ON THE EFFECT OF TIME VARIABILITY OF THE WIND ON RATES OF AEOLIAN SAND TRANSPORT

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ABSTRACT

Mean transport rates over prolonged periods are often calculated by inserting the mean wind speed in the Bagnold formula or in one of the several other published transport rate formulae. The error made, when calculating transport rates in this way for a wind that varies with time, is studied. It is found that the calculated transport rate can differ significantly from the correct transport rate when the time variability of the wind is large, or when the wind speed is close to the threshold, where sand movement ceases. The effects of the inertia of the saltation process and the autocorrelation of the wind speed on transport rate calculations are also considered.

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INTRODUCTION

Wind conditions in the field differ considerably from those in a wind tunnel. In particular, the wind in the field is typically fluctuating strongly, whereas in a wind tunnel it is almost constant. It is therefore difficult to apply results from wind tunnel experiments under field conditions. One problem is how to apply transport rate formulae obtained from wind tunnel studies in the field.

Theoretical as well as empirical investigations have established that the transport rate responds quickly to wind speed variations; see Lee (1987), Anderson and Haff (1988, 1991), McEwan and Willetts (1991, 1993), and Butterfield (1991). Therefore, the non-linearity of all transport rate formuae implies that a considerable error can be made when simply inserting the mean wind speed in a transport rate formula. The present paper is an attempt to estimate the magnitude of this error.

A FIRST ANALYSIS

Several formulae have been published for the transport rate of sand by wind; see e.g. Greeley and Iversen (1985). A transport rate formula based on the present understanding of the physics of wind blown sand was derived in Sørensen (1991). A discussion of this and other transport rate formulae can be found in McEwan and Willetts (1994). In the present paper, the modest aim is to get a feel for the magnitude of the effect of the time variability of the wind on transport rates measured over prolonged periods. To this end, it is sufficient to use the classical Bagnold formula (Bagnold, 1941) that the transport rate is proportional to the friction velocity cubed. This formula certainly explains most of the variation of the transport rate with wind speed, and all other formulae can be thought of as refinements of Bagnold's formula. From Bagnold's formula it follows that the transport rate is proportional to U^3, where U is the mean wind speed at a fixed height in the logarithmic layer.

As mentioned in the introduction, the transport rate responds quickly to wind speed variations. Let us, to simplify matters, first assume that the transport rate at time t is proportional to $U(t)^3$, where $U(t)$ is the instantaneous wind speed at time t (at a fixed height in the logarithmic layer). The wind speed $U(t)$, and hence the transport rate, will fluctuate randomly as time goes. We will assume that it is a stationary stochastic process, i.e. that the wind regime does not change while measurements of the sand transport rate are made. If the amount of sand transported past a particular point is measured over a long period, for instance by means of a trap, the transport rate obtained from such an experiment will be proportional to $E(U^3)$. Here $E(\cdot)$ denotes mathematical expectation (ensemble mean). The usual way of calculating the transport rate over a prolonged period is to insert a time average of the wind speed in the Bagnold formula. This gives, apart from a sampling error, the same as is obtained by inserting $(E(U))^3$ in the Bagnold formula.

This might, however, be way off the true transport rate, which is obtained by inserting $E(U^3)$ in the formula. Indeed,

$$E(U^3) = (E(U))^3 + 3E(U)\mathrm{Var}(U) + E((U - E(U))^3), \tag{0.1}$$

where $\mathrm{Var}(U)$ denotes the variance of the U. From this a formula follows for the relative error (RE) between the true transport rate and the one calculated by inserting the mean wind speed in the Bagnold formula:

$$\mathrm{RE} = \frac{E(U^3) - (E(U))^3}{(E(U))^3} = 3\gamma^2 + \gamma^3\rho, \tag{0.2}$$

where $\gamma = \sqrt{\mathrm{Var}(U)}/E(U)$ is the coefficient of variation of the distribution of the wind speed, while $\rho = E((U - E(U))^3)/\mathrm{Var}(U)^{3/2}$ is the skewness of this distribution. We see that if the wind is very gusty, i.e. if γ is large, the transport rate calculated by inserting the mean wind speed in the Bagnold formula will underestimate the actual transport rate considerably.

If the distribution of the wind speed is symmetric, the skewness is zero. Therefore, if the wind varies in a symmetric way around its mean value $E(U)$, which is usually the case, at least to a good approximation, we have the following simple expression for the relative error:

$$\mathrm{RE} = \frac{3\mathrm{Var}(U)}{E(U)^2} = 3\gamma^2. \tag{0.3}$$

INFLUENCE OF THE THRESHOLD

The analysis in the previous section did not take into account the fact that there exists a threshold wind speed below which no transport takes place. As is well-known from field observations, the sand is mainly transported during bursts of movement separated by quiet periods when the mean wind speed is near the threshold speed. This will, of course, have a profound effect on the mean transport rate measured over a time sufficient long that it includes both types of periods.

To study this effect, we assume that the transport rate is proportional to $U(t)^3$ when $U(t)$ is larger than the critical value u_c while it is zero when $U(t) < u_c$. Let $f(u)$ denote the probability density function of the random variable $U(t)$, and let $\bar{u}$ be the mean wind speed. Then the relative error by inserting the mean wind speed in the Bagnold equation (without a threshold) is

$$\begin{aligned}\mathrm{RE} &= \bar{u}^{-3}\int_{u_c}^{\infty} u^3 f(u)du - 1 \\ &= 3\gamma^2 + \gamma^3\rho - \bar{u}^{-3}\int_0^{u_c} u^3 f(u)du.\end{aligned} \tag{0.4}$$

As one would expect, the influence of the threshold is to reduce RE, and thus it works in the direction opposite to the effect discussed in the previous section. Near the threshold wind speed, the mean wind speed Bagnold equation will therefore either overestimate or underestimate the actual sand transport dependent on which of the two effects is the stronger.

A crude upper bound for the last correction term in formula (4) is

$$\bar{u}^{-3}\int_0^{u_c} u^3 f(u)du \leq (\frac{u_c}{\bar{u}})^3 P(U(t) \leq u_c). \tag{0.5}$$

An estimate of the probability $P(U(t) \leq u_c)$ is the fraction of time the wind speed is below the threshold value u_c. To get a feel for the magnitude of this correction, let us for a while assume that the distribution of the wind speed is Gaussian with variance v (and ignore the fact that there is then a small probability that the wind speed becomes negative). Then $u_c \leq \bar{u} - a\sqrt{v}$ implies that $P(U(t) \leq u_c) \leq \Phi(-a)$, where Φ denotes the cumulative distribution function of the standard normal distribution. The number a will typically be positive, but the formula holds also when a is negative. From a table of the standard normal distrubution, it is seen that if u_c is one standard deviation below $\bar{u}$, then the relative error caused by the threshold is smaller than $0.16(u_c/\bar{u})^3$; when u_c is two standard deviation below $\bar{u}$, the relative error is smaller than $0.025(u_c/\bar{u})^3$.

EFFECT OF THE RESPONSE TIME

There is considerable evidence, theoretical as well as empirical, that the transport rate is not determined by the instantaneous wind speed only, as assumed so far. Rather, the saltation process has a certain response time; see Anderson and Haff (1988, 1991), McEwan and Willetts (1991, 1993), Butterfield (1991), and Shao and Raupack (1992). It is therefore of interest to study to what extend this response time modifies the conclusions above.

A simple way of building the effect of past wind speeds on the transport rate into the transport rate formula is to assume that the transport rate at time t is proportional to $\tilde{U}(t)^3$, where

$$\tilde{U}(t) = \int_0^\infty U(t-s)\varphi(s)ds. \qquad (0.6)$$

Here φ is a positive function satisfying

$$\int_0^\infty \varphi(s)ds = 1. \qquad (0.7)$$

The function φ expresses the weight with which past wind speeds influence the present transport rate. Presumably, $\varphi(s)$ decreases rapidly as s increases. It might also be zero when s is sufficiently large. A simple choise would be $\varphi(s) = e^{-s/T}/T$, where T represents the response time of the saltation system.

As in the previous sections, we assume that the wind speed is a stationary stochastic process and denote the mean wind speed by $\bar{u}$ and the variance of the wind speed by v. Then $\tilde{U}$ is also stationary, and its mean value is

$$\begin{aligned} E(\tilde{U}(t)) &= \int_0^\infty E(U(t-s))\varphi(s)ds \quad (0.8) \\ &= \bar{u}\int_0^\infty \varphi(s)ds = \bar{u}. \end{aligned}$$

If we ignore the effect of the threshold, the relative error made by using the Bagnold formula with $\bar{u}$ inserted is, therefore, given by a formula similar to (2), where γ and ρ are replaced by the coefficient of variation and the skewness, respectively, of the distribution of $\tilde{U}(t)$. To simplify matters, let us assume that the skewness of the distribution of $\tilde{U}(t)$ is zero. Then we need only calculate the variance of $\tilde{U}(t)$. First we find $E(\tilde{U}(t)^2)$:

$$\begin{aligned} &E(\tilde{U}(t)^2) \qquad (0.9) \\ &= E\left(\int_0^\infty \int_0^\infty U(t-s)U(t-w)\varphi(s)\varphi(w)dsdw\right) \\ &= 2\int_0^\infty \int_0^w E(U(t-s)U(t-w))\varphi(s)\varphi(w)dsdw. \end{aligned}$$

In order to continue, we need to make an assumption about the autocorrelation of the wind speed process. Let us first make the simple assumption that the autocorrelation function is $e^{-\rho s}$; i.e. the correlation between wind speeds s time units apart is $e^{-\rho s}$. This implies that the Eulerian macro time-scale of the turbulence is $\tau = \rho^{-1}$. With this assumption,

$$E(U(t)U(s)) = \bar{u}^2 + ve^{-\rho(t-s)} \qquad (0.10)$$

for $t \geq s$. If we, moreover, take $\varphi(s) = e^{-s/T}/T$, we find that

$$\begin{aligned} &E(\tilde{U}(t)^2) \qquad (0.11) \\ &= \bar{u}^2 + 2v\int_0^\infty e^{-\rho w}\varphi(w)\int_0^w e^{\rho s}\varphi(s)dsdw \\ &= \bar{u}^2 + \frac{v}{T\rho+1}, \end{aligned}$$

and hence that the variance of $\tilde{U}(t)$ is

$$\mathrm{Var}(\tilde{U}(t)) = \frac{v}{T\rho+1} = \frac{v}{T/\tau+1}. \qquad (0.12)$$

Using formula (3) with U replaced by $\tilde{U}$, we find that

$$\mathrm{RE} = \frac{3\gamma^2}{T/\tau+1}, \qquad (0.13)$$

where $\gamma = \sqrt{v}/\bar{u}$, as earlier, is the coefficient of variation of the instantaneous wind speed.

The response time T of the saltation is about 1-2 seconds, while the time-scale τ of the turbulence is typically larger, so the correction with T/τ will usually not drastically change conclusions based on formula (3). Note, that the effect found in Section 2 tends to be somewhat diminished.

The exponential autocorrelation function used above is a rather crude approximation. It has turned out that a much better fit to empirical time series of wind data can be obtained by an autocorrelation function of the type

$$(\chi e^{-\rho_1 s} + e^{-\rho_2 s})/(\chi+1), \qquad (0.14)$$

where $\chi > 0$; see Barndorff-Nielsen, Jensen and Sørensen (1990, 1993, 1995), where models of this type were studied and shown to provide a very good fit to a data set recorded on the beach at Ferring on the Danish west coast. For more information on the field experiment and an extensive study of other aspects of the data, see Mikkelsen (1988, 1989). A correlation function of the form (14) expresses the fact that there are typically more than one correlation time scale in turbulence data. Incidentally, this is also the case of many other types of data, for instance, time series of prices in financial markets. The two length scales are $\tau_1 = \rho_1^{-1}$ and

$\tau_2 = \rho_2^{-1}$. The usual Eulerian macro time-scale is now $\tau = (\chi\tau_1 + \tau_2)/(\chi + 1)$.

If we repeat the calculations above using the autocorrelation function (14), we find that

$$\begin{aligned} \mathrm{RE} &= \frac{3\gamma^2}{\chi+1}\left[\frac{\chi}{T\rho_1+1} + \frac{1}{T\rho_2+1}\right] \quad (0.15) \\ &= \frac{3\gamma^2}{\chi+1}\left[\frac{\chi}{T/\tau_1+1} + \frac{1}{T/\tau_2+1}\right]. \end{aligned}$$

By inserting $T = 1$ and the values of ρ_1, ρ_2, and χ estimated from the Ferring data in Barndorff-Nielsen et al. (1993), it is found that $\mathrm{RE} = 3\gamma^2 \cdot 0.81$.

Under the assumption made in this section that the transport rate is proportional to $\tilde{U}(t)^3$, the effect of the threshold u_c can be studied in analogy to the analysis in Section 3. The correction term is as the last term in (4) with f replaced by the probability density function of $\tilde{U}(t)$.

DISCUSSION AND CONCLUSION

What lies behind the problems discussed in this paper, is the well-known fact that for a random variable X, $E(f(X))$ is usually not equal to $f(E(X))$ when the function f is not linear. If f is convex, then $E(f(X)) \geq f(E(X))$. This result is known as Jensen's inequality. The relative error discussed in Section 2 is due to the fact that $E(U^3) \geq (E(U))^3$, because the function $f(u) = u^3$ is convex (for $u \geq 0$). We will therefore refer to it as the Jensen effect. The relative error caused by the Jensen effect is positive, i.e. the actual transport rate is larger than what is predicted by inserting the mean wind speed in the Bagnold formula. In most cases, the coefficient of variation γ will presumably satisfy $0 < \gamma < 1$, implying that the relative error due to the Jensen effect will not be huge. If, for instance, $\gamma = \frac{1}{3}$, then formula (3) predicts an error of 11 per cent. For $\gamma = \frac{1}{2}$, it is 38 per cent. In cases where $\gamma \geq 1$, the effect predicted by formula (3) is dramatic.

The error caused by the threshold wind speed is in the direction opposite to that caused by the Jensen effect. It might therefore, in fortunate cases, to some extent cancel the Jensen effect, but it might also be larger than the Jensen effect and thus cause the transport rate to be overestimated. When the mean wind speed, $\bar{u}$, is much larger than the threshold speed, u_c, the effect is negligible. If $\bar{u} = u_c$ and if we assume that the probability distribution of the wind speed is symmetric about $\bar{u}$, the upper limit to the effect of the threshold given by formula (5) is 0.5, i.e. 50 per cent. If $\bar{u} < u_c$, the error made is very considerable, but it is doubtful that anyone would use the Bagnold formula without a threshold in this situation. Note, that it is rather easy to estimate the probability $P(U(t) \leq u_c)$, which appears in the upper limit given by formula (5), from wind data or by field observation. It can be estimated by the fraction of time, where the wind speed is below the threshold value, or the fraction of time, where there is no sand movement.

The effect of the saltation response time and the autocorrelation of the wind is also to reduce the Jensen effect. In contrast to what was found for the threshold effect, the reduction is here by a factor, so RE cannot become negative. If the saltation response time T is one second and the Eulerian time scale of the turbulence is five seconds, then by formula (13), the Jensen effect is reduced by the factor 0.83. These choises are not unrealistic, and the result is in accordance with the factor 0.81 found in Section 4 by formula (15) using estimates based on the Ferring data.

Several other effects have, on purpose, not been taken into account in the analysis made in this paper. One example is the effect of the changes in the wind direction. The analysis could presumably be extended to include some other effects. However, the aim of this paper was to obtain simple and easily interpretable expressions by considering a few main effects only and by using a simple, but realistic, transport rate formula.

When measuring wind speeds in an area over a prolonged period in order to estimate the sand transport (for instance, in connection with construction work), it is standard to record mean wind speeds over certain periods and insert these in a transport rate formula. By suitably averaging the numbers obtained for the different periods, an estimate of the sand transport is obtained. Formula (3) suggests that in such investigations an estimate of the variance in each period should also be stored. Then it would be possible to correct results obtained by inserting the mean values in the Bagnold formula. The results in Sections 3 and 4 indicate that such a correction could to some extent result in overestimation of the amount of sand transported, but use of the Bagnold formula without correction for the Jensen effect could badly underestimate the transport rate. Of course, more elaborate correction could be made using the formulae in Sections 3 and 4 if the necessary information is available or can be estimated theoretically.

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