Extremes and sums of regularly varying observations

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based on the joint work with

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Stationary regularly varying sequences

- A stationary time series $(X_n)_n$ is said to be **regularly varying** if random vectors

$$(X_0, \ldots, X_k) \quad k \geq 0$$

are regularly varying for each k.

A random vector $\mathbf{X}$ is **regularly varying** with tail index α if

- $\|\mathbf{X}\|$ is regularly varying, ie $P(\|\mathbf{X}\| > u) = u^{-\alpha}L(u)$,
- and for $x \to \infty$

$$\frac{\mathbf{X}}{\|\mathbf{X}\|}\bigg| \|\mathbf{X}\| > x \xrightarrow{d} \Theta$$

Or alternatively for $x \to \infty$

$$\left(\frac{\|\mathbf{X}\|}{x}, \frac{\mathbf{X}}{\|\mathbf{X}\|}\right)\Bigg| \, \|\mathbf{X}\| > x \xrightarrow{d} (R, \Theta)$$

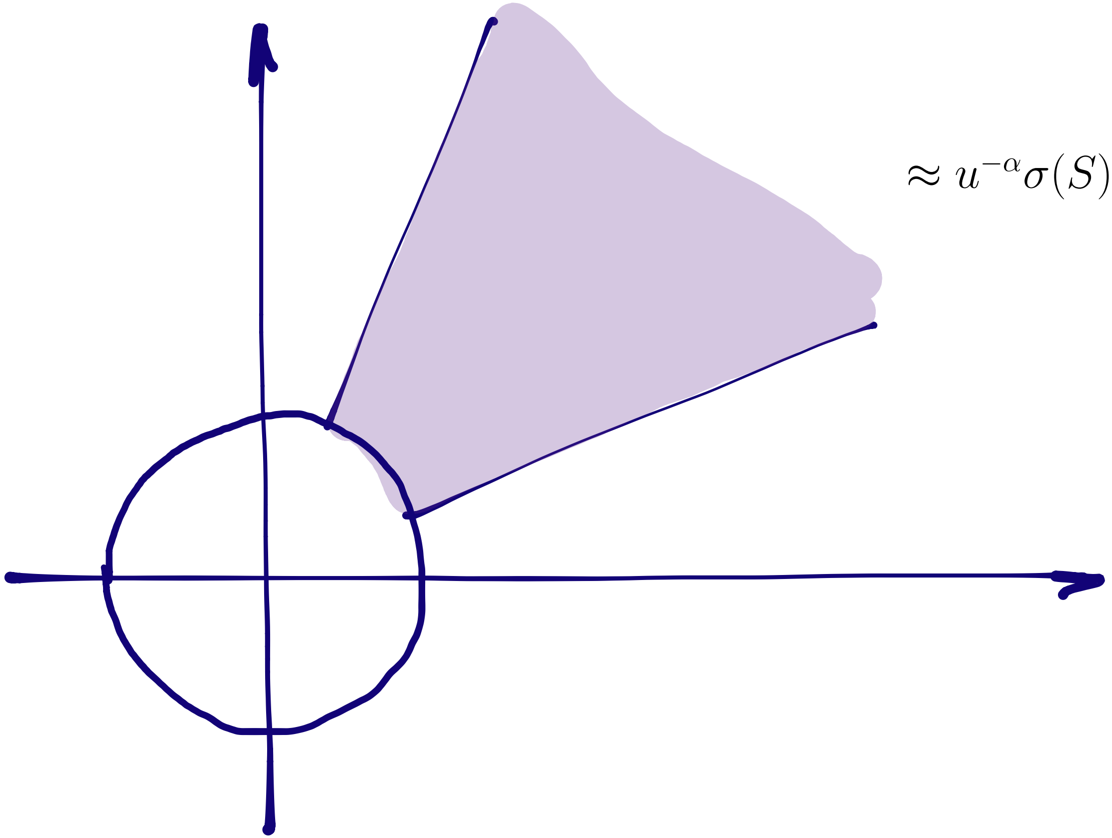
with R Pareto(α) and independent of $\Theta \sim \sigma$ on $\mathbb{S}^{k-1}$, thus

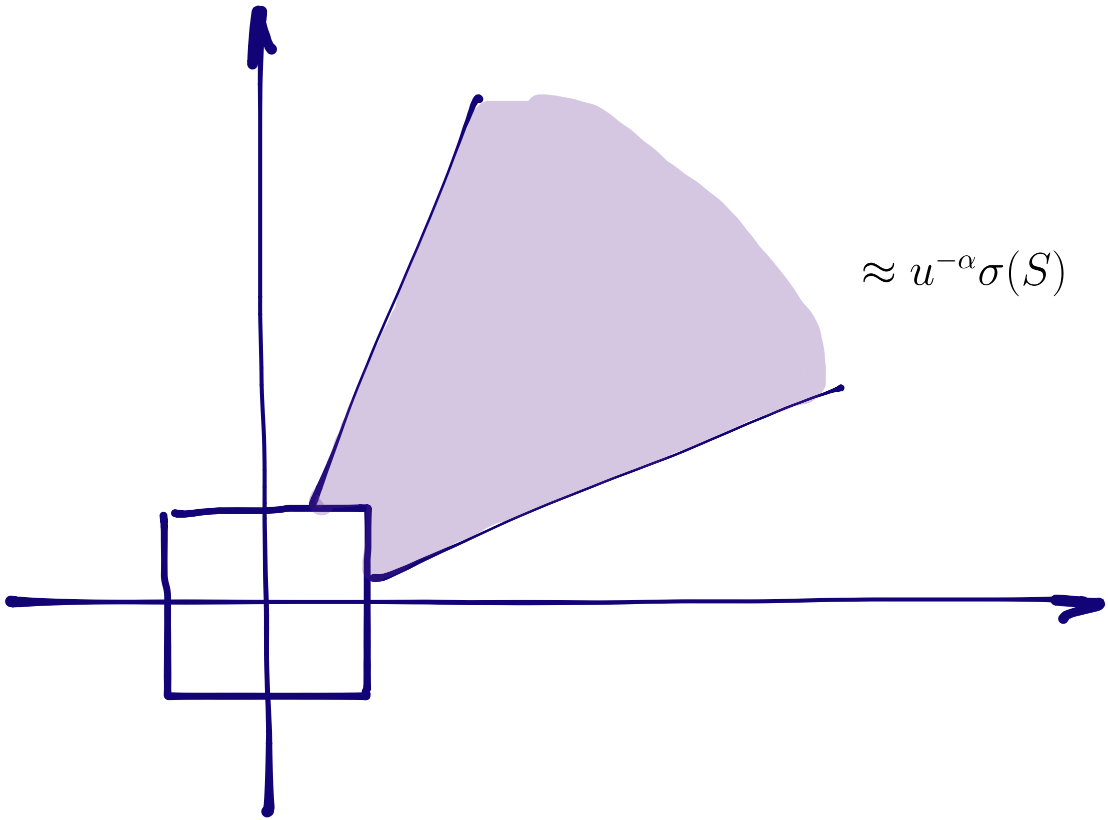
$$\frac{\mathbf{X}}{x}\Bigg| \, \|\mathbf{X}\| > x \xrightarrow{d} R \cdot \Theta$$

We write $\mathbf{X} \sim$ RV(α, σ).

In dimension one

$$\Theta \sim \begin{pmatrix} -1 & 1 \\ q & p \end{pmatrix} .$$


$\approx u^{-\alpha}\sigma(S)$


$\approx u^{-\alpha}\sigma(S)$

Regular variation is further equivalent to

$$\frac{P(\mathbf{X}/x \in \cdot\,)}{P(\|\mathbf{X}\| > x)} \stackrel{v}{\to} \mu'(\,\cdot\,)\,,$$

as $x \to \infty$ on $\overline{\mathbb{R}}^k \setminus \{0\}$.

Note: definition is independent on the choice of norm on $\mathbb{R}^k$ and the normalizing event can be altered, so for instance using $\|\mathbf{X}\| = \max\{|X_1|, \ldots, |X_k|\}$, this is equivalent to

$$\frac{P(\mathbf{X}/x \in \cdot\,)}{P(|X_0| > x)} \stackrel{v}{\to} C\mu'(\,\cdot\,) =: \mu(\,\cdot\,)\,,$$

as $x \to \infty$ on $\bar{\mathbb{R}}^k \setminus \{0\}$.

Therefore for a stationary regularly varying sequence (X_t) as $x \to \infty$

$$\left(\frac{|X_0|}{x}, \frac{(X_0,\ldots,X_k)}{|X_0|}\right)\Bigg|\, |X_0| > x \stackrel{d}{\to} (R, (\theta_0,\ldots,\theta_k))$$

with $R \sim$ Pareto(α) and independent of $(\theta_0,\ldots,\theta_k)$ which is not necessarily $\mathbb{S}^k$–valued any more.

Thus, for each $k > 0$

$$\begin{aligned}\frac{(X_0,\ldots,X_k)}{x}\Bigg|\, |X_0| > x &\stackrel{d}{\to} R(\theta_0,\ldots,\theta_k) \\ &=: (Y_0,\ldots,Y_k)\end{aligned}$$

Clearly

$$|\theta_0| = 1 \quad \text{and} \quad |Y_0| = R$$

By construction, distributions of $(\theta_0, \ldots, \theta_k)$ and $(Y_0, \ldots, Y_k)$ satisfy Kolmogorov's consistency critera and therefore there exists a **tail process**

$$(Y_t)_{t \in \mathbb{Z}}$$

such that

$$\left. \left(\frac{X_t}{x} \right)_{t \in \mathbb{Z}} \right| |X_0| > x \xrightarrow{d} (Y_t)_{t \in \mathbb{Z}}$$

and a **spectral tail process**

$$(\theta_t)_{t \in \mathbb{Z}}$$

independent of $|Y_0|$ such that

$$(Y_t)_t \stackrel{d}{=} |Y_0| (\theta_t)_t$$

Note there exists a sequence (a_n) such that

$$nP(|X_0| > a_n u) \to u^{-\alpha}$$

for $u > 0$ and

$$\left(\frac{X_t}{a_n}\right)_{t\in\mathbb{Z}} \Bigg| \, |X_0| > a_n \xrightarrow{d} (Y_t)_{t\in\mathbb{Z}}.$$

Examples (for simplicity, all nonnegative i.e. $\theta_0 = 1$)

a) $X_t = X \sim$ RV(α), $\quad \theta_t = 1$, for all t.

b) X_t iid RV(α), $\quad \theta_t = 0$, for $t \neq 0$.

c) $X_t = Z_t \vee Z_{t-1}$, Z_t iid RV(α), $\quad \theta_t = 0$, for $|t| \geq 2$

$$(\theta_{-1}, \theta_0, \theta_1) \sim \begin{cases} (1,1,0) & \text{with prob. } 1/2 \\ (0,1,1) & \text{with prob. } 1/2 \end{cases}$$

d) $X_t = Z_t + \frac{1}{2} Z_{t-1}$, Z_t iid RV(α), $\quad \theta_t = 0$, for $|t| \geq 2$

$$(\theta_{-1}, \theta_0, \theta_1) \sim \begin{cases} (0, 1, \frac{1}{2}) & \text{with prob.} \quad p = 1/(1 + (1/2)^\alpha) \\ (2, 1, 0) & \text{with prob.} \quad 1 - p \end{cases}$$

e) $X_t = A_t X_{t-1} + B_t$, with (A_t, B_t) iid satisfying Kesten's (1973) conditions, $X_t \sim$ RV(α) and for $t = 0, 1, 2, \ldots$

$$\theta_t = A_t A_{t-1} \cdots A_1 \,.$$

There is a subtle and somewhat startling connection between the past and the future of the tail process, so for instance

$$P(\theta_{-t} \neq 0) = E|\theta_t|^\alpha \,.$$

Point processes

Point process is a random Radon point measure, i.e. mapping

$$N : \Omega \to M_p$$

where M_p denotes a set of point measures on some fixed state space $\mathbb{E}$. For $m \in M_p$

$$m = \sum_i \delta_{x_i}.$$

Hence

$$\int f dm = \sum_i f(x_i) =: f(m)$$

However, M_p needs topology and even more desperately, a σ–algebra.

Vague topology is introduced by

$$m_n \xrightarrow{v} m$$

if

$$f(m_n) \to f(m).$$

for all f cont. with compact supp.

Poisson point process (PRM) N with intensity measure μ satisfies

▷ $N(A) \sim \text{Poisson}(\mu(A))$ for all A,

▷ $N(A_1), N(A_2), \ldots, N(A_k)$ are independent for all disjoint $A_1, A_2, \ldots, A_k$

Note

$$N_n \xrightarrow{d} N$$

again means

$$Ef(N_n) \xrightarrow{d} Ef(N)$$

for any bounded f continuous in vague topology.

Distribution of the point process

$$N = \sum_i \delta_{P_i}$$

is uniquely determined by **Laplace functionals** of the form

$$E^{-f(N)} = Ee^{-\sum_i f(P_i)},$$

for $f \in C_K^+$

Laplace functionals of $N \sim$ PRM(μ) take form

$$E^{-f(N)} = \exp\left[-\int_E (1 - e^{f(x)}) d\mu(x)\right] ,$$

for $f \in C_K^+$

For our stationary and regularly varying sequence (X_i) consider

$$N_n = \sum_{i=1}^{n} \delta_{(i/n, X_i/a_n)},$$

on the space

$$\mathbb{E} = [0,1] \times \overline{\mathbb{R}} \setminus \{0\}.$$

To check

$$N_n \stackrel{d}{\to} N = \sum_i \delta_{(T_i, P_i)}$$

one can use Laplace functionals and show

$$Ee^{-f(N_n)} \stackrel{d}{\to} Ee^{-f(N)}$$

ie

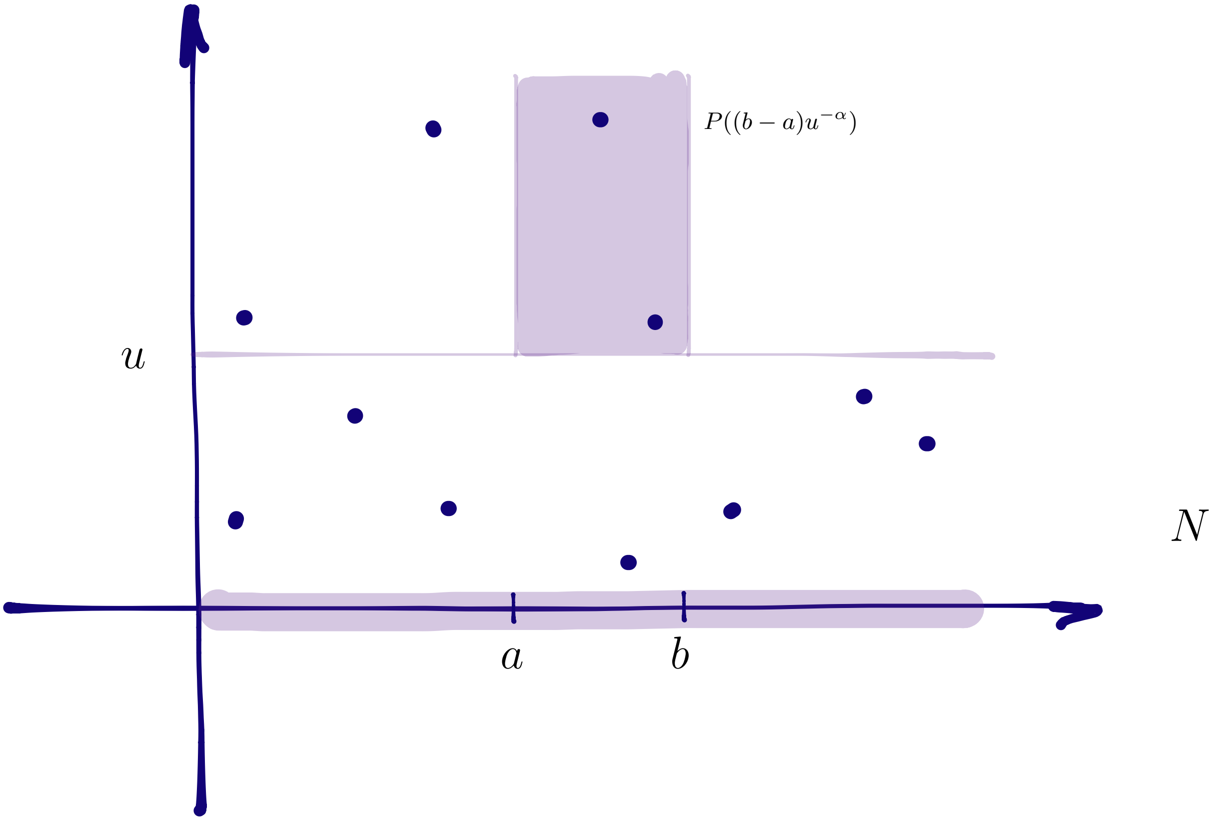
$$E \exp\left\{-\sum_i f\left(\frac{i}{n}, \frac{X_i}{a_n}\right)\right\} \stackrel{d}{\to} E \exp\left\{-\sum_i f\left(T_i, P_i\right)\right\}$$

for all f nonnegative continuous with relatively compact support ie $f \in C_K^+$.

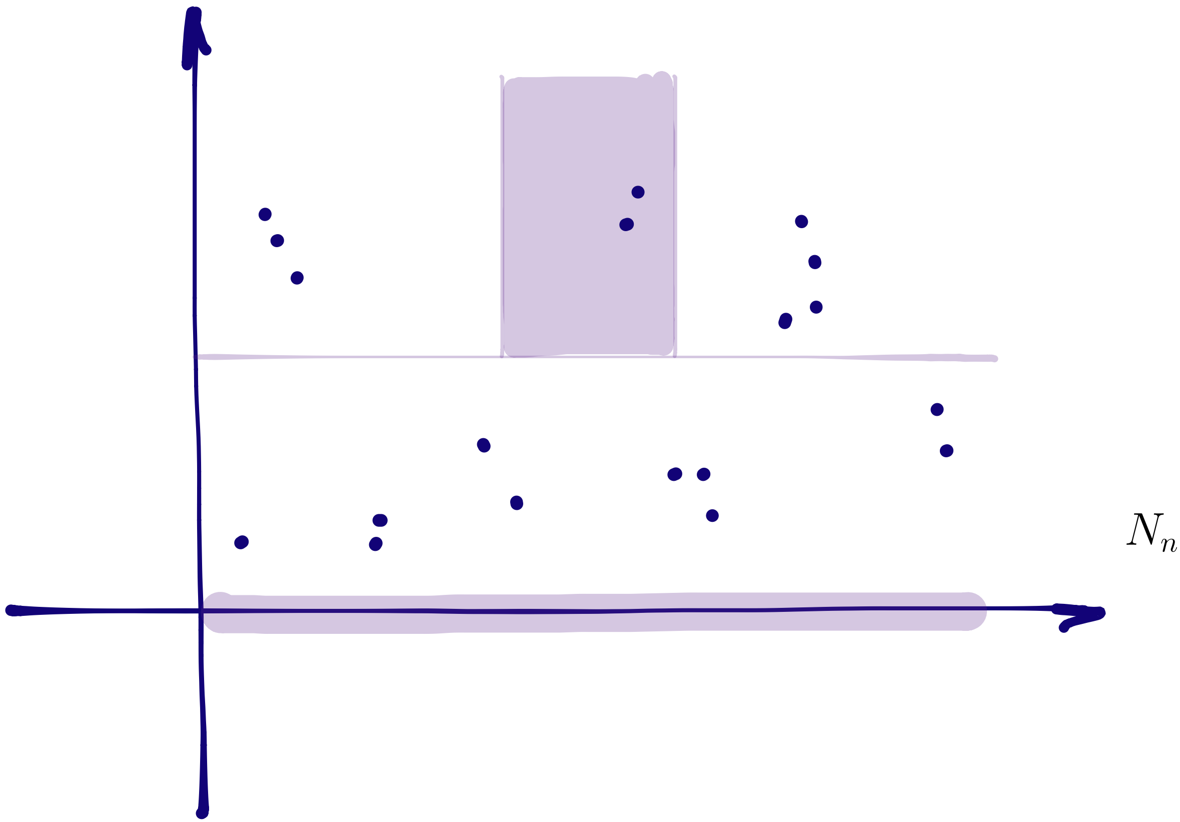
Theorem For iid $\mathbf{X}_t$, $\mathbf{X}_0 \sim \mathrm{RV}(\alpha, \sigma)$ is equivalent to

$$\sum_1^n \delta_{\frac{i}{n}, \frac{\mathbf{X}_i}{a_n}} \stackrel{d}{\to} N\,,$$

where N is PRM(Leb$\times\mu$).


$P((b-a)u^{-\alpha})$
u
N
a
b

Extremes of dependent sequences cluster


N_n

Main idea: try to break the series into "nearly independent blocks" of size

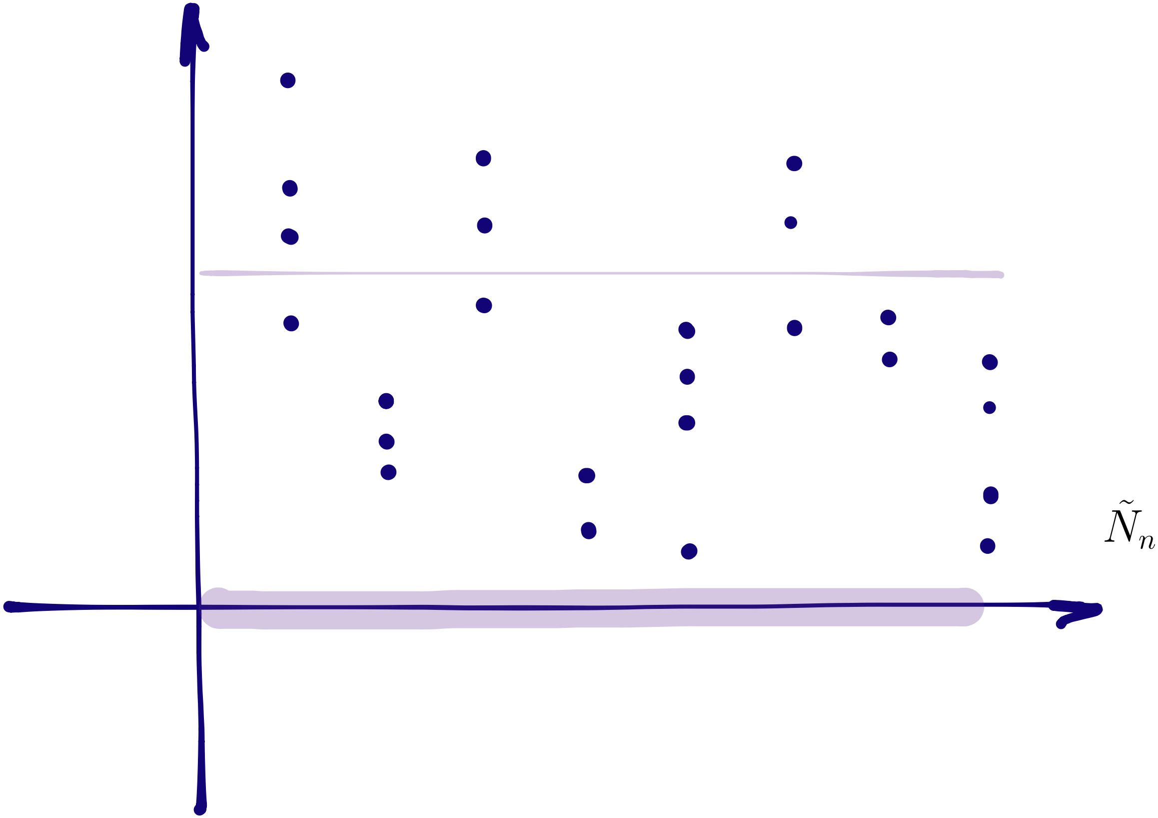
$$r_n, \quad r_n \to \infty, \quad \frac{n}{r_n} \to \infty\,,$$

so that for $k_n = \lfloor n/r_n \rfloor$

$$N_n \overset{d}{\approx} N_1^{r_n} + \cdots + N_k^{r_n} =: \tilde{N}_n$$

for independent

$$N_j^{r_n} \overset{d}{=} \sum_{i=1}^{r_n} \delta_{(jr_n/n, X_i/a_n)}$$


$\tilde{N}_n$

Weak dependence condition

or $\mathcal{A}'(a_n)$ condition

WDC Following Davis and Hsing (1995) we introduce the following condition (implied by strong mixing): for some r_n and all f as above

$$Ee^{-f(N_n)} + o(1) = Ee^{-f(\tilde{N}_n)} = Ee^{-[f(N_1^{r_n})+\cdots+f(N_k^{r_n})]} = \prod_{j=1}^{k_n} Ee^{-f(N_j^{r_n})}$$

Therefore it is sufficient to study clusters with a fixed time coordinate. Note $f \in C_K^+$ has a support on $|x| > \varepsilon$ for some $\varepsilon > 0$, thus

$$
\begin{aligned}
Ee^{-f(N^{rn})} &= Ee^{-\sum_1^{rn} f(X_i/a_n)} \\
&= P\left(M_{r_n} \le a_n\varepsilon\right) \\
&+ E\left(e^{-\sum_1^{rn} f(X_i/a_n)}\middle| M_{r_n} > a_n\varepsilon\right) \cdot P\left(M_{r_n} > a_n\varepsilon\right)
\end{aligned}
$$

where $M_{r_n} = \max\{|X_1|, \ldots, |X_{r_n}|\}$.

Understanding asymptotics of extremes boils down to understanding behavior of the two terms on the rhs.

Or at least

- $E\left(\sum_1^{r_n} \mathbb{I}_{|X_i|>a_n} \middle| M_{r_n} > a_n\right)$
- $P\left(\sum_1^{r_n} \mathbb{I}_{|X_i|>a_n} \geq k \middle| M_{r_n} > a_n\right)$

First, we need to restrict dependence within the cluster

Finite mean cluster size condition

or anti-clustering condition

FCC The high level exceedances are not clustering for "too long":

$$\lim_{m\to\infty} \limsup_{n\to\infty} P\left(\bigvee_{m\le|i|\le r_n} |X_i| > a_n u \,\middle|\, |X_0| > a_n u \right) = 0\,, \quad u > 0\,. \quad (1)$$

Clusters via tail process

Under FCC the tail process satisfies

$$|Y_m| \stackrel{P}{\to} 0, \quad \text{as } |m| \to \infty.$$

Just note

$$P(|Y_m| \geq \varepsilon) = \lim_{n\to\infty} P(|X_m|/a_n > \varepsilon \big| |X_0| > a_n)$$

and take $\lim_{m\to\infty}$

Moreover,

$$k_n P(M_{r_n} > a_n) \to \theta > 0.$$

where θ is ...

Extremal index

of the sequence $|X_t|$

Really

$$\frac{1}{k_n P(M_{r_n} > a_n)} \sim \frac{nP(|X_0| > a_n)}{k_n P(M_{r_n} > a_n)} \sim \frac{k_n E\left(\sum_1^{r_n} \mathbb{I}_{|X_i|>a_n}\right)}{k_n P(M_{r_n} > a_n)}$$

$$= E\left(\sum_1^{r_n} \mathbb{I}_{|X_i|>a_n} \middle| M_{r_n} > a_n\right) \to \frac{1}{\theta}$$

Alternatively (O'Brien)

$$\begin{aligned}\theta &= \lim_{n\to\infty} P(|X_1|,\dots,|X_{r_n}| \le a_n \,|\, |X_0| > a_n) \\ &= \lim_{n\to\infty} P(|X_{-r_n}|,\dots,|X_{-1}| \le a_n \,|\, |X_0| > a_n)\end{aligned}$$

which gives

$$\theta = P(\bigvee_{i\ge 1} |Y_i| \le 1) = P(\bigvee_{i\le -1} |Y_i| \le 1)$$

Examples

a) X_t iid RV(α), $\quad \theta = 1$.

b) $X_t = Z_t \vee Z_{t-1}$, $\quad \theta = 1/2$.

c) $X_t = A_t X_{t-1} + B_t$, as above

$$\begin{aligned} \theta &= P(\sup_{t \geq 1} A_t \cdots A_1 |Y_0| \leq 1) \\ &= E\left(1 - \sup_{t \geq 1} [A_t \cdots A_1]^\alpha\right)_+ \end{aligned}$$

cf de Haan et al (1989)

Observe that θ also has the following property

$$P\left(\frac{M_n}{a_n} \leq x\right) \to e^{-\theta x^{-\alpha}}$$

although

$$nP(|X_0| > a_n x) \to x^{-\alpha}$$

Formally

$$k_n P(M_{r_n} > a_n x) \to \theta x^{-\alpha}$$

follows from WDC, FCC and stationarity, since

$$\begin{aligned}
& k_n \sum_{j=1}^{r_n} P(|X_j| > a_n x,\ |X_1|, \ldots, |X_{j-1}| \le a_n x) \\
&= k_n \sum_{j=m+1}^{r_n} P(|X_j| > a_n x,\ |X_{j-1}|, \ldots, |X_{j-m}| \le a_n x) + o(1) \\
&= k_n (r_n - m) P(|X_0| > a_n x,\ |X_1|, \ldots, |X_m| \le a_n x) + o(1) \\
&= n P(|X_0| > a_n x,\ M_m \le a_n x) + o(1) \\
&= P(M_m \le a_n x \mid |X_0| > a_n x) \cdot n P(|X_0| > a_n x) + o(1) \\
&\to \theta x^{-\alpha}.
\end{aligned}$$

Note

$$P\left(\sum_{1}^{r_n} \mathbb{I}_{|X_i|>a_n} \geq k \middle| M_{r_n} > a_n\right)$$

$$= \frac{\sum_{i=1}^{r_n-m} P\left(|X_i| > a_n \text{ and } \sum_{j=i}^{i+m} \mathbb{I}_{|X_j|>a_n} = k\right)}{P(M_{r_n} > a_n)} + o(1)$$

$$= \frac{r_n P(|X_0| > a_n) P\left(\sum_0^m \mathbb{I}_{|X_j|>a_n} = k \middle| |X_0| > a_n\right)}{P(M_{r_n} > a_n)} \frac{k_n}{k_n} + o(1)$$

$$\to \frac{1}{\theta} P(\sum_{1}^{\infty} \mathbb{I}_{|Y_j|>1} = k)$$

$$= \frac{1}{\theta}\left(P(\sum_{0}^{\infty} \mathbb{I}_{|Y_j|>1} \geq k) - P(\sum_{0}^{\infty} \mathbb{I}_{|Y_j|>1} \geq k+1)\right)$$

Similarly, if $f(x) = 0$ for $|x| \leq 1$

$$E\left(e^{-\sum_1^{r_n} f(X_i/a_n)} \middle| M_{r_n} > a_n\right)$$

$$\to \frac{1}{\theta}\left(Ee^{-\sum_0^\infty f(Y_i)} - Ee^{-\sum_1^\infty f(Y_i)} \mathbb{I}_{\sup_{j\geq 1}|Y_j|\geq 1}\right)$$

In general

$$E\left(e^{-\sum_1^{r_n} f(X_i/a_n)} \middle| M_{r_n} > a_n\right)$$

$$\to E\left(e^{-\sum_{-\infty}^\infty f(Y_i)} \middle| \sup_{j<0} |Y_j| \leq 1\right) = E\left(e^{-\sum_{-\infty}^\infty f(Z_i)}\right)$$

Formally, this means there is a point process

$$\sum \delta_{Z_i} \stackrel{d}{=} \left(\sum \delta_{Y_i}\right) \,\Big|\, \sup_{j<0} |Y_j| \le 1$$

such that

$$\sum_{i=1}^{r_n} \delta_{X_i/a_n} \Big| M_{r_n} > a_n \xrightarrow{d} \sum_{i=1}^{\infty} \delta_{Z_i}$$

Main result

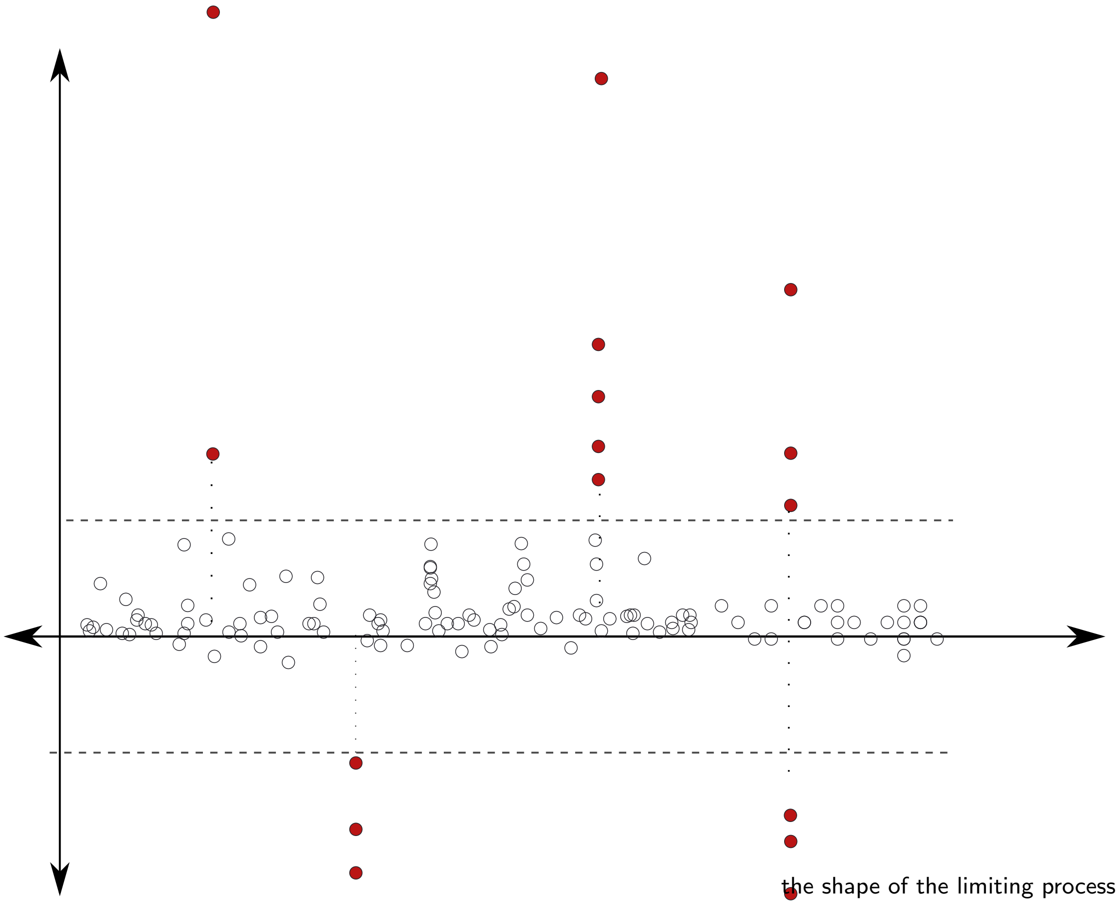
krizmanić, segers, b. (2012)

Theorem Under WDC and FCC, for every $u > 0$ and as $n \to \infty$,

$$N_n \xrightarrow{d} N = \sum_{i,j} \delta_{(T_i^{(u)}, uZ_{ij})}\Big|_{\mathbb{E}_u},$$

in $\mathbb{E}_u = [0,1] \times \{x : |x| > u\}$, where

1. $\sum_i \delta_{T_i^{(u)}}$ is a homogeneous Poisson process on $[0,1]$ with intensity $\theta u^{-\alpha}$;
2. $(\sum_j \delta_{Z_{ij}})_i$ is an iid sequence of point processes in $\mathbb{E}$, independent of $\sum_i \delta_{T_i^{(u)}}$


the shape of the limiting process

If you are willing to forget time coordinate

davis, hsing (1995)

Theorem

$$N'_n = \sum_{i=1}^{n} \delta_{\frac{X_i}{a_n}} \xrightarrow{d} N' = \sum_{i,j} \delta_{P_i Q_{ij}} \,,$$

in $\mathbb{E}$, where

1. $\sum_i \delta_{P_i}$ is a Poisson process on $(0, \infty]$ with intensity $d(-u^{-\alpha})$;
2. $(\sum_j \delta_{Q_{ij}})_i$ is an iid sequence of point processes in $\mathbb{E}$, independent of $\sum_i \delta_{P_i}$

Functional limit theorems

Recall

- A stationary time series $(X_n)_n$ is said to be **regularly varying** if random vectors

$$(X_0, \dots, X_k) \quad k \geq 0$$

are regularly varying for each k.

There exists a **tail process**

$$(Y_t)_{t\in\mathbb{Z}}$$

such that

$$\left(\frac{X_t}{x}\right)_{t\in\mathbb{Z}}\Bigg| |X_0| > x \xrightarrow{d} (Y_t)_{t\in\mathbb{Z}}$$

and a **spectral tail process**

$$(\theta_t)_{t\in\mathbb{Z}}$$

independent of $|Y_0|$ such that

$$(Y_t)_t \stackrel{d}{=} |Y_0|(\theta_t)_t$$

For our stationary and regularly varying sequence (X_i) consider

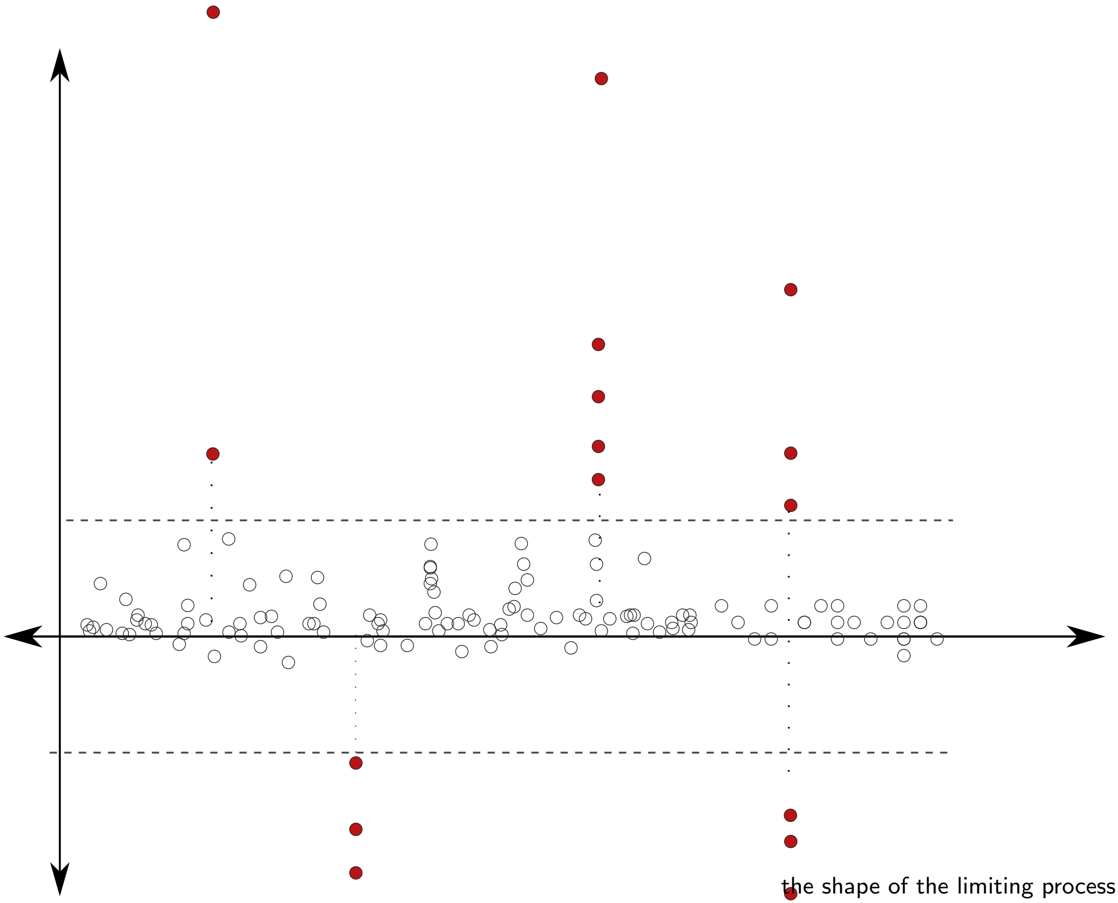
$$N_n = \sum_{i=1}^{n} \delta_{(i/n, X_i/a_n)},$$

Then under weak dependence conditions

$$N_n \xrightarrow{d} N$$

Where

- $N \sim$ PRM($Leb \times \mu$) in iid case
- N has clusters governed by the tail process in general


the shape of the limiting process

For a (stationary) sequence $X_1, X_2, \ldots$, partial sums

$$S_n = X_1 + \cdots + X_n, \quad n \in \mathbb{N}$$

form a **random walk**.

For iid steps with $\mu = EX_1$ and $\sigma^2 = \mathrm{var} X_1 < \infty$ it satisfies **central limit theorem**, i.e. with $W \sim N(0, \sigma^2)$

$$\frac{1}{\sqrt{n}}(S_n - n\mu) \xrightarrow{d} W$$

There are other possible limits for (S_n), these are so-called **stable distributions**.

Recall, Y is **stable** if for iid $Y_1, Y_2, \ldots \stackrel{d}{=} Y$ and any n there exist a_n, b_n such that

$$Y_1 + \cdots + Y_n \stackrel{d}{=} a_n Y + b_n.$$

Then $a_n = n^{1/\alpha}$, with $\alpha \in (0, 2]$ so we call Y α–stable.

Brownian motion $(W_t)_t$ is a random process satisfying

- $W_0 = 0$ a.s.
- path $t \mapsto W_t$ is a.s. continuous
- increments $W_t - W_s,\ s < t$ on disjoint intervals are independent and satisfy

$$W_t - W_s \sim N(0, t-s)$$

It is also an example of **Lévy process**, which

- start at zero
- have càdlàg paths
- and stationary independent increments.

Brownian bridge is given by

$$B_t = W_t - tW_1, \quad t \in [0, 1].$$

Scaling limits for random walks and e.d.f. of iid sequence $X_1, X_2, \dots$ with $b_n = n\mu$ are

$$\frac{1}{\sqrt{n}}\left(S_{\lfloor nt \rfloor} - b_n t\right) \xrightarrow{d} (W_t) \tag{2}$$

or

$$\sqrt{n}\left(\hat{F}_n - F\right) \xrightarrow{d} B \tag{3}$$

Recall, in a space $\mathcal{S}$

$$V_n \xrightarrow{d} V$$

stands for

$$Ef(V_n) \rightarrow Ef(V)$$

for all f bounded, continuous on $\mathcal{S}$.

Recall, on metric space $\mathcal{S}$, $f : \mathcal{S} \to \mathbb{R}$ is continuous if

$$s_n \to s,$$

implies

$$f(s_n) \to f(s)\,.$$

Intuitively, stronger metric $\Longrightarrow$ there are fewer convergent sequences $\Longrightarrow$ there are more continuous functions $\Longrightarrow$ stronger notion of convergence in distribution.

Naturally, in our case $\mathcal{S} = D[0,1]$, but we need to pick topology

very carefully.

Uniform topology (used by Donsker) causes all kinds of trouble.

Two ways around this

Shorohod gave

▷ several very intuitive metrics on $D[0,1]$

▷ among them, everybody's favourite is J_1.

Dudley suggested

▷ don't use σ-algebra generated by open sets

▷ or even check $Ef(V_n) \to Ef(V)$ for some smaller, but sufficiently rich class of functions.

We study the process

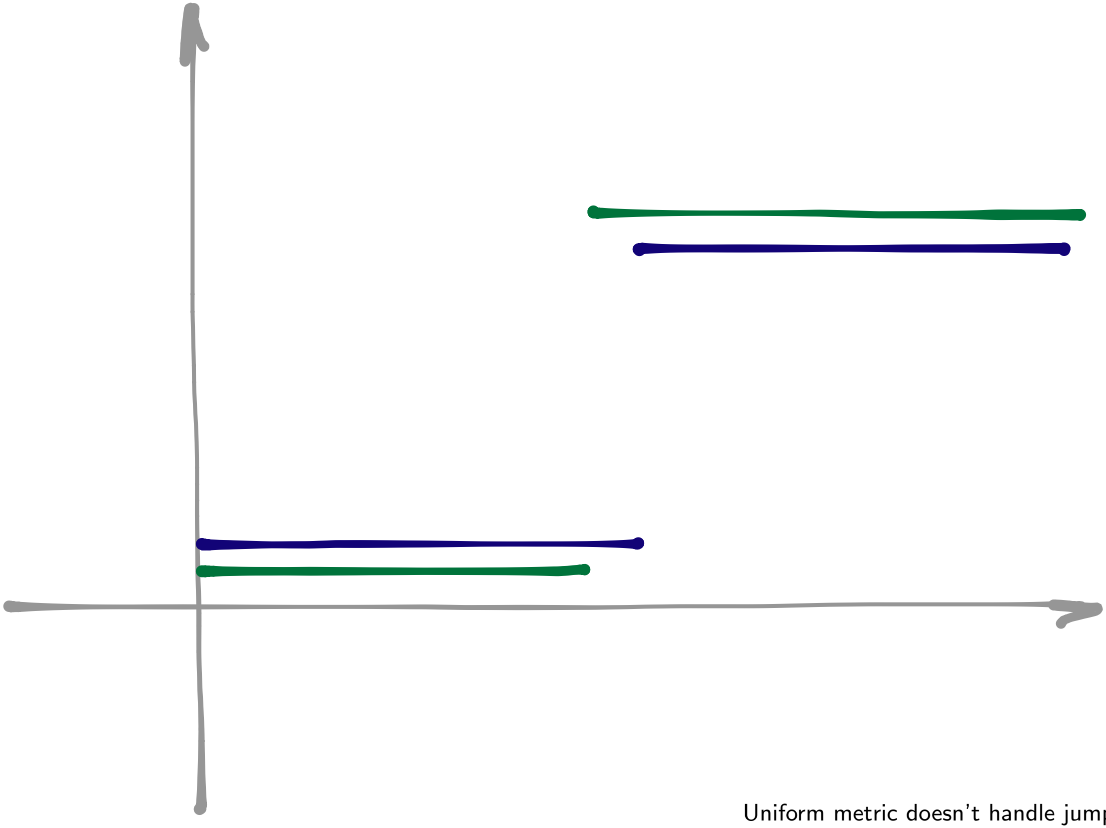
$$S_{\lfloor nt \rfloor} = \sum_{i=1}^{\lfloor nt \rfloor} X_i \,, \quad t \in [0,1].$$

It can be viewed as a random element in the space of cadlag functions $D[0,1]$, which needs to be equipped with topology.

For two càdlàg functions f and g on $[0,1]$ set

$$d_{J_1}(f,g) = \inf_{\lambda} \max\{\|f \circ \lambda - g\|_\infty, \|\lambda - \mathsf{id}\|_\infty\}$$

where the infimum is taken over increasing and continuous mappings $\lambda : [0,1] \to [0,1]$.



Uniform metric doesn't handle jumps very well

Functional limit theorem

in J_1 metric

skorohod

Theorem

For (X_i) iid RV(α) with $\alpha \in (0,2)$ and an α–stable Lévy process V_α

$$\frac{S_{\lfloor nt \rfloor} - \lfloor nt \rfloor\, b_n}{a_n} \xrightarrow{d} V_\alpha(t) \qquad (n \to \infty),$$

in $D[0,1]$ endowed with the J_1 topology.

A few words about the proof

For point measures

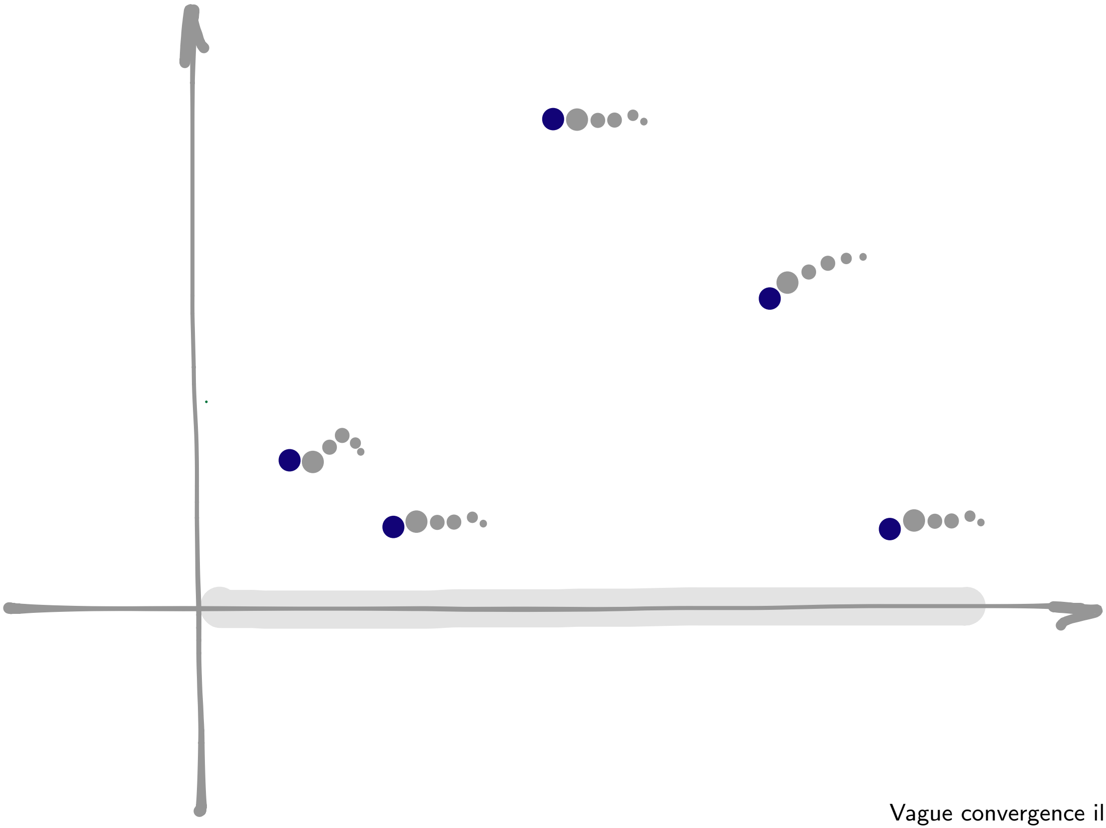
$$m_n \xrightarrow{v} m \in M_p$$

means that on each compact K with $m(\partial K) = 0$, there is n_0, such that for $n \geq n_0$

$$m_n\big|_K = \sum_{i=1}^{k} \delta_{x_i^n} \quad \text{and} \quad m\big|_K = \sum_{i=1}^{k} \delta_{x_i}$$

and

$$(x_1^n, \ldots, x_k^n) \to (x_1, \ldots, x_k)$$



Vague convergence illustrated

- Introduce the sum functional $m \to \Psi_m$ mapping M_p to $D[0,1]$

$$\Psi_m(t) = \sum_{t_i \le t} x_i, \qquad \text{where } m = \sum_{i=1}^{\infty} \delta_{t_i, x_i}$$

Clearly

$$\Psi_{N_n}(t) = S_{\lfloor nt \rfloor} = \sum_{i=1}^{\lfloor nt \rfloor} X_i \,, \quad t \ge 0.$$

- Apply contin. map. thm. to show that on $K = [0,1] \times (-\varepsilon, \varepsilon)^c$

$$m_n \stackrel{v}{\to} m \implies \Psi_{m_n}\Big|_K \to \Psi_m\Big|_K$$

in appropriate metric, whenever $m \in M' \subset M_p$.

In this case M' is a set of point measures

$$m = \sum_{i=1}^{\infty} \delta_{t_i, x_i}$$

such that

$$m(\{t\} \times (-\varepsilon, \varepsilon)^c) \leq 1$$

and

$$m([0,1], \times \{-\varepsilon, \varepsilon\}) = 0$$

- But the limiting point process is PRM, thus

$$P(N \in M') = 1$$

- Apply standard approximation argument to let $\varepsilon \to 0$.

What about dependent steps?

The sums were considered by many: e.g. Denker & Jakubowski (1989), Davis & Hsing (1995), Davis & Mikosch (1998), Bartkiewicz et al. (2009)

Partial results exist on the functional level too

If dependence is very week, that is if $\theta = 1$ and extremes do not cluster at all, the convergence result still holds. – Leadbetter and Rootzén (1988); Tyran-Kamińska (2009).

For **moving average processes**, e.g.

$$X_n = c_0 Z_n + c_1 Z_{n-1} + \cdots + c_m Z_{n-m}$$

things can go wrong, and J_1 topology does not work – Avram & Taqqu (1992).

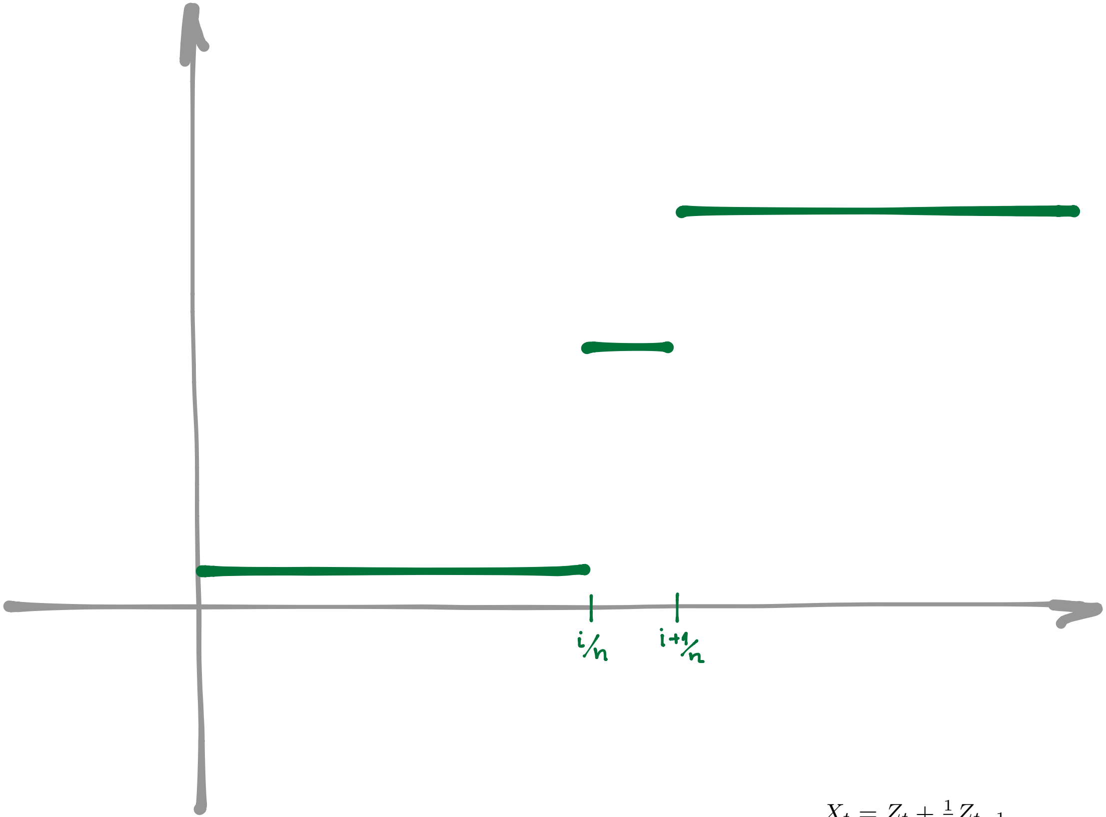
They can still save it under additional assumption that all the coefficients c_i have the same sign using M_1 topology.

Examples

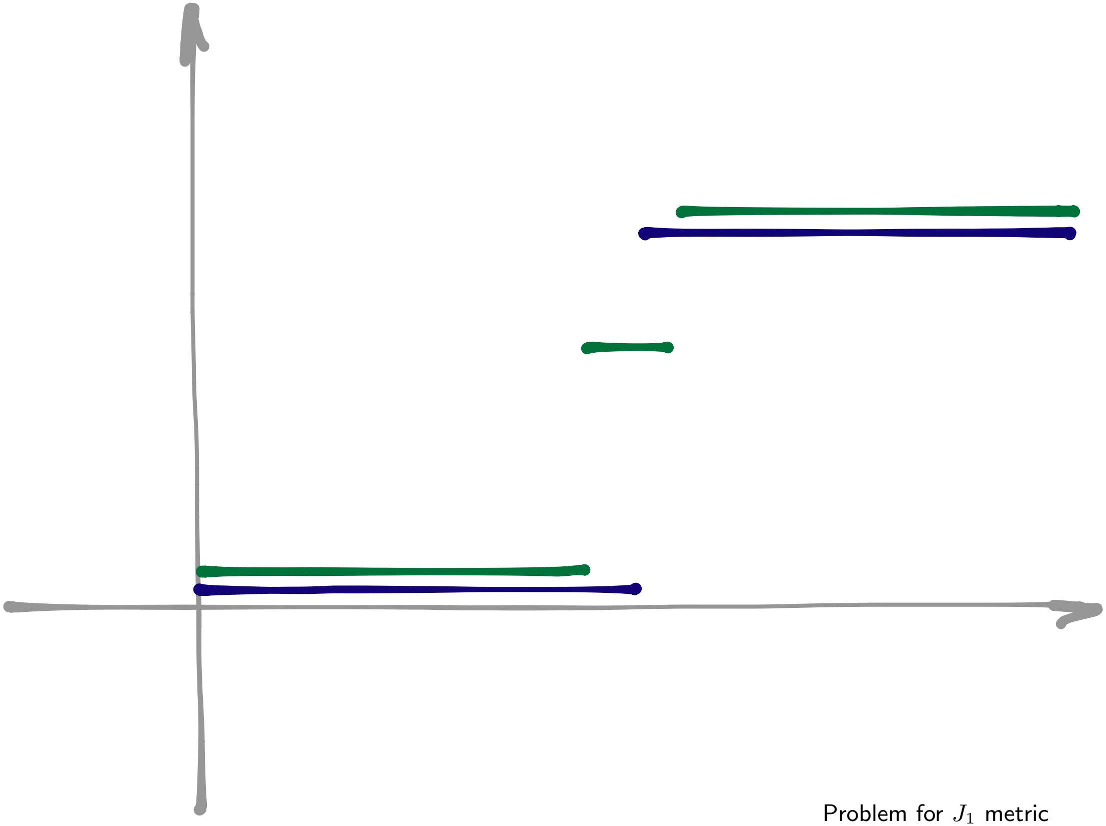
a) $X_t = Z_t \vee Z_{t-1}$, Z_t iid RV(α),

b) $X_t = Z_t + \frac{1}{2}Z_{t-1}$, Z_t iid RV(α),

c) $X_t = A_t X_{t-1} + B_t$, with (A_t, B_t) iid nonnegative satisfying Kesten's conditions.



$X_t = Z_t + \frac{1}{2} Z_{t-1}$



Problem for J_1 metric

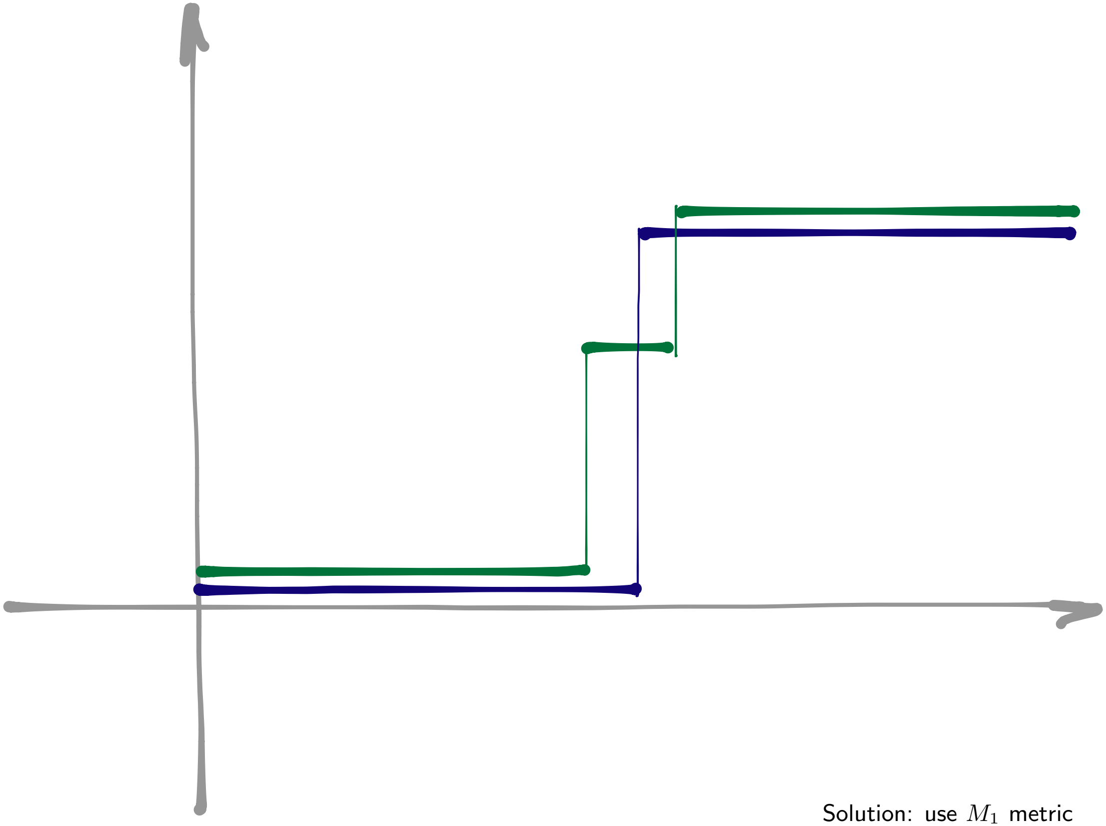
Solution: use M_1 metric, distance between functions f and g is measured by comparing **completed graphs** Γ_f and Γ_g

$$\Gamma_f = \{(t,x) : x = f(t) \text{ or } x \in [f(t-), f(t)]\}$$

and

$$d_{M_1}(f,g) = \inf_{\lambda_f, \lambda_g} \max \|\lambda_f - \lambda_g\|_\infty$$

where infimum is taken over continuous and increasing parametrizations of λ_f, λ_g.



Solution: use M_1 metric

Assumptions

some old, one new

Assume (X_n) is a stationary regularly varying sequence with $\alpha < 2$ satisfying WDC and FFC

suppose further

- its tail process has no two values of the opposite signs a.s.

Functional limit theorem

in M_1 metric

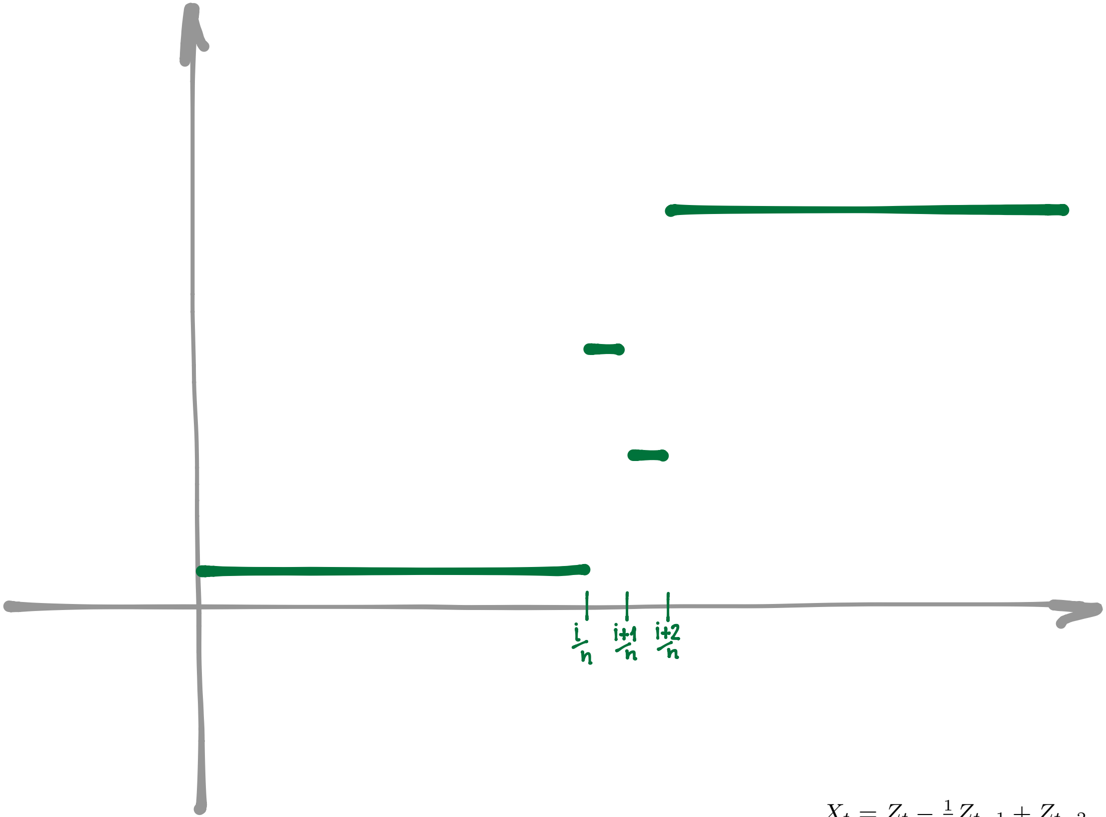
krizmanić, segers, b. (2012)

Theorem

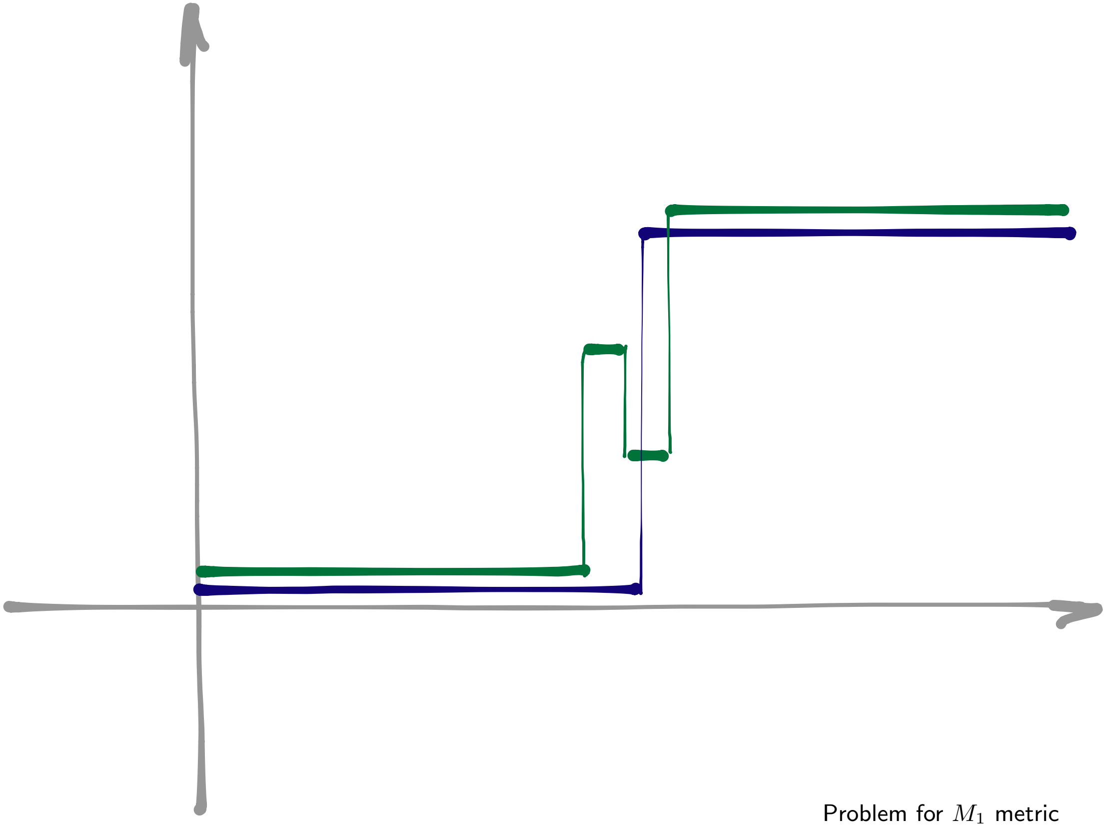
Under the assumptions, there is an α–stable Lévy process V_α such that

$$\frac{S_{\lfloor nt \rfloor} - \lfloor nt \rfloor \, b_n}{a_n} \xrightarrow{d} V_\alpha(t) \qquad (n \to \infty),$$

in $D[0,1]$ endowed with the M_1 topology.



$X_t = Z_t - \frac{1}{2}Z_{t-1} + Z_{t-2}$



Problem for M_1 metric

Assumptions

Assume (X_n) is a stationary regularly varying MA sequence with $\alpha < 2$ satisfying

$$X_n = c_0 Z_n + c_1 Z_{n-1} + \cdots + c_m Z_{n-m}$$

such that for all j

$$0 \leq \sum_{i=1}^{j} c_i \leq \sum_{i=1}^{m} c_i$$

Functional limit theorem

in M_2 metric

krizmanić, b. (2013)

Theorem

Under the assumptions, there is an α–stable Lévy process V_α such that

$$\frac{S_{\lfloor nt \rfloor} - \lfloor nt \rfloor \, b_n}{a_n} \xrightarrow{d} V_\alpha(t) \qquad (n \to \infty),$$

in $D[0,1]$ endowed with the M_2 topology.

To remember

A stationary regularly varying sequence (X_t)

- $\triangleright$ has a tail process (Y_t)
- $\triangleright$ the clusters of extremes can be described by (Y_t)
- $\triangleright$ point processes N_n have a limit characterized by (Y_t)
- $\triangleright$ random walks with steps (X_t) have an α–stable limit for $\alpha \in (0,2)$ but in strange topologies on $D[0,1]$.

Related

an incomplete list

- Extremogram (Davis, Mikosch)
- Cluster functionals (Yun, Segers)
- Large deviations (Mikosch, Wintenberger)
- Markov chains, duality and time change formula (Rootzén, Segers, Janssen)
- Other failure sets and Banach spaces (Hult, Lindskog, Segers, Meinguet)