \begin{cases} \operatorname{PGL}_2(\mathbb{F}_\ell), & \text{if } p \text{ is a quadratic residue modulo } \ell, \\ \operatorname{PSL}_2(\mathbb{F}_\ell), & \text{if } p \text{ is a quadratic non-residue modulo } \ell. \end{cases}$$

The set K is the set of quaternions

$$q = a + ib + jc + ijd,$$

where a, b, c, and d are integers such that $a > 0$ and odd, such that b, c, and d are even, and such that the reduced norm

$$\operatorname{Nrd}(q) = q\bar{q} = a^2 + b^2 + c^2 + d^2$$

is equal to p. It has $p+1$ elements. The group $G = \{1, g\}$ acts on K by

$$g \cdot q = \bar{q},$$

and this action is free. Finally, the map $f\colon K \to V$ is defined by

$$f(q) = \text{class of } \begin{pmatrix} a + \varepsilon b & c + \varepsilon d \\ -c + \varepsilon d & a - \varepsilon b \end{pmatrix},$$

where $\varepsilon \in \mathbb{F}_\ell$ is a square root of -1.

(1) Show that $f\colon K \to V$ is G-equivariant.[2]
(2) Find the order of V, as a function of ℓ.
(3) Choose two (small) primes p and ℓ that satisfy the assumptions above and write down the set K in this case.

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[1] We will use the same notation as in Problem set 2.
[2] Recall that G acts on V by $g \cdot v = v^{-1}$.