A Fourier Analysis of Extremal Events

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Overview

Extremogram and Regularly Varying Sequence

- A sequence (X_t) is regularly varying if $(X_0, \ldots, X_h)$ is regularly varying, that is,

$$nP(a_n^{-1}(X_0, \ldots, X_h) \in C) \;\to\; \mu_h(C)\,, \quad h \geq 0\,.$$

  Then it follows $\mu_h(tC) = t^{-\alpha}\mu_h(C)$, $t > 0$.
  $P(|X_0| > a_n) = 1/n$.

- For a d-dimensional strictly stationary time series (X_t), the extremogram at lag $h \geq 1$

$$\begin{aligned}\rho(h) &= \lim_{x\to\infty} P(x^{-1}X_h \in A | x^{-1}X_0 \in A) \\ &= \frac{\mu_h(A \times \mathbb{R}^{h-1} \times A)}{\mu_0(A)}\,.\end{aligned}$$

Examples of Regularly Varying Sequences

Assumptions: Dimension $d = 1$ and $A = (1, \infty)$.

Example:

- An iid sequence (Z_t) with Z_t student(α)-distributed. Extremograms: $\rho(0) = 1$ and $\rho(h) = 0$ for $h \geq 1$.
- ARMA process with iid regularly varying noise.
- Max-stable processes.
- GARCH with normally distributed noise.

Estimation of Extremogram

The sample extremogram is defined as:

$$\hat{\rho}(h) = \frac{(\mathbf{m_n/n}) \sum_{t=1}^{n-h} I_{\{a_m^{-1}X_t>1, a_m^{-1}X_{t+h}>1\}}}{(\mathbf{m_n/n}) \sum_{t=1}^{n} I_{\{a_m^{-1}X_t>1\}}} \to \rho(h)\,.$$

where $m_n \to \infty$ and $m_n/n \to 0$ as $n \to \infty$. (See Davis and Mikosch (2009))

$$(m_n/n) \sum_{t=1}^{n} I_{\{a_m^{-1}X_t>1, a_m^{-1}X_{t+h}>1\}} \xrightarrow{L^2} \mu_h(1,\infty)\,.$$

For $h > 0$, *(pre-asymptotic central limit theorem)*

$$\left(\frac{n}{m_n}\right)^{1/2} (\widehat{\rho}(h) - P(X_h > a_m | X_0 > a_m)) \xrightarrow{d} N(0, \sigma^2(h))$$

Example: ACF vs. Extremogram

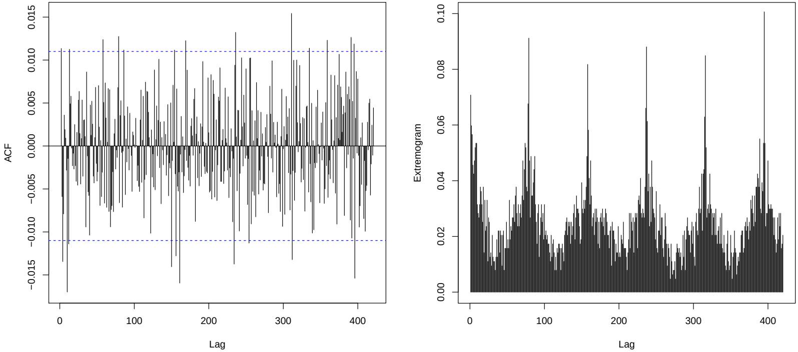


Figure : 5-min stock price of Bank of America. The sample size is about $32,000$.

Spectral Density and Periodogram

- *Spectral density:* $f(\lambda) = 1 + 2\sum_{h=1}^{\infty} \rho(h) \cos(h\lambda)$.
- *Periodogram:*

$$
\begin{aligned}
I_n(\lambda) &= \frac{(m_n/n)\left|\sum_{t=1}^{n} I_t e^{-it\lambda}\right|^2}{(m_n/n)\sum_{t=1}^{n} I_t} \\
&= \frac{0.5\left[(\alpha_n(\lambda))^2 + (\beta_n(\lambda))^2\right]}{(m_n/n)\sum_{t=1}^{n} I_t},
\end{aligned}
$$

where $\lambda \in (0, \pi)$ and $I_t = I_{\{X_t/a_{m_n} > 1\}}$,

$$
\begin{aligned}
\alpha_n(\lambda) &= (2m_n/n)^{1/2} \sum_{t=1}^{n} I_t \cos(\lambda t), \\
\beta_n(\lambda) &= (2m_n/n)^{1/2} \sum_{t=1}^{n} I_t \sin(\lambda t).
\end{aligned}
$$

Central Limit Theorem

Theorem

Assume that (X_t) satisfies some mild conditions.

1. *For any fixed frequencies $0 < \lambda_1 < \cdots < \lambda_N < \pi$ for some $N \geq 1$, then as $n \to \infty$,*

$$\begin{aligned} & \big(\alpha_n(\lambda_j), \beta_n(\lambda_j)\big)_{j=1,\ldots,N} \\ (1) \qquad & \xrightarrow{d} \big(\alpha(\lambda_j), \beta(\lambda_j)\big)_{j=1,\ldots,N} \sim N(\mathbf{0}, \Sigma_N)\,, \end{aligned}$$

where $\Sigma_N = \mathrm{diag}\big(f(\lambda_1), f(\lambda_1), \ldots, f(\lambda_N), f(\lambda_N)\big)$.

2. (1) *remains valid for distinct Fourier frequencies $\lambda_{j_n} \to \lambda_j \in (0, \pi)$ as $n \to \infty$.*

Asymptotic Distribution of Periodogram Ordinates

Corollary

1. *Consider any fixed frequencies* $0 < \lambda_1 < \cdots < \lambda_N < \pi$ *for some* $N \geq 1$. *As* $n \to \infty$,

$$\big(I_n(\lambda_j)\big)_{j=1,\ldots,N} \xrightarrow{d} \big(f(\lambda_j)E_j\big)_{j=1,\ldots,N};$$

where (E_j) *is a sequence of iid standard exponential random variables.*

2. *Analog results hold for any distinct Fourier frequencies* $\lambda_{j_n} \to \lambda_j \in (0, \pi)$.

Example: ARMA$(1, 1)$

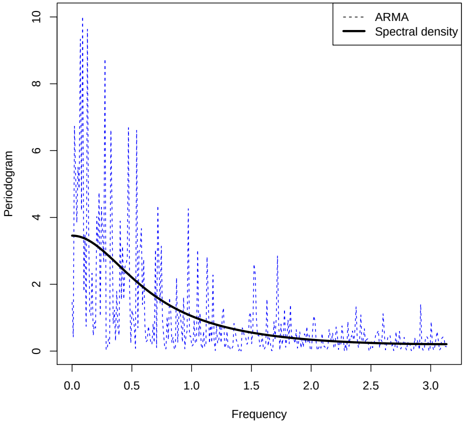


Figure : $X_t = 0.8X_{t-1} + Z_t + 0.1Z_{t-1}$. The sample size is $31,747$.

Two Methods

1. **Lag Window Estimator:**

$$\widetilde{f}_{nA}(\lambda) = 1 + 2\sum_{h=1}^{r_n} \cos(\lambda h)\, \widehat{\rho}(h)$$

for fixed $\lambda \in (0, \pi)$.

2. **Discrete Spectral Average Estimator:**

$$\widehat{f}_n(\lambda) = \sum_{|u| \le s_n} W_n(u) I_n(\lambda_{nu}),$$

where $\lambda_{nu} = \lambda_n + 2\pi u/n \in (0, \pi)$, for $1 \le u < n$, and $(W_n(\cdot))$ and (s_n) satisfies certain conditions.

Conditions on $(W_n(\cdot))$ and (s_n)

$$\widehat{f}_n(\lambda) = \sum_{|u| \le s_n} W_n(u) I_n(\lambda_{nu}),$$

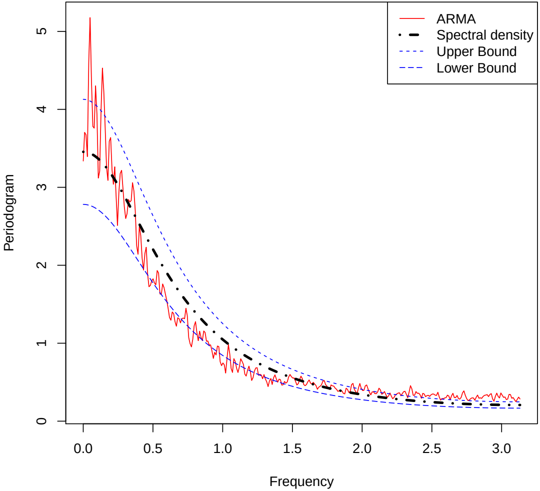
where $\lambda_{nu} = \lambda_n + 2\pi u/n \in (0, \pi)$.

- $s_n \to \infty$ and $s_n/n \to 0$ as $n \to \infty$.
- $\sum_{|u| \le s_n} W_n(u) = 1$ (Unbiased).
- $\sum_{|u| \le s_n} W_n^2(u) \to 0$ (Consistency).

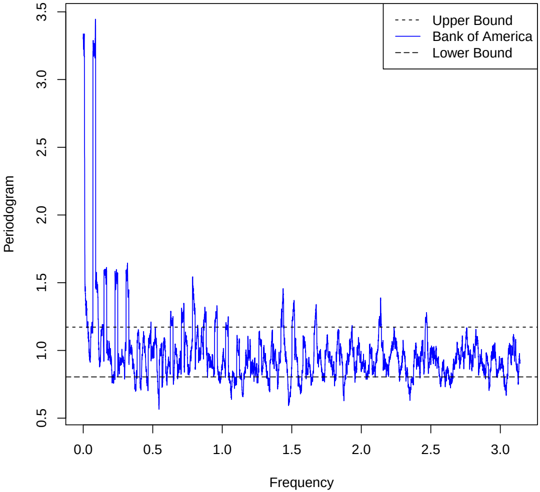
Results:

$$\sum_{|u| \le s_n} W_n(u) I_n(\lambda_{nu}) \xrightarrow{L^2} f(\lambda).$$

Example: ARMA$(1,1)$



Example: Bank of America



Central Limit Theorem

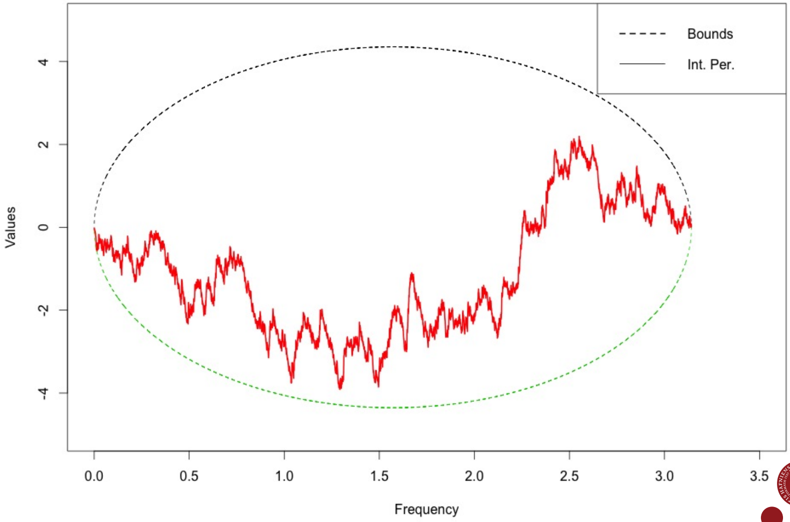
Assumption: (X_t) is an i.i.d regularly varying sequence. $\gamma_n(h) = (m/n)\sum_{t=1}^{n-h} I_{\{X_t/a_m>1\}} I_{\{X_{t+h}/a_m>1\}}$ and $p_0 = P(X_0/a_m > 1)$.

- For $h \geq 1$, $\sqrt{n}\big(\gamma_n(i) - E\gamma_n(i)\big)_{i=1,\ldots,h} \xrightarrow{d} N(\mathbf{0}, \overline{\Sigma}_h)$, where $\overline{\Sigma}_h = (\sigma_{ij})_{i,j=1,\ldots,h}$ has finite entries given by $\sigma_{ii} = 1$ and $\sigma_{ij} = 0$ for $i \neq j$.
- Define $J_n(x) = x\gamma_n(0) + 2\sum_{h=1}^{n-1} \gamma_n(h)\sin(hx)/h$. We have

$$\sqrt{n}\Big(J_n(x) - x\gamma_n(0) - \big(EJ_n(x) - xmp_0\big)\Big) \xrightarrow{d} G(x),$$

where the limit process is given by the infinite series $G(x) = 2\sum_{h=1}^{\infty} Z_h \sin(hx)/h$ and (Z_h) is an i.i.d normal sequence with mean 0 and variance 1.

Integrated Periodogram



Goodness-of-fit Tests

1. Grenander-Rosenblatt test:

$$\sqrt{n} \sup_{x\in[0,\pi]} \left| J_n(x) - x\gamma_n(0) - \left(EJ_n(x) - xmp_0\right) \right|$$

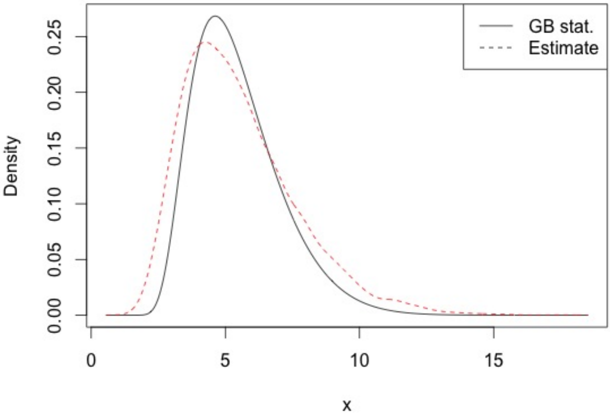
$$\stackrel{d}{\to} \sup_{x\in[0,\pi]} |G(x)| \,.$$

2. ω^2-statistic or Cramér-von Mises test:

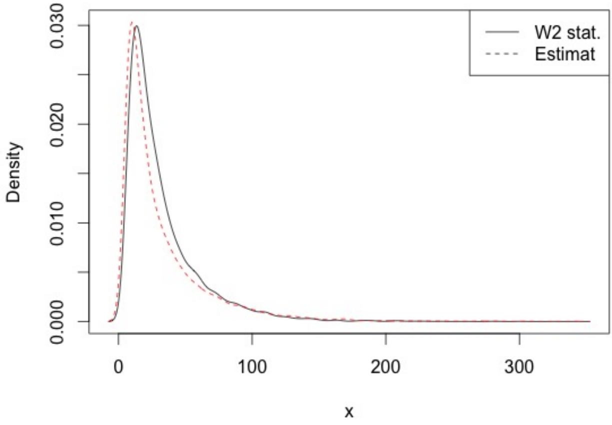
$$n \int_0^x \left(J_n(\lambda) - x\gamma_n(0) - \left(EJ_n(x) - xmp_0\right) \right)^2 d\lambda$$

$$\stackrel{d}{\to} \int_0^x G^2(\lambda)\, d\lambda \,.$$

Grenander-Rosenblatt Statistic with Threshold 93.3%



ω^2-statistic with Threshold 95.6%



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