ALLEN HATCHER: ALGEBRAIC TOPOLOGY

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All references are to the 2002 printed edition.

Chapter 0

Ex. 0.2. Define $H\colon (\mathbf{R}^n - \{0\}) \times I \to \mathbf{R}^n - \{0\}$ by
$$H(x,t) = (1-t)x + \frac{t}{|x|}x,$$
$x \in \mathbf{R}^n - \{0\}$, $t \in I$. It is easily verified that H is a homotopy between the identity map and a retraction onto S^{n-1}, i.e. a deformation retraction.

Ex. 0.3. First a few results which make things easier.

Lemma 1. *Let f_0, f_1 and f_2 be maps $X \to Y$. If $f_0 \simeq f_1$ and $f_1 \simeq f_2$ then $f_0 \simeq f_2$.*

Proof. Let $F_0\colon X \times I \to Y$ be a homotopy between f_0 and f_1, and $F_1\colon X \times I \to Y$ a homotopy between f_1 and f_2.

Define $F\colon X \times I \to Y$ by
$$F(x,t) = \begin{cases} F_0(x,2t), & t \in [0,1/2] \\ F_1(x,2t-1), & t \in [1/2,1]. \end{cases}$$
If $t = 1/2$ then $F_0(x,2t) = F_0(x,1) = f_1(x) = F_1(x,0) = F_1(x,2t-1)$, i.e. the map F is well-defined. By the pasting lemma, F is continuous. Since $F(x,0) = F_0(x,0) = f_0(x)$ and $F(x,1) = F_1(x,1) = f_2(x)$, F is a homotopy between f_0 and f_2. □

Lemma 2. *If $f_0, f_1\colon X \to Y$ are homotopic and $g_0, g_1\colon Y \to Z$ are homotopic then $g_0f_0, g_1f_1\colon X \to Z$ are homotopic.*

Proof. Let $F\colon X \times I \to Y$ be a homotopy between f_0 and f_1, and let $G\colon Y \times I \to Z$ be a homotopy between g_0 and g_1.

One proof: Now the composite $g_0F\colon X \times I \to Z$ is a homotopy between g_0f_0 and g_0f_1, and the composite $G(f_1 \times \mathrm{id}_I)\colon X \times I \to Z$ is a homotopy between g_0f_1 and g_1f_1. By lemma 1, $g_0f_0 \simeq g_1f_1$.

Another proof: The map $G(F \times \mathrm{id}_I)(\mathrm{id}_X \times \Delta)\colon X \times I \to Z$ is continuous, where $\Delta\colon I \to I \times I$ is the diagonal map, that is, $\Delta(t) = (t,t)$. Since
$$G(F \times \mathrm{id}_I)(\mathrm{id}_X \times \Delta)(x,0) = G(F \times \mathrm{id}_I)(x,0,0) = G(F(x,0),0) = g_0f_0(x)$$
and
$$G(F \times \mathrm{id}_I)(\mathrm{id}_X \times \Delta)(x,1) = G(F \times \mathrm{id}_I)(x,1,1) = G(F(x,1),1) = g_1f_1(x),$$
g_0f_0 and g_1f_1 are homotopic. □

(a). Suppose $f_0\colon X \to Y$ is a homotopy equivalence with homotopy inverse $f_1\colon Y \to X$, and $g_0\colon Y \to Z$ is a homotopy equivalence with homotopy inverse $g_1\colon Z \to Y$.

Using lemma 2, $f_1g_1g_0f_0 \simeq f_1\,\mathrm{id}_Y\,f_0 = f_1f_0 \simeq \mathrm{id}_X$ and $g_0f_0f_1g_1 \simeq g_0\,\mathrm{id}_Y\,g_1 = g_0g_1 \simeq \mathrm{id}_Z$. In other words, $g_0f_0\colon X \to Z$ is a homotopy equivalence.

Since being homotopy equivalent clearly is reflexive and symmetric, homotopy equivalence among spaces is an equivalence relation.

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(b). Trivially, $f \simeq f$ for any map $f\colon X \to Y$. Let $f_0, f_1\colon X \to Y$ be homotopic, i.e. there exists a homotopy $F\colon X \times I \to Y$ between f_0 and f_1. Now $G(x,t) = F(x, 1-t)$ is a homotopy with $G(x,0) = F(x,1) = f_1(x)$ and $G(x,1) = F(x,0) = f_0(x)$, i.e. a homotopy between f_1 and f_0.

Thus, the relation of homotopy among maps between two fixed spaces is reflexive, symmetric and transitive, the latter by lemma 1, i.e. an equivalence relation.

(c). Let $f_0\colon X \to Y$ be a homotopy equivalence with homotopy inverse $g_0\colon Y \to X$. If $f_0 \simeq f_1$, then, by lemma 2, $\mathrm{id}_X \simeq g_0 f_0 \simeq g_0 f_1$ and $\mathrm{id}_Y \simeq f_0 g_0 \simeq f_1 g_0$. Thus, f_1 is a homotopy equivalence with g_0 as homotopy inverse.

Remarks. Homotopy inverses are unique up to homotopy:

Lemma 3. *If $f\colon X \to Y$ is a homotopy equivalence with homotopy inverses $g_0, g_1\colon Y \to X$ then $g_0 \simeq g_1$.*

Proof. $g_0 = g_0 \,\mathrm{id}_Y \simeq g_0 f g_1 \simeq \mathrm{id}_X g_1 = g_1$. □

Using lemma 2, there is a *homotopy category of topological spaces* whose objects are topological spaces and whose morphisms are homotopy classes. Furthermore, there is a covariant functor from the category of topological spaces to the homotopy category that sends a map to its homotopy class. A homotopy equivalence is an equivalence in the homotopy category.

Ex. 0.9. Let X be a contractible space, that is, id_X is nullhomotopic, i.e., the identity map is homotopic to a constant map c. Furthermore, let $r\colon X \to A$ be a retraction onto the subspace A. Finally, let $i\colon A \to X$ be the inclusion map.

One proof: By lemma 2, $\mathrm{id}_A = ri = r\,\mathrm{id}_X\, i \simeq rci$, where the latter map is a constant map. Hence A is contractible.

Another proof: Let $f\colon X \times I \to X$ be a nullhomotopy of id_X. Clearly, $rf|_{A\times I}\colon A \times I \to A$ is a nullhomotopy of id_A.

Ex. 0.10.

Lemma 4. *For a space X, the following are equivalent:*

(i) *X contractible.*
(ii) *Every map $f\colon X \to Y$ for all Y is nullhomotopic.*
(iii) *Every map $g\colon Y \to X$ for all Y is nullhomotopic.*

Proof. *(i)* $\Rightarrow$ *(ii)* : If $h\colon X \times I \to X$ be a homotopy from the identity to a constant map, then $fh\colon X \times I \to Y$ is a homotopy from f to a constant map.

(ii) $\Rightarrow$ *(iii)* : If $h\colon X \times I \to X$ be a homotopy from the identity to a constant map, then $h(g \times \mathrm{id})\colon Y \times I \to X$ is a homotopy from g to a constant map.

(iii) $\Rightarrow$ *(i)* : The identity map on X is nullhomotopic. □

Ex. 0.12. Let X and Y be spaces, and let $\pi_0(X)$ and $\pi_0(Y)$ denote the sets of path components of X and Y, respectively.

Recall that, if $f\colon X \to Y$ and A is a path component of X, then $f(A)$ is path connected and there is a unique path component of Y containing $f(A)$. Furthermore, path components are either disjoint or equal. Thus, f induces a well-defined map $f_*\colon \pi_0(X) \to \pi_0(Y)$ which sends a path component A of X to the unique path component $f_*(A)$ of Y containing $f(A)$. Clearly, $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_0(X)}$.

Lemma 5. *Let $f, g\colon X \to Y$. If $f \simeq g$ then $f_* = g_*$.*

Proof. Let A be a path component of X, and let $h\colon X \times I \to Y$ be a homotopy between f and g. Since $f(A), g(A) \subset h(A \times I)$ and $h(A \times I)$ is path connected, $f(A)$ and $g(A)$ is contained in the same path component of Y, that is, $f_*(A) = g_*(A)$. □

Lemma 6. *If $f\colon X \to Y$ and $g\colon Y \to Z$ then $(gf)_* = g_* f_*$.*

Proof. Let A be a path component of X. Since $gf(A) \subset g(f_*(A)) \subset g_* f_*(A)$, $(gf)_*(A) = g_* f_*(A)$. □

Thus, in the realm of categories, there is a functor from the category of topological spaces to the category of sets sending a space X to the set of path components $\pi_0(X)$, and a map $f\colon X \to Y$ to $f_*\colon \pi_0(X) \to \pi_0(Y)$.

Lemma 7. *If $f\colon X \to Y$ is a homotopy equivalence, then f_* is bijective.*

Proof. If $g\colon Y \to X$ be a homotopy inverse of f, then $\mathrm{id}_{\pi_0(X)} = (gf)_* = g_*f_*$ and $\mathrm{id}_{\pi_0(Y)} = (fg)_* = f_*g_*$. □

Lemma 8. *If $f\colon X \to Y$ is a homotopy equivalence, then $f|_A\colon A \to f_*(A)$ is a homotopy equivalence for all path components A of X.*

Proof. Let $g\colon Y \to X$ be a homotopy inverse of f, and let A be a path component of X. Furthermore, let h_1 be a homotopy from gf to id_X and h_2 a homotopy from fg to id_Y.

Since $h_1(A \times \{1\}) = A$ and $A \times I$ is path connected, $h_1(A \times I) \subset A$. Thus, $h_1|_{A\times I}\colon A \times I \to A$ is a homotopy from $(gf)|_A = (g|_{f_*(A)})(f|_A)$ to id_A.

Similarly, if B is a path component of Y, then $h_2|_{B\times I}\colon B \times I \to B$ is a homotopy from $(fg)|_B = (f|_{g_*(B)})(g|_B)$ to id_B.

In particular, if $B = f_*(A)$, then $(f|_{g_*(f_*(A))})(g|_{f_*(A)}) = (f|_A)(g|_{f_*(A)})$ to $\mathrm{id}_{f_*(A)}$, that is, $f|_A\colon A \to f_*(A)$ is a homotopy equivalence with homotopy inverse $g|_{f_*(A)}\colon f_*(A) \to A$. □

The arguments above are easily modified to prove the equivalent result about components instead of path components, using that the continuous image of a connected space is connected. The details are left to the reader. Write $\pi_0'(X)$ for the set of components of X, and $f_*'\colon \pi_0'(X) \to \pi_0'(Y)$ for the map induced by f.

Lemma 9. *If $f\colon X \to Y$ is a homotopy equivalence and the components and path components of X coincide, then the components and path components of Y coincide.*

Proof. Let $B' \in \pi_0'(Y)$ and $B \in \pi_0(B')$, that is, B is path component of Y contained in B'. By assumption and the results above, $B' \simeq g_*'(B') = g_*(B) \simeq B$. Hence, $|\pi_0(B')| = |\pi_0(B)| = |\{B\}| = 1$, that is, B' has exactly one path component. □

Ex. 0.16. See example 1B.3.

Ex. 0.17. One idea is to attach the core circle of the Möbius band to a boundary circle of the annulus (the region between two concentric circles), see figure 1. The CW complex consists of four 0-cells, seven 1-cells and three 2-cells.

A deformation retraction of the Möbius band onto its core circle gives the annulus, and a deformation retraction of the annulus onto its boundary circle c gives the Möbius band.

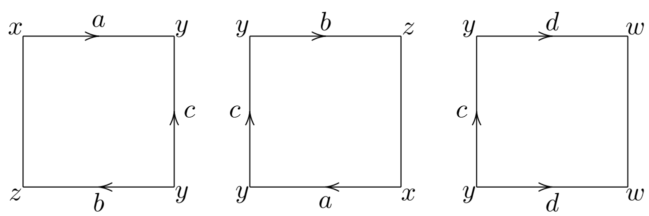


FIGURE 1. A 2-dim CW complex containing an annulus and a Möbius band.

Ex. 0.20. By collapsing the closed (and contractible) disk where the klein bottle, immersed in $\mathbf{R}^3$, intersects itself and inserting two strings, A (inside the sphere) and B, we get a space which is homotopy equivalent to the space in figure 2. By collapsing the contractible arcs C and D on the sphere we get $S^2 \vee S^1 \vee S^1$.

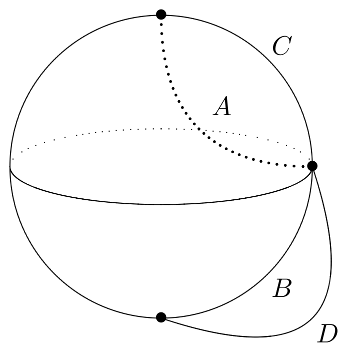


FIGURE 2. A space homotopy equivalent to the klein bottle immersed in $\mathbf{R}^3$.

Chapter 1

Ex. 1.1.5.

Lemma 10. *For a space X, the following are equivalent:*

(a) *Every map $S^1 \to X$ is nullhomotopic.*
(b) *Every map $S^1 \to X$ extends to a map $D^2 \to X$.*
(c) *$\pi_1(X, x_0) = 0$ for all $x_0 \in X$.*

Proof. Let $i\colon S^1 \to D^2$ be the inclusion map.

$(a) \Rightarrow (b)$: Suppose $f\colon S^1 \to X$ is nullhomotopic, i.e., there is a homotopy $h\colon S^1 \times I \to X$ from a constant map, $S^1 \mapsto x_0$, to f.

One proof: Observe that h is a partial homotopy of the constant map $D^2 \mapsto x_0$. Since (D^2, S^1) has the homotopy extension property, h extends to a homotopy $\widetilde{h}\colon D^2 \times I \to X$ such that the restriction of $\widetilde{h}$ to $S^1 \times \{1\}$ is f.

Another proof (thanks to Nicolai and Rune): Clearly, h extends to a map

$$\widetilde{h}\colon D^2 = (D^2 \times I) - (\mathrm{Int}(D^2) \times [0,1[) \to X,$$

by letting $\widetilde{h}$ be the constant map on $D^2 \times \{1\}$.

$(b) \Rightarrow (c)$: Let $f\colon (S^1, s_0) \to (X, x_0)$ be a loop in X, and let $\widetilde{f}\colon D^2 \to X$ be an extension of f to D^2. Furthermore, let $h\colon D^2 \times I \to D^2$ be the deformation retraction of D^2 onto s_0 along the lines through s_0. In particular, h does not move s_0, i.e., $h(s_0, t) = s_0$.

Now,

$$\widetilde{h}\colon\ S^1 \times I \xrightarrow{i \times \mathrm{id}} D^2 \times I \xrightarrow{h} D^2 \xrightarrow{\widetilde{f}} X$$

is a homotopy from $\widetilde{h}(s,0) = \widetilde{f}h(s,0) = f(s)$ to $\widetilde{h}(s,1) = \widetilde{f}h(s,1) = \widetilde{f}(s_0) = x_0$, and $\widetilde{f}h(s_0,t) = \widetilde{f}(s_0) = x_0$. Thus, $\widetilde{h}$ is a homotopy of loops from f to the constant loop.

$(c) \Rightarrow (a)$: Clear.

□

Lemma 11. *A space X is simply-connected if and only if all maps $S^1 \to X$ are homotopic.*

Proof. Suppose X is simply-connected. Since X is path connected and $\pi_1(X, x_0) = 0$ for all x_0 in X, then, by lemma 10, all maps $S^1 \to X$ are homotopic to any constant map $S^1 \to X$.

Conversely, suppose all maps $S^1 \to X$ are homotopic. Then, all maps $S^1 \to X$ are homotopic to any constant map. In particular, all constant maps are homotopic, i.e., X is path connected. By lemma 10, $\pi_1(X, x_0) = 0$ for all x_0 in X. □

Ex. 1.1.6. Consider maps $(I, \partial I) \to (X, x_0)$ as maps $(I/\partial I, \partial I/\partial I) = (S^1, s_0) \to (X, x_0)$. Let $\Phi\colon \pi_1(X, x_0) \to [S^1, X]$ be the map that sends a based homotopy class of a map $S^1 \to X$ to its unbased homotopy class.

Suppose X is path connected.

Lemma 12. Φ *is surjective.*

Proof. It suffices to prove that any map $f\colon S^1 \to X$ is homotopic to a map $(S^1, s_0) \to (X, x_0)$.

Let $h\colon I \to X$ be a path from $f(s_0)$ to x_0, that is, h is a partial homotopy of f on the subset $\{s_0\}$. Since (S^1, s_0) has the HEP, there exists $\varphi\colon S^1 \times I \to X$ such that the diagram

$$\begin{array}{ccc} S^1 \times \{0\} \cup \{s_0\} \times I & \xrightarrow{f \cup h} & X \\ \Big\downarrow{\scriptstyle i} & \nearrow{\scriptstyle \varphi} & \\ S^1 \times I & & \end{array}$$

commutes, where i is the inclusion map. Now, $\varphi|_{S^1 \times \{0\}} = f$ and $\varphi(s_0, 1) = h(1) = x_0$, i.e., $\Phi([\varphi|_{S^1 \times \{1\}}]) = [f]$. □

Lemma 13. $\forall\, [f], [g] \in \pi_1(X, x_0)\colon \Phi([f]) = \Phi([g]) \Leftrightarrow \exists\, [h] \in \pi_1(X, x_0)\colon [h][f][h]^{-1} = [g]$.

Proof. "$\Rightarrow$": Since $\Phi([f]) = \Phi([g])$, there is a homotopy $\varphi_t\colon S^1 \to X$ from f to g. Let $h\colon S^1 \to X$ be the loop $h(t) = \varphi_t(s_0)$. By 1.19, the diagram

$$\begin{array}{ccc} & & \pi_1(X, x_0) \\ & \nearrow{\scriptstyle f_*} & \Big\downarrow{\scriptstyle \beta_h} \\ \pi_1(S^1, s_0) & & \\ & \searrow{\scriptstyle g_*} & \\ & & \pi_1(X, x_0) \end{array}$$

commutes. In particular,

$$[g] = g_*[\mathrm{id}_{S^1}] = \beta_h f_*[\mathrm{id}_{S^1}] = \beta_h[f] = [h][f][h]^{-1}.$$

"$\Leftarrow$": For $t \in I$, let $h_t\colon I \to X$ be the path $h_t(s) = h((1-s)t + s)$, that is, a path from $h(t)$ to $h(1) = x_0$. Observe that $h_0 = h$ and $h_1 = x_0$. Now, $\varphi_t = h_t \cdot f \cdot \overline{h}_t$, where $\cdot$ denotes pah composition, is a homotopy with $\varphi_0 = h_0 \cdot f \cdot \overline{h}_0 = h \cdot f \cdot \overline{h}$ and $\varphi_1 = f$ (draw a picture of this homotopy), i.e., $\Phi([f]) = \Phi([h \cdot f \cdot \overline{h}]) = \Phi([g])$.

□

Ex. 1.1.9. Since A_1, A_2 and A_3 is compact they have finite measure. Every point $s \in S^2$ determines a unit vector i $\mathbf{R}^3$ and hence a direction, so we can regard s as a unit vector in $\mathbf{R}^3$.

For each A_i choose a plane P_i with s as a normal, such that P_i divides A_i in two pieces of equal measure. It is intuitively clear that P_i exists, by continuously sliding P_i along the line determined by s, and is unique, but the proof is omitted.

Let $d_i(s)$, $i = 1, 2$, denote the Euclidean distance between P_3 and P_i in the direction determined by s. The situation is illustrated in figure 3. In the situation pictured below are $d_1(s) < 0$ and $d_2(s) > 0$. We want to proof that there exists $s \in S^2$, such that $d_i(s) = 0$, $i = 1, 2$, hence the three planes coincide and the result follows.

Now define $f : S^2 \to \mathbf{R}^2$ by $f(s) = (d_1(s), d_2(s))$. Clearly, f is continuous. Note that f is an odd map, i.e. $f(-s) = -f(s)$. By the Borsuk-Ulam Theorem, there exists $s_0 \in S^2$ such that $f(s_0) = f(-s_0)$. But this means that $d_i(s_0) = -d_i(s_0)$, hence $d_i(s_0) = 0$, $i = 1, 2$, as desired.

Finally, since the Borsuk-Ulam Theorem hold for continuous maps $S^n \to \mathbf{R}^n$, this argument easily generalises to $\mathbf{R}^n$ for a hyperplane of dimension $n - 1$. Hence the result also holds i $\mathbf{R}^n$.

Ex. 1.1.10. Let $f\colon I \to X \times \{y_0\}$ and $g\colon I \to \{x_0\} \times Y$ be loops based at $x_0 \times y_0$. Furthermore, let $\cdot$ denote path composition. By definition,

$$f \cdot g(s) = \begin{cases} f(2s) \times y_0, & t \in [0, 1/2] \\ x_0 \times g(2s-1), & t \in [1/2, 1] \end{cases}$$

and

$$g \cdot f(s) = \begin{cases} x_0 \times g(2s), & t \in [0, 1/2] \\ f(2s-1) \times y_0, & t \in [1/2, 1]. \end{cases}$$

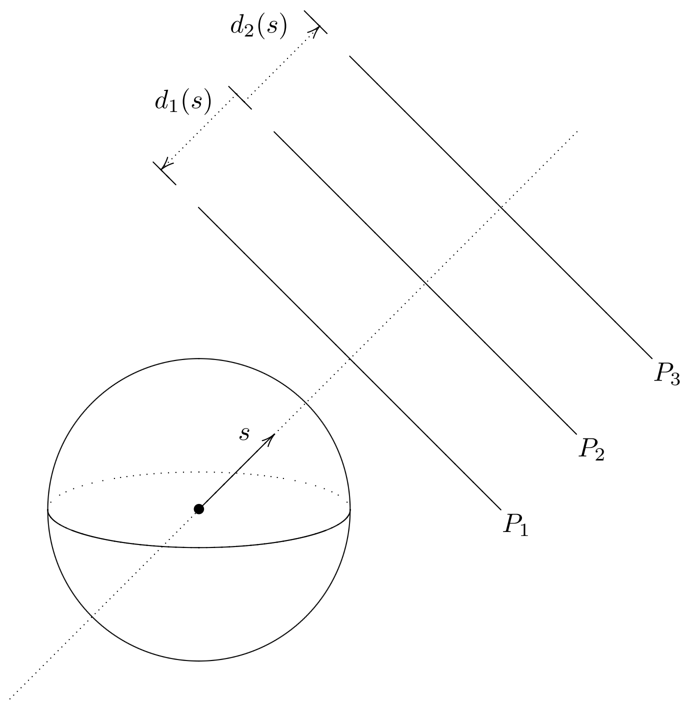


FIGURE 3. The Ham Sandwich theorem.

Define $x\colon I \times I \to X$ and $y\colon I \times I \to Y$ by

$$x(s,t) = \begin{cases} x_0, & s \in [0, t/2] \\ f(2s-t), & s \in [t/2, (1+t)/2] \\ x_0, & s \in [(1+t)/2, 1] \end{cases}$$

and

$$y(s,t) = \begin{cases} y_0, & s \in [0, (1-t)/2] \\ g(2s+t-1), & s \in [(1-t)/2, (2-t)/2] \\ y_0, & s \in [(2-t)/2, 1], \end{cases}$$

where we regard f and g as maps $I \to X$ and $I \to$ respectively. See figure 4 for an illustration of these maps. By the pasting lemma, x and y are continuous.

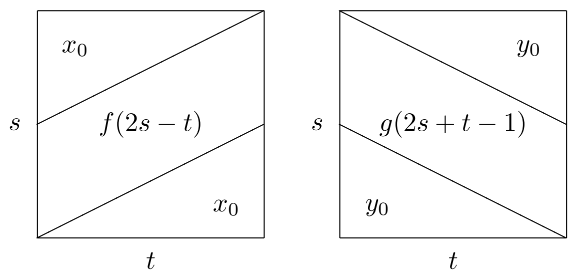


FIGURE 4. The maps x (left) and y (right).

Now $h\colon I \times I \to X \times Y$, $h(s,t) = x(s,t) \times y(s,t)$, is continuous with $h(0,t) = h(1,t) = x_0 \times y_0$, $h(s,0) = x(s,0) \times y(s,0) = f \cdot g(s)$ and $h(s,1) = x(s,1) \times y(s,1) = g \cdot f(s)$. Thus, h is a base point preserving homotopy between $f \cdot g$ and $g \cdot f$.

Ex. 1.1.11. Let X be a space with base point x_0, and let X_0 be the path component of X containing x_0. Let $i\colon X_0 \to X$ be the inclusion map. Consider the homomorphism $i_*\colon \pi_1(X_0, x_0) \to \pi_1(X, x_0)$.

i_* is surjective: Let $f\colon I \to X$ be a loop based at x_0. Since $f(I) \subset X_0$, the corestriction of f to X_0 is a loop in X_0, hence $i_*[f] = [if] = [f]$, i.e. i_* is surjective.

i_* is injective: Let $g, h\colon I \to X_0$ be loops based at x_0. Suppose $i_*[g] = [ig]$ and $i_*[h] = [ih]$ are homotopic as loops in X, i.e. there exists base point preserving homotopy between ig and ih. The image of this homotopy is path connected, hence contained in X_0, hence corestriction gives a homotopy between g and h, i.e. $[g] = [h]$. In other words, i_* is injective.

Summarizing, i_* is an isomorphism.

Ex. 1.1.12. Any endomorphism of the abelian group $\mathbf{Z}$ is multiplication by n for some $n \in \mathbf{Z}$. Recall that n equals the image of the generator 1.

Let $n \in \mathbf{Z}$. Consider S^1 as the unit circle i $\mathbf{C}$ with base point 1. Let $f, g\colon S^1 \to S^1$ be the maps given by $f(z) = z^n$ and $g(z) = z^{-n}$.

Recall, c.f. theorem 1.7, that the path $\omega_1(t) = e^{2\pi i t}$, $t \in I$, generates $\pi_1(S^1, 1)$. Note that $f\omega_1(t) = e^{2\pi i n t} = \omega_n(t)$, i.e. n times the generator of $\pi_1(S^1, 1)$, c.f. theorem 1.7. Hence f_* is multiplication by n via the isomorphism in theorem 1.7. Similarly, g_* is multiplication by $-n$.

Ex. 1.1.13. Let X be a space with base point x_0, and let A be a path connected subspace of X containing x_0. Let $i\colon A \to X$ be the inclusion map. Let $\cdot$ denote path composition. Consider the homomorphism $i_*\colon \pi_1(A, x_0) \to \pi_1(X, x_0)$.

Suppose i_* is surjective. Let $g\colon I \to X$ be a path in X with end points, $g(0) = x_1$ and $g(1) = x_2$, in A. Since A is path connected, there exists a path f in X contained in A from x_0 to x_1 and a path h in X contained in A from x_2 to x_0. Now $f \cdot g \cdot h$ is a loop based at x_0, i.e. $[f \cdot g \cdot h] \in \pi_1(X, x_0)$. Since i_* is surjective, there exists a loop l in A based at x_0 such that $i_*[l] = [il] = [f \cdot g \cdot h]$. Hence $[g] = [\overline{f} \cdot il \cdot \overline{h}]$ as paths in X from x_1 to x_2, and $\overline{f} \cdot il \cdot \overline{h}$ is a path in X contained A.

Conversely, suppose every path in X with end points in A is path homotopic to a path in A. In particular, every loop in X based at x_0 is path homotopic to a loop in A, i.e. i_* is surjective.

Ex. 1.1.15. Let $\cdot$ denote path composition. First a lemma:

Lemma 14. *If $f\colon X \to Y$ is a map and if g and h are paths in X with $g(1) = h(0)$, then $f(g \cdot h) = fg \cdot fh$.*

Proof. This follows immediately from the definition of path composition. □

Now $f\colon X \to Y$ be a map and $h\colon I \to X$ a path from x_0 to x_1. Let $g\colon I \to X$ be a loop based at x_1. Note that $\overline{fh} = f\overline{h}$. By lemma 14,

$$\beta_{fh} f_*[g] = [fh \cdot fg \cdot \overline{fh}] = [f(h \cdot g) \cdot f\overline{h}] = [f(h \cdot g \cdot \overline{h})],$$

and

$$f_* \beta_h [g] = [f(h \cdot g \cdot \overline{h})].$$

Hence the diagram

$$\begin{array}{ccc} \pi_1(X, x_1) & \xrightarrow{\beta_h} & \pi_1(X, x_0) \\ \big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle f_*} \\ \pi_1(Y, f(x_1)) & \xrightarrow{\beta_{fh}} & \pi_1(Y, f(x_0)) \end{array}$$

commutes.

Ex. 1.1.16.

(e). Choose the base point $x_0 \in A \subset X$ to be the point on the boundary of X corresponding to the identification of the two points on the boundary of the disk. Let $i\colon A \to X$ be the inclusion map. We will give two proofs:

(1) Consider the loops a and b in A illustrated in figure 5. Note that $A = S^1_a \vee S^1_b$, and X deformation retracts onto S^1_a. The map $r = \mathrm{id}_{S^1_a} \vee x_0\colon S^1_a \vee S^1_b \to S^1_a$ is a retraction. In particular, $r(a) = a$ and $r(b) = x_0$, i.e., $r_*[a] = [a]$ and $r_*[b] = 0$. Thus, $[a] \neq [b]$ in $\pi_1(A, x_0)$.

Clearly, a and b are path homotopic in X. Thus $[a] = [b]$ in $\pi_1(X, x_0)$. Now $i_*([a]) = i_*([b])$, but $[a] \neq [b]$, i.e., i_* is not injective. By 1.17, A is not a retract of X.

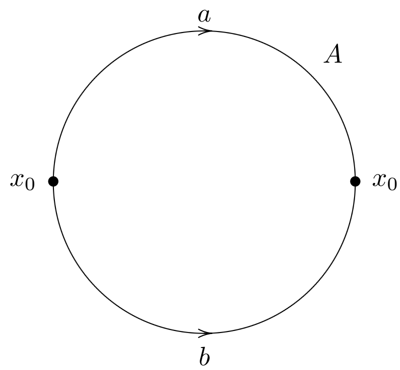


Figure 5.

(2) Since $A = S^1 \vee S^1$ and $X = S^1$, $\pi_1(A, x_0) = \mathbf{Z} * \mathbf{Z}$ and $\pi_1(X, x_0) = \mathbf{Z}$. Since $\mathbf{Z} * \mathbf{Z}$ is not abelian and all subgroups of $\mathbf{Z}$ are abelian, there is no injective homomorphism $\mathbf{Z} * \mathbf{Z} \to \mathbf{Z}$. In particular, the map $i_*\colon \pi_1(A, x_0) \to \pi_1(X, x_0)$ is not injective. By 1.17, A is not a retract of X.

(f). Note that X deformation retracts onto its core circle C. Choose a base point $x_0 \in A \subset X$, see figure 6. Furthermore, $\pi_1(A, x_0) = \mathbf{Z} = \langle [\mathrm{id}_A] \rangle$ and $\pi_1(X, x_0) = \mathbf{Z} = \langle [\mathrm{id}_C] \rangle$. Now, $i_*[\mathrm{id}_A] = [\mathrm{id}_C]^2$, since going once around the boundary cirle, A, corresponds in X to go twice around C.

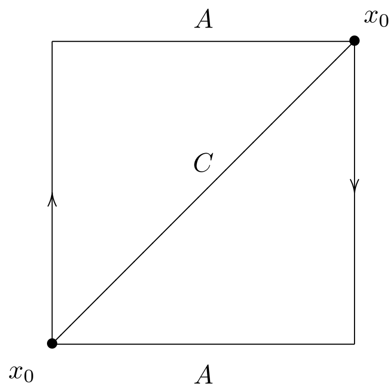
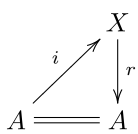


Figure 6.

Suppose $r\colon X \to A$ is a retraction, that is, the diagram



commutes, where $i\colon A \to X$ is the inclusion map. Applying π_1 gives the commutative diagram

$$\begin{array}{ccc} & & \pi_1(X, x_0) \\ & \overset{i_*}{\nearrow} & \big\downarrow {\scriptstyle r_*} \\ \pi_1(A, x_0) & = & \pi_1(A, x_0). \end{array}$$

Thus, $r_* i_* [\mathrm{id}_A] = r_*[\mathrm{id}_C]^2 = [\mathrm{id}_A]$, which is a contradiction; there is no homomorphism $\mathbf{Z} \to \mathbf{Z}$ mapping twice a generator to a generator. Recall, every homomorphism $\mathbf{Z} \to \mathbf{Z}$ is $z \mapsto nz$ for some nonnegative integer n.

Ex. 1.2.2. We use induction on the number n of open convex sets. Suppose $n = 1$. Any convex set in $\mathbf{R}^m$ is simply connected, c.f. example 1.4.

Suppose $n > 1$ and $Y = X_1 \cup \cdots \cup X_{n-1}$ is simply connected. We want to use Van Kampen's theorem.

$X_1 \cup \cdots \cup X_{n-1}$ is path connected: Let $x, y \in X_1 \cup \cdots \cup X_{n-1}$, i.e. there exists $1 \le i, j \le n-1$ such that $x \in X_i$ and $y \in X_j$. Choose $z \in X_i \cap X_j \neq \emptyset$. Since X_k is path connected for all k, there exists a path in X_i from x to z and a path in X_j from z to y. Hence $X_1 \cup \cdots \cup X_{n-1}$ is path connected.

$(X_1 \cup \cdots \cup X_{n-1}) \cap X_n$ is path connected: Let $x, y \in (X_1 \cup \cdots \cup X_{n-1}) \cap X_n$, i.e. there exists $1 \le i, j \le n-1$ such that $x \in X_i$ and $y \in X_j$. Choose $z \in X_i \cap X_j \cap X_n \neq \emptyset$. Since X_k is path connected for all k, there exists a path in X_i from x to z and a path in X_j from z to y. Hence $(X_1 \cup \cdots \cup X_{n-1}) \cap X_n$ is path connected.

Now $X_1 \cup \cdots \cup X_{n-1}$ and X_n are open path connected sets, and $(X_1 \cup \cdots \cup X_{n-1}) \cap X_n$ is path connected and not empty. Choose a base point x_0 in $(X_1 \cup \cdots \cup X_{n-1}) \cap X_n$. By Van Kampen's theorem, $\pi_1(X, x_0) \simeq (\pi_1(X_1 \cup \cdots \cup X_{n-1}) * \pi_1(X_n))/N$ for some normal subgroup N. By induction, $\pi_1(X_1 \cup \cdots \cup X_{n-1})$ is trivial. So $\pi_1(X_1 \cup \cdots \cup X_{n-1})$ and $\pi_1(X_n)$ are both trivial, i.e. X is simply connected.

Ex. 1.2.3. Consider $\mathbf{R}^n$ with base point x_0. Let $x_1, \ldots, x_m$ be m distinct points in $\mathbf{R}^n - \{x_0\}$.

We use induction on the number m of distint points. By example 1.15, $\mathbf{R}^n - \{x_1\}$ is simply connected if $n \ge 3$.

Suppose $m > 1$ and $\mathbf{R}^n - \{x_1, \ldots, x_{m-1}\}$ is simply connected. It suffices to show that any loop f in $\mathbf{R}^n - \{x_1, \ldots, x_m\}$ is homotopic to a loop in $\mathbf{R}^n - \{x_1, \ldots, x_{m-1}\}$. This follows as in the proof of 1.14.

There is a more general result concerning manifolds. Recall that a manifold is a locally euclidian second countable Hausdorff space. Lemma 15 and the fact that an open subspace of a manifold is a manifold gives another proof of the result above.

Lemma 15. *Let M be a n-manifold, and let x_0 and x_1 be distint points in M. If $n \ge 3$ then $\pi_1(M, x_0) \simeq \pi_1(M - \{x_1\}, x_0)$.*

Ex. 1.2.4. Let X be the union of n lines through the origin, and let $x_0 \in \mathbf{R} - X$.

$\mathbf{R}^3 - \{0\}$ deformation retracts onto S^2, c.f. ex. 0.2. Restriction of this deformation retraction gives a deformation retraction of $\mathbf{R}^3 - X$ onto $S^2 - (X \cap S^2)$. By proposition 1.17, $\pi_1(\mathbf{R}^3 - X, x_0) \simeq \pi_1(S^2 - (X \cap S^2), x_0)$.

Note that $S^2 - (X \cap S^2)$ is S^2 with $2n$ holes. By stereographic projection, S^2 with a hole is $\mathbf{R}^2$, hence $S^2 - (X \cap S^2)$ is $\mathbf{R}^2$ with $2n-1$ holes, which deformation retracts onto a wedge of $2n-1$ circles. Thus, $\pi_1(\mathbf{R}^3 - X, x_0)$ is the free product of $2n-1$ copies of $\mathbf{Z}$.

Ex. 1.2.6. Let Y be obtained from a path connected subspace X by attaching n-cells for a fixed $n \ge 3$. Let $i\colon X \to Y$ be the inclusion map. Let the base point x_0 be in X.

Claim 16. *The induced map $i_*\colon \pi_1(X, x_0) \to \pi_1(Y, x_0)$ is an isomorphism.*

Proof. Modify the proof of 1.26 by replacing 2-cells with n-cells. Since $n \ge 3$, A_α is simply connected, i.e. $A \cap B = \cup A_\alpha$ is simply connected. Furthermore, B is still contractible. □

Let A be a discrete subspace of $\mathbf{R}^n$, $n \geq 3$. Choose a base point $x_0 \in \mathbf{R}^n - A$.

Claim 17. *The complement* $\mathbf{R}^n - A$ *is simply connected.*

Proof. For each $x \in A$, there exists an open n-ball B_x such that $\overline{B}_x$ is homeomorphic to D^n and $B_x \cap A = \{x\}$. Clearly $X = \mathbf{R}^n - \bigcup_{x \in A} B_x$ deformation retracts onto $\mathbf{R}^n - A$.

Now let Y be the space obtained by attachning n-cells to X via homeomorphisms $\varphi_x \colon \partial D^n \to \partial B_x$ for each $x \in A$. Now X is path connected and $Y = \mathbf{R}^n$. By claim 16, $\pi_1(\mathbf{R}^n - A, x_0) \simeq \pi_1(X, x_0) \simeq \pi_1(Y, x_0) = 0$. □

Ex. 1.2.7. One CW complex structure on X consists of one 0-cell, x_0, one 1-cell, a, and one 2-cell. The attachment map is aa^{-1}. By 1.26, $\pi_1(X, x_0) = \langle\, a \mid aa^{-1} = 1 \,\rangle = \langle\, a \,\rangle = \mathbf{Z}$.

Another CW complex consists of one 0-cell, x_0, two 1-cells, a, b, and two 2-cells, c, d. The attachment map of both 2-cells is ab. Again by 1.26, $\pi_1(X, x_0) = \langle\, a, b \mid ab = 1 \,\rangle = \langle\, a \,\rangle = \mathbf{Z}$.

Ex. 1.2.9. For later reference we give some basic properties about the abelianization of a group.

Let G be a group, let A be a subset of G. We write group composition multiplicatively and denote the identity element by 1. Let $\mathcal{H}$ be the set of subgroups of G containing A. Now, the subgroup

$$\langle A \rangle = \bigcap_{H \in \mathcal{H}} H$$

of G is called the subgroup of G generated by A. By construction, $\langle\, A \,\rangle$ is the smallest subgroup of G containing A. It is easily verified that any element of $\langle\, A \,\rangle$ can be written as a product $a_1 \cdots a_n$, $a_1, \ldots, a_n \in A$, $n \in \mathbf{Z}_+$, repetitions are allowed.

The commutator of two elements a and b in G is the element $aba^{-1}b^{-1}$ and is denoted $[a, b]$. Clearly, $[a, b] = 1$ iff a and b commute. Let G' denote the subgroup of G generated by the set of commutators in G, i.e., $G' = \langle\, \{\, [a, b] \mid a, b \in G \,\} \,\rangle$.

For any a, b and g in G, $g[a, b]g^{-1} = [gag^{-1}, gbg^{-1}]$. Let $[a_1, b_1] \cdots [a_n, b_n]$, $a_1, b_1 \ldots, a_n, b_n \in G$, $n \in \mathbf{Z}_+$, be any element of G'. Now,

$$g[a_1, b_1] \cdots [a_n, b_n]g^{-1} = g[a_1, b_1]g^{-1}g \cdots g^{-1}g[a_n, b_n]g^{-1} = [ga_1g^{-1}, gb_1g^{-1}] \cdots [ga_ng^{-1}, gb_ng^{-1}]$$

for any g in G. Hence, $g[a_1, b_1] \cdots [a_n, b_n]g^{-1} \in G'$, i.e. G' is a normal subgroup of G.

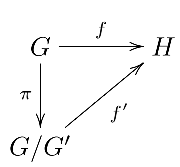
Theorem 18. *If H is a normal subgroup of G, then G/H is abelian iff $G' \subset H$.*

Proof. Suppose G/H is abelian. Let $a, b \in G$. Now $[a][b] = [b][a]$ in G/H, i.e. $[a][b][a]^{-1}[b]^{-1} = [aba^{-1}b^{-1}] = [1]$, i.e., $aba^{-1}b^{-1} \in H$. Hence, $G' \subset H$.

Conversely, suppose $G' \subset H$, i.e. $aba^{-1}b^{-1} \in H$ for all $a, b \in G$. Hence $[a][b][a]^{-1}[b]^{-1} = [1]$ in G/H, i.e., $[a]$ and $[b]$ commutes. □

In other words, G' is the smallest normal subgroup of G with abelian factor group or G/G' is the largest abelian factor group of G. Clearly, G abelian iff G' is trivial. Let $\pi \colon G \to G/G'$ denote the residue homomorphism, i.e., π maps an element g in G to the residue class $[g]$ in G/G'.

Theorem 19. *If $f \colon G \to H$ be a homomorphism into an abelian group H, then there exists a unique homomorphism $f' \colon G/G' \to H$ such that*



commutes.

Proof. Let $a, b \in G$. Since H abelian, $f([a, b]) = f(aba^{-1}b^{-1}) = f(a)f(b)f(a)^{-1}f(b)^{-1} = 1$, i.e. $G' \subset \operatorname{Ker} f$. By the homomorphism theorem, f factors uniquely through the factor group G/G'. □

For any group G we define the abelianization G_{ab} of G to be G/G'. Theorem 1 says that G_{ab} is the largest abelian factor group og G. Theorem 19 says that any homomorphism from G into an abelian group factors uniquely through G_{ab}.

Exercise 1.2.9. A CW complex structure on M_h' consists of one 0-cell, x_0, $2h+1$ 1-cells, $a_1, b_1, \dots, a_h, b_h, C$, and one 2-cell. Hence, the 1-skeleton is a wedge of $2g+1$ circles. The attachment map of the 2-cell is $[a_1, b_1] \cdots [a_h, b_h]C^{-1}$. (It might be instructive to draw the situation for $h = 2$ using the usual visualization of the 1-skeleton of M_2 as a polygon with 8 edges and the circle C as a circle inside the polygon intersecting the 1-skeleton at one of the points representing x_0. Remember to choose the orientations in accordance with the attachment map or adjust the attachment map appropriately.)

Now, by 1.26,

$$\pi_1(M_h', x_0) = \langle\, a_1, b_1, \dots a_h, b_h, C \mid [a_1, b_1] \cdots [a_h, b_h]C^{-1} = 1 \,\rangle.$$

Since

$$(\dagger) \qquad [a_1, b_1] \cdots [a_h, b_h] = C,$$

another presentation of the fundamental group is

$$\pi_1(M_h', x_0) = \langle\, a_1, b_1, \dots, a_h, b_h \,\rangle,$$

the free product of $2h$ copies of $\mathbf{Z}$.

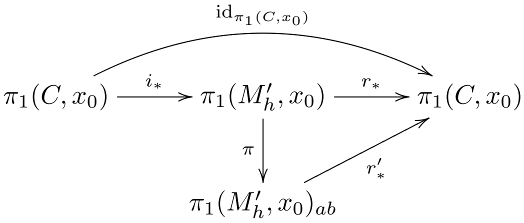
Suppose $r\colon M_h' \to C$ is a retraction. Let $i\colon C \to M_h'$ be the inclusion map. As always, $ri = \mathrm{id}_C$ and $r_* i_* = \mathrm{id}_{\pi_1(C,x_0)}$. Note that $\pi_1(C, x_0) = \langle\, C \,\rangle = \mathbf{Z}$, and $i_*(C) = C = [a_1, b_1] \cdots [a_h, b_h]$, by $(\dagger)$. We will derive a contraction to the existence of the retraction r in two almost identical ways:

(1) Since $\pi_1(C, x_0)$ is abelian,

$$r_* i_*(C) = r_*([a_1, b_1] \cdots [a_h, b_h]) = [r_*(a_1), r_*(b_1)] \cdots [r_*(a_h), r_*(b_h)] = 1,$$

i.e., $r_* i_*$ is the trivial homomorphism, a contradiction.

(2) Since $\pi_1(C, x_0)$ is abelian, there exists a homomorphism $r_*'\colon \pi_1(M_h', x_0)_{ab} \to \pi_1(C, x_0)$ such that the diagram



commutes, c.f. theorem 19. Since C is a product of commutators in $\pi_1(M_h', x_0)$, c.f. $(\dagger)$, $\pi i_*(C) = 1$. Hence, $r_* i_* = r_*' \pi i_*$ is the trivial homomorphism, a contradiction.

In particular, there is no retraction $M_g \to C$, since restriction would give a retraction $M_h' \to C$.

The usual CW complex structure on M_g consists of one 0-cell, $2g$ 1-cells, $a_1, b_1, \dots, a_g, b_g$ and one 2-cell. The 1-skeleton is the wedge

$$\bigvee_{i=1}^{g} (S^1_{a_i} \vee S^1_{b_i}).$$

The attachment map of the 2-cell is $[a_1, b_1] \cdots [a_g, b_g]$. Collapsing

$$\bigvee_{i=2}^{g} (S^1_{a_i} \vee S^1_{b_i})$$

gives a quotient map $q\colon M_g \to M_1$. The map $r\colon M_1 = S^1 \times S^1 \to S^1 \times \{s_0\} = C'$, $s_0 \in S^1$, defined by $r(x, y) = (x, s_0)$ is a retraction. Now $rq\colon M_g \to C'$ is a retraction.

Ex. 1.2.20. Let X be $\bigcup_{n=1}^{\infty} C_n$, and let $Y = \bigvee_{\infty} S^1$.

$\pi_1(X) = *_{n=1}^{\infty}\pi_1(C_n)$: Let U be an open ball with center 0 and radius less than 1. Then $V = U \cap X$ is contractible. Let $A_n = V \cup (C_n - \{0\})$. Then A_n is open, being the union of two open sets, and $\bigcup_{n=1}^{\infty} A_n = X$. By van Kampen's theorem,

$$\pi_1(X, 0) = *_{n=1}^{\infty}\pi_1(C_n, 0).$$

Note that the closure $\overline{X}$ of X in $\mathbf{R}^2$ is $X \cup (\{0\} \times \mathbf{R})$.

X and Y are homotopy equivalent: Observe that $\mathbf{R}^2 - (2\mathbf{Z}_+ - 1)$ deformation retracts onto $\overline{X}$, which is homotopy equivalent to X. See figure 7 for an illustration of this deformation retraction, where the arrows illustrates the deformation.

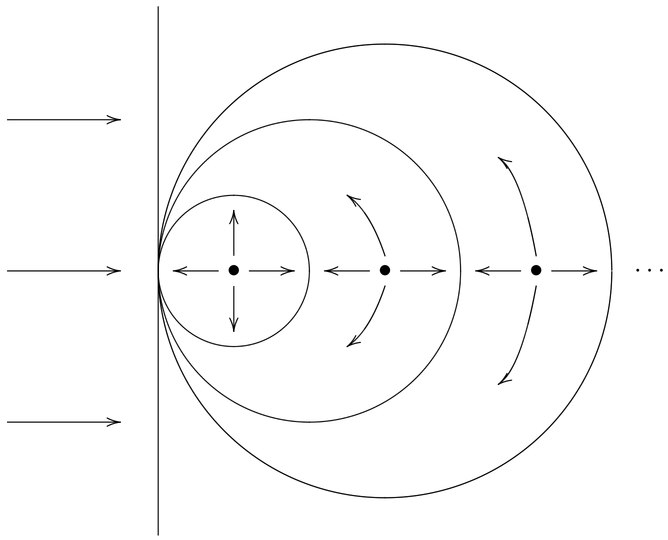


FIGURE 7. The deformation retraction of $\mathbf{R}^2 - (2\mathbf{Z}_+ - 1)$ onto $\overline{X}$.

But $\mathbf{R}^2 - (2\mathbf{Z}_+ - 1)$ also deformation retracts onto $Z = \bigcup_{n=1}^{\infty} S_n$, where $S_n \subset \mathbf{R}^2$ is the circle with center $2n - 1$ and radius 1. See figure 8 for an illustration of this deformation retraction, where the arrows illustrates the deformation. By collapsing the union of the southern hemispheres of the circles, which is contractible, Z is homotopy equivalent to Y.

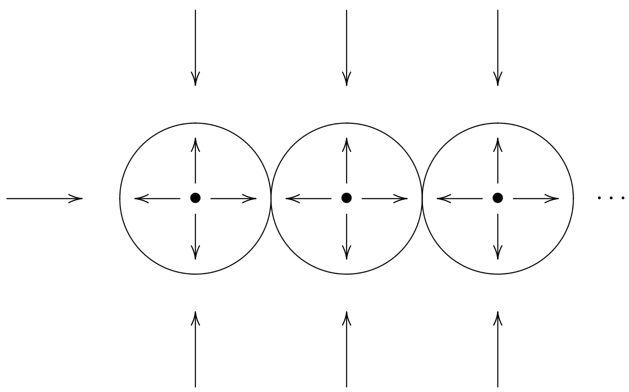


FIGURE 8. The deformation retraction of $\mathbf{R}^2 - (2\mathbf{Z}_+ - 1)$ onto Z.

Recall that a space is first countable if every point x has a countable neighborhood basis, that is, for each x there is a countable set $\{U_n\}$ of open neighborhoods of x such that for any neighborhood V of x there is an n with $U_n \subset V$.

Lemma 20. *The infinite wedge of circles is not first countable.*

Proof. Let $s_0 \in S^1$. Consider the quotient map

$$q\colon \coprod_\infty S^1 = \mathbf{Z}_+ \times S^1 \to \mathbf{Z}_+ \times S^1/\mathbf{Z}_+ \times \{s_0\} = \bigvee_\infty S^1,$$

where

$$q(\{n\} \times \{s\}) = \begin{cases} \{n\} \times \{s\}, & s \neq s_0 \\ \mathbf{Z}_+ \times \{s_0\}, & s = s_0, \end{cases}$$

where $\mathbf{Z}_+ \times \{s_0\}$ denotes the subset of $\coprod_\infty S^1$, as well as the common point in $\bigvee_\infty S^1$.

Recall that U is open in $\bigvee_\infty S^1$ iff $q^{-1}(U)$ open in $\coprod_\infty S^1$ iff $(\{n\} \times S^1) \cap q^{-1}(U)$ open in $\{n\} \times S^1$ for all n.

Let $\{U_n\}_{n\in\mathbf{Z}_+}$ be a countable set of open neighborhoods of $\mathbf{Z}_+ \times \{s_0\} \subset \bigvee_\infty S^1$. Now,

$$V_n = q^{-1}(U_n) \cap (\{n\} \times S^1)$$

is an open neighborhood of $\{n\}\times\{s_0\}$ in $\{n\}\times S^1$. Observe that $\{n\}\times\{s_0\} \subsetneq V_n$, since $\{n\}\times\{s_0\}$ is closed in $\{n\} \times S^1$ which is connected, that is, $\emptyset$ and $\{n\} \times S^1$ are the only clopen subsets of $\{n\} \times S^1$. Hence, there exists $x_n \in V_n - (\{n\} \times \{s_0\})$.

Now,

$$W_n = V_n - \{x_n\} = ((\mathbf{Z}_+ \times S^1) - \{x_n\}) \cap V_n$$

is an open proper subset of V_n containing $\{n\} \times \{s_0\}$. Thus, $W = \bigcup_n W_n$ is open in $\mathbf{Z}_+ \times S^1$. Clearly, $q^{-1}q(W) = W$ (by the definition of q). Hence, $q(W)$ is an open neighborhood of $\mathbf{Z}_+\times\{s_0\}$.

If $U_n \subset q(W)$, then

$$V_n = q^{-1}(U_n) \cap (\{n\} \times S^1) \subset W \cap (\{n\} \times S^1) = W_n,$$

which is a contradiction. Hence, the infinite wedge of circles is not first countable at $\mathbf{Z}_+\times\{s_0\}$. □

An immediate consequence is that $\bigvee_\infty S^1$ does not embed in any first countable space, in particular $\mathbf{R}^2$. Thus, X and Y are not homeomorphic.

Ex. 1.3.1. Let $p\colon \widetilde{X} \to X$ be a covering space, and let A be a subspace of X. Furthermore, let $x_0 \in A$, and let U be an open neighborhood of x_0 in X which is evenly covered by p, that is, $p^{-1}(U)$ is a disjoint union of open sets V_α each of which is mapped homeomorphically onto U. Then $U \cap A$ is an open neighborhood of x_0 in A, $V_\alpha \cap p^{-1}(A)$ are disjoint open sets in $\widetilde{X}$ with union $p^{-1}(U\cap A)$, and each $V_\alpha \cap p^{-1}(A)$ are mapped homeomorphically onto $U \cap A$ by p. In other words, x_0 has an open neighborhood which is evenly covered by the restriction of p.

Ex. 1.3.2. Let $p_1\colon \widetilde{X}_1 \to X_1$ and $p_2\colon \widetilde{X}_2 \to X_2$ be covering spaces. Let $(x_1, x_2) \in X_1 \times X_2$, and let U_1 and U_2 be open neighborhoods of x_1 and x_2, respectively, which are evenly covered by p_1 and p_2, respectively. Now, $(p_1 \times p_2)^{-1}(U_1 \times U_2)$ is the disjoint union of products of open sets in $p_1^{-1}(U_1)$ and $p_2^{-1}(U_2)$, and each is mapped homeomorphically onto $U_1 \times U_2$ by $p_1 \times p_2$.

Ex. 1.3.4. Let $X \subset \mathbf{R}^3$ be the union of the unit sphere S^2 in $\mathbf{R}^3$ and the diameter D connecting $(-1, 0, 0)$ and $(1, 0, 0)$. For $t \in \mathbf{R}$, let $\tau_t\colon \mathbf{R}^3 \to \mathbf{R}^3$ be translation by t along the first axis, that is, $\tau_t(x, y, z) = (x + t, y, z)$.

Let $\widetilde{X} \subset \mathbf{R}^3$ be the union

$$\bigcup_{n\in\mathbf{Z}} (\tau_{4n}(S^2) \cup \tau_{4n+1}(D)),$$

see figure 9, which is simply connected, since homotopy equivalent to the wedge of countable infinite circles (why?).

Define $p_X\colon \widetilde{X} \to X$ to be the inverse translation on each translated sphere, and the inverse translation followed by a reflection on each translated diameter. The map p_X is clearly a covering map.

Let $Y \subset \mathbf{R}^3$ be the union of S^2 and a circle intersection it in two points. For convenience, let the two points be the north pole and the south pole.

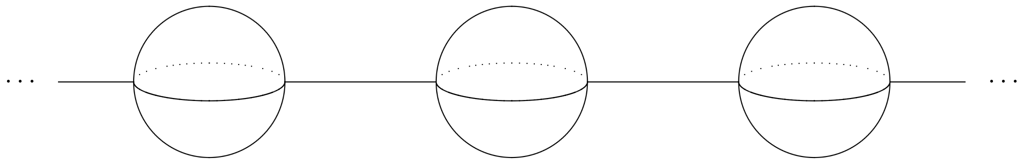


FIGURE 9. The universal covering space of X.

Consider the space $\widetilde{Y}$ in figure 10, which is simply connected, since homotopy equivalent to the wedge of countably infinite spheres (why?). Let $p_Y : \widetilde{Y} \to Y$ be the map that maps a sphere in $\widetilde{Y}$ to the sphere in Y, maps the horizontal arcs to the circle segment inside (or outside) the sphere, and maps the vertical lines to the circle segment outside (or inside) the sphere. Clearly, p_Y is a covering map.

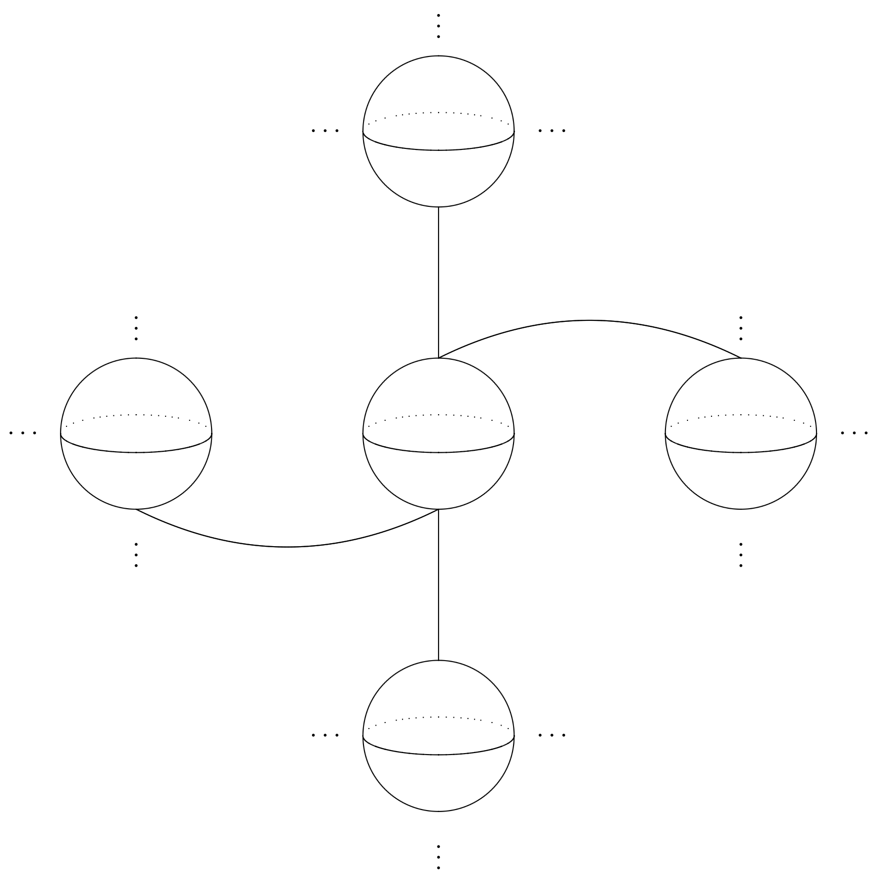


FIGURE 10. The universal covering space of Y.

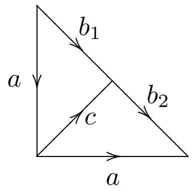
Ex. 1.3.9. Let X be a space with finite $\pi_1(X)$. Consider a map $f\colon X \to S^1$. Since $f_*(\pi_1(X))$ is a finite subgroup of $\pi_1(S^1) = \mathbf{Z}$ and $\mathbf{Z}$ contains no nontrivial finite subgroups, $f_*\colon \pi_1(X) \to \pi_1(S^1)$ is the trivial homomorphism.

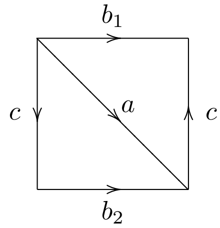
Let $p\colon \mathbf{R} \to S^1$ be the universal covering space. By 1.33, f lifts to the universal covering space, i.e. there exists a map $\widetilde{f}\colon X \to \mathbf{R}$ such that $f = p\widetilde{f}$.

Since $\mathbf{R}$ is contractible, $\mathrm{id}_{\mathbf{R}}$ is homotopic to a constant map. Thus, $g = \mathrm{id}_{\mathbf{R}}\, g \simeq *g = *$ for any map $g\colon Y \to \mathbf{R}$ and any space Y. This also follows from ex. 0.10. Now, $f = p\widetilde{f} \simeq p* = *$.

Chapter 2

Ex. 2.1.1. Consider the subdivision pictured in the left figure below.





Rearrangement gives the Δ-complex on the right, which is the Möbius band.

Ex. 2.1.4. Let X be the triangular parachute. The Δ-complex structure, see figure 11, has one 0-simplex v, three 1-simplices a, b, c, and one 2-simplex T.

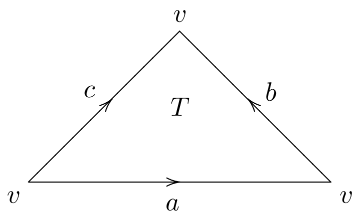


Figure 11. The triangular parachute.

The boundary maps are $\partial_1(a) = \partial_1(b) = \partial_1(c) = v - v = 0$, and $\partial_2(T) = b - c + a$. The simplicial chain complex is

$$0 \longrightarrow \mathbf{Z}\{T\} \xrightarrow{\partial_2} \mathbf{Z}\{a, b, c\} = \mathbf{Z}\{a, b, b - c + a\} \xrightarrow{\partial_1} \mathbf{Z}\{v\} \longrightarrow 0.$$

The nontrivial homology groups are $H_0^\Delta(X) = \mathbf{Z}$,

$$H_1^\Delta(X) = \operatorname{Ker}\partial_1 / \operatorname{Im}\partial_2 = \mathbf{Z}\{a, b, b - c + a\}/\mathbf{Z}\{b - c + a\} = \mathbf{Z}^2,$$

and $H_2^\Delta(X) = \operatorname{Ker}\partial_2 = 0$.

Ex. 2.1.9. Let X be the Δ-complex obtained from Δ^n by identifying all faces of the same dimension.

Observe that $H_m^\Delta(X) = 0$ for $m > n$, and, by the identifications, that $\Delta_m(X) = \mathbf{Z}\{[v_0, \dots, v_m]\}$ for $0 \le m \le n$. The simplicial chain complex is

$$0 \longrightarrow \mathbf{Z}\{[v_0, \dots, v_n]\} \xrightarrow{\partial_n} \mathbf{Z}\{[v_0, \dots, v_{n-1}]\} \longrightarrow \cdots \longrightarrow \mathbf{Z}\{[v_0, v_1]\} \xrightarrow{\partial_1} \mathbf{Z}\{[v_0]\} \longrightarrow 0.$$

For $0 < m \le n$,

$$\begin{aligned}\partial_m([v_0, \dots, v_m]) &= \sum_{i=0}^{m} (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_m] \\ &= [v_0, \dots, v_{m-1}] \sum_{i=0}^{m} (-1)^i \\ &= \begin{cases} 0, & m \text{ odd} \\ [v_0, \dots, v_{m-1}], & m \text{ even,} \end{cases}\end{aligned}$$

where the second equality follows by the identification of all faces of the same dimension. Hence, ∂_m is a isomorphism when m is even (mapping generator to generator), and the zero map when m is odd.

For $0 < m < n$,

$$\operatorname{Ker}\partial_m = \operatorname{Im}\partial_{m+1} = \begin{cases} 0, & m \text{ even} \\ \Delta_m(X), & m \text{ odd}, \end{cases}$$

that is, $H_m^{\Delta}(X) = 0$. Furthermore,

$$H_0^{\Delta}(X) = \Delta_0(X)/\operatorname{Im}\partial_1 = \mathbf{Z},$$

and

$$H_n^{\Delta}(X) = \operatorname{Ker}\partial_n = \begin{cases} 0, & n \text{ even} \\ \mathbf{Z}, & n \text{ odd}. \end{cases}$$

Ex. 2.1.11. Let $r\colon X \to A$ be a retract, and let $i\colon A \to X$ be the inclusion map. Then $ri = \mathrm{id}_A$. Applying the covariant functor $H_n(-)$ gives that $H_n(ri) = H_n(r)H_n(i) = \mathrm{id}_{H_n(A)}$, i.e., $H_n(i)$ is injective.

Ex. 2.1.16. Let X be a space, let $\{X_\alpha\}$ be the path components of X, and let

$$\cdots \longrightarrow C_1(X) \xrightarrow{\partial_1} C_0(X) \longrightarrow 0.$$

be the singular chain complex of X. Recall that the 0-simplices are the points of X, and the 1-simplices are the paths in X.

Lemma 21. *Let $x, y \in X$. Then $y - x \in \operatorname{Im}\partial_1$ iff x and y lie in the same path component of X.*

Proof. Suppose $\sigma\colon I \to X$ is a path with $y - x = \partial_1(\sigma) = \sigma(1) - \sigma(0)$. Since the points of X is a basis of $C_0(X)$, $\sigma(0) = x$ and $\sigma(1) = y$, that is, σ is a path from x to y.

Conversely, suppose $\sigma\colon I \to X$ is a path from x to y. Then $\partial_1(\sigma) = \sigma(1) - \sigma(0) = y - x$. □

For a point x in X let $[x]$ denote its coset in $H_0(X) = C_0(X)/\operatorname{Im}\partial_1$. By lemma 21, $[x] = [y]$ in $H_0(X)$ iff x and y lie in the same path component. In particular, if X is path connected, then every point in X generates $H_0(X)$. In other words, $H_0(X)$ is free abelian with basis the path components of X, i.e., the map $p_X\colon H_0(X) \to \mathbf{Z}\{X_\alpha\}$, defined by letting $p_X([x])$ be the unique path component of X containing x, is an isomorphism.

Now, let A be a subspace of X, and let $\{A_\beta\}$ be the set of path components of A. Observe that

$$X_\alpha \cap A = \bigcup_{\{\beta \mid A_\beta \subset X_\alpha\}} A_\beta.$$

As usual, let $i\colon A \to X$ be the inclusion map. By definition, $i_*\colon H_0(A) \to H_0(X)$ maps a coset $[x] \in H_0(A)$, $x \in A \subset X$, to the coset $[i(x)] = [x] \in H_0(X)$.

The diagram

$$\begin{array}{ccc} H_0(A) & \xrightarrow{i_*} & H_0(X) \\ {\scriptstyle p_A}\downarrow\cong & & {\scriptstyle p_X}\downarrow\cong \\ \mathbf{Z}\{A_\beta\} & \xrightarrow{i'} & \mathbf{Z}\{X_\alpha\} \end{array}$$

commutes, where $i'\colon \mathbf{Z}\{A_\beta\} \to \mathbf{Z}\{X_\alpha\}$ is defined by letting $i'(A_\beta)$ be the unique path component of X containing A_β. The description of i_* via i' gives the following result.

Lemma 22.
1. *$i_*\colon H_0(A) \to H_0(X)$ is surjective iff A meets each path component of X iff $X_\alpha \cap A$ is nonempty for all α.*
2. *$i_*\colon H_0(A) \to H_0(X)$ is injective iff each path component of X contains at most one path component of A iff $X_\alpha \cap A$ is path connected for all α.*

(a). Since

$$H_0(A) \xrightarrow{i_*} H_0(X) \longrightarrow H_0(X, A) \longrightarrow 0$$

is exact, $H_0(X, A) = 0$ iff $i_*\colon H_0(A) \to H_0(X)$ is surjective iff A meets each path component of X, c.f. lemma 22(1).

(b). Since

$$H_1(A) \xrightarrow{i_*} H_1(X) \longrightarrow H_1(X,A) \longrightarrow H_0(A) \xrightarrow{i_*} H_0(X)$$

is exact, $H_1(X,A) = 0$ iff $i_*\colon H_1(A) \to H_1(X)$ is surjective and $i_*\colon H_0(A) \to H_0(X)$ is injective iff $i_*\colon H_1(A) \to H_1(X)$ is surjective and each path component of X contains at most one path component of A, c.f. lemma 22(2).

Ex. 2.1.18. Since $\mathbf{R}$ is contractible, the long exact sequence of reduced homology groups gives the exact sequence

$$0 = \widetilde{H}_1(\mathbf{R}) \longrightarrow \widetilde{H}_1(\mathbf{R},\mathbf{Q}) = H_1(\mathbf{R},\mathbf{Q}) \longrightarrow \widetilde{H}_0(\mathbf{Q}) \longrightarrow \widetilde{H}_0(\mathbf{R}) = 0,$$

that is, $H_1(\mathbf{R},\mathbf{Q}) = \widetilde{H}_0(Q)$.

Recall that $\widetilde{H}_0(\mathbf{Q}) = \operatorname{Ker}\varepsilon$, where ε is the map induced by the augmentation map, which maps a finite linear combination of 0-simplices to the sum of the coefficients, in the augmented singular chain complex. In other words, there is a short exact sequence

$$0 \longrightarrow \widetilde{H}_0(\mathbf{Q}) \longrightarrow H_0(\mathbf{Q}) \xrightarrow{\varepsilon} \mathbf{Z} \longrightarrow 0.$$

Since $\mathbf{Q}$ is totally disconnected, $H_0(\mathbf{Q}) = \oplus_{\mathbf{Q}}\mathbf{Z}$ with basis the 0-simplices in $\mathbf{Q}$, that is, the set consisting of maps $\sigma_q\colon \Delta^0 \to \mathbf{Q}$, $\Delta^0 \mapsto q$, $q \in \mathbf{Q}$. Thus, the kernel of ε consists of finite integer linear combinations of 0-simplices such that the sum of the coefficients is zero. Hence, the set

$$\{\, \sigma_0 - \sigma_q \mid q \in \mathbf{Q} \,\}$$

is a basis for the kernel of ε.

Ex. 2.1.29. By 2.3, 2.14 and 2.25,

$$H_n(S^1 \times S^1) = H_n(S^2 \vee S^1 \vee S^1) = \begin{cases} \mathbf{Z}, & n = 0 \\ \mathbf{Z}^2, & n = 1 \\ \mathbf{Z}, & n = 2 \\ 0, & n \geq 3. \end{cases}$$

The universal covering space of $S^1 \times S^1$ is $\mathbf{R} \times \mathbf{R}$, c.f. ex. 1.3.2, which is contractible. In particular, $H_n(\mathbf{R} \times \mathbf{R}) = 0$ for all $n \neq 0$.

The universal covering space $p\colon E \to S^2 \vee S^1 \vee S^1$ is the universal covering space of $S^1 \vee S^1$, c.f. 1.45, with a S^2 attached at each vertex.

We will prove that $\mathbf{R}^2$ and E do not have the same homology in two ways:

(1) Since S^2 is simply connected, the inclusion $i\colon S^2 \to S^2 \vee S^1 \vee S^1$ lifts to E, that is, there exists $j\colon S^2 \to E$ such that $i = pj$. In particular, the diagram

$$\begin{array}{ccc} & & H_2(E) \\ & \overset{j_*}{\nearrow} & \downarrow p_* \\ H_2(S^2) & \xrightarrow{i_*} & H_2(S^2 \vee S^1 \vee S^1) \end{array}$$

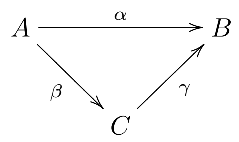
commutes. The map $S^2 \vee S^1 \vee S^1 \to S^2$ mapping $S^1 \vee S^1$ to the base point is a retraction. By ex. 2.1.11, $i_*\colon \mathbf{Z} = H_2(S^2) \to H_2(S^2 \vee S^1 \vee S^1)$ is injective. Hence, $H_2(E) \neq 0 = H_2(\mathbf{R} \times \mathbf{R})$.

(2) Any tree is contractible: For any vertex v, there is a unique path γ_v from a fixed base point to v. Now, the homotopy that at time t sends v to $\gamma_v(t)$ is a deformation retraction of the tree onto the base point fixing the base point.

The homology of E: Let G be the free on two generators. Since the universal covering space of $S^1 \vee S^1$ is contractible, being a tree, E is homotopy equivalent to $\bigvee_{g \in G} S^2$. Thus, $\widetilde{H}_n(E)$ is $\bigoplus_{g \in G} \mathbf{Z}$ for $n = 2$ and zero otherwise. In particular, the homology of E and $\mathbf{R}^2$ are not equal.

Ex. 2.1.30. Recall that composites of isomorphisms are isomorphisms.

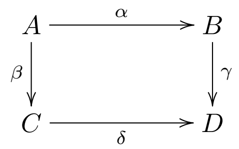
Consider the commutative diagram



There are three cases:

(1) If α, β are isomorphisms, then $\gamma = \alpha\beta^{-1}$ is an isomorphism.
(2) If α and γ are isomorphisms, then $\beta = \gamma^{-1}\alpha$ is an isomorphism.
(3) If β and γ are isomorphisms, then $\alpha = \gamma\beta$ is an isomorphism.

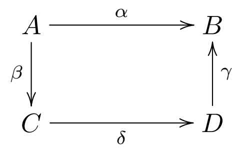
Consider the commutative diagram



There are four cases:

(1) If α, β and γ are isomorphisms, then $\delta = \gamma\alpha\beta^{-1}$ is an isomorphism.
(2) If α, β and δ are isomorphisms, then $\gamma = \delta\beta\alpha^{-1}$ is an isomorphism.
(3) If α, γ and δ are isomorphisms, then $\beta = \delta^{-1}\gamma\alpha$ is an isomorphism.
(4) If β, γ and δ are isomorphisms, then $\alpha = \gamma^{-1}\delta\beta$ is an isomorphism.

Consider the commutative diagram



There are four cases:

(1) If α, β and γ are isomorphisms, then $\delta = \gamma^{-1}\alpha\beta^{-1}$ is an isomorphism.
(2) If α, β and δ are isomorphisms, then $\gamma = \alpha\beta^{-1}\delta^{-1}$ is an isomorphism.
(3) If α, γ and δ are isomorphisms, then $\beta = \delta^{-1}\gamma^{-1}\alpha$ is an isomorphism.
(4) If β, γ and δ are isomorphisms, then $\alpha = \gamma\delta\beta$ is an isomorphism.

Ex. 2.2.2.

Lemma 23. *For any map $f\colon S^{2n} \to S^{2n}$ there exists $x \in S^{2n}$ such that $f(x) = x$ or $f(x) = -x$, that is, either f of $-f$ has a fixed point.*

Note that $-f = af$, where a is the antipodal map.

Proof. Since

$$(\forall\, x \in S^{2n}\colon f(x) \neq x) \overset{(g)}{\Rightarrow} \deg(f) = (-1)^{2n+1} = -1$$

and

$$\begin{aligned}(\forall\, x \in S^{2n}\colon f(x) \neq -x) &\Leftrightarrow (\forall\, x \in S^{2n}\colon (-f)(x) \neq x) \\ &\overset{(g)}{\Rightarrow} \deg(-f) = (-1)^{2n+1} = -1 \\ &\overset{(d)}{\Rightarrow} -1 = \deg(-f) = \deg(a)\deg(f) = (-1)^{2n+1}\deg(f) = -\deg(f) \\ &\Rightarrow \deg(f) = 1,\end{aligned}$$

either f or $-f$ has a fixed point. □

Lemma 24. *Any map $g\colon \mathbf{R}P^{2n} \to \mathbf{R}P^{2n}$ has a fixed point.*

Proof. Let $p\colon S^{2n} \to \mathbf{R}\mathrm{P}^{2n}$ be the universal covering space identifying antipodal points, c.f. 1.43. By the lifting criterion, c.f. 1.33, there exists $f\colon S^{2n} \to S^{2n}$ such that the diagram

$$\begin{array}{ccc} S^{2n} & \xrightarrow{f} & S^{2n} \\ \downarrow p & & \downarrow p \\ \mathbf{R}\mathrm{P}^{2n} & \xrightarrow{g} & \mathbf{R}\mathrm{P}^{2n} \end{array}$$

commutes. By lemma 23, there exists $x \in S^{2n}$ such that $f(x) = \pm x$. Now, $gp(x) = pf(x) = p(\pm x) = p(x)$, i.e., g fixes $p(x)$. □

Lemma 25. *There exists maps $\mathbf{R}P^{2n-1} \to \mathbf{R}P^{2n-1}$ without fixed points.*

Before proving this result we give a short survey of projective linear maps. Regard $\mathbf{R}\mathrm{P}^n$ as the quotient space of $\mathbf{R}^{n+1}$ by identifying lines through 0. Let $[-]\colon \mathbf{R}^{n+1} \to \mathbf{R}\mathrm{P}^n$ be the quotient map.

A linear map $T \in GL(n+1, \mathbf{R})$ induces a *projective linear map* $\widetilde{T}\colon \mathbf{R}\mathrm{P}^n \to \mathbf{R}\mathrm{P}^n$ defined by $\widetilde{T}([v]) = [T(v)]$, $v \in \mathbf{R}^{n+1}$. This is clearly well-defined, if $[v] = [w]$, then $v = \lambda w$, $\lambda \neq 0$, and $\widetilde{T}([v]) = [T(v)] = [\lambda T(w)] = [T(w)] = \widetilde{T}([w])$.

Let I be the identity matrix, and for a group G let $Z(G)$ denote its center.

Lemma 26. $Z(GL(n, \mathbf{R})) = \{\, \lambda I \mid \lambda \neq 0 \,\}$.

Proof. Let $M \in Z(GL(n, \mathbf{R}))$. In particular, M commutes with the elementary matrices, see the solution of ex. 2.2.7. Thus, for any $c \neq 0$, multiplying the ith row of M by c is equal to multiplying the ith column of M by c, that is, M is a diagonal matrix. Furthermore, since interchanging the ith and jth row of M is equal to interchanging the ith and jth column of M, the ith diagonal entry of M is equal to jth diagonal entry of M. In other words, $M = \lambda I$ for some $\lambda \neq 0$.

Clearly, λI, $\lambda \neq 0$, commutes with all matrices. □

Lemma 27. *If $S, T \in GL(n+1, \mathbf{R})$, then $\widetilde{S} = \widetilde{T}$ iff there exists $\lambda \neq 0$ such that $T = \lambda S$.*

Proof. Suppose $\widetilde{S} = \widetilde{T}$. Let $\{e_1, \ldots, e_{n+1}\}$ be the standard basis of $\mathbf{R}^{n+1}$. Since $\widetilde{T}([e_i]) = \widetilde{S}([e_i])$, $T(e_i) = \lambda_i S(e_i)$. Hence, $S^{-1}T(e_i) = \lambda_i e_i$, that is, $S^{-1}T = \mathrm{diag}(\lambda_1, \ldots, \lambda_{n+1})$.

Let $v = e_1 + \cdots e_{n+1}$. Since $\widetilde{S} = \widetilde{T}$, $S^{-1}T(v) = \lambda v$ for some $\lambda \neq 0$. Now,

$$\lambda v = S^{-1}T(v) = \lambda_1 e_1 + \cdots \lambda_{n+1} e_{n+1},$$

that is, $\lambda = \lambda_1 = \cdots = \lambda_{n+1}$. In other words, $S^{-1}T = \lambda I$.

Conversely, suppose $T = \lambda S$, $\lambda \neq 0$. Then $\widetilde{T}([v]) = [T(v)] = [\lambda S(v)] = [S(v)] = \widetilde{S}([v])$. □

The set $\mathrm{Aut}(\mathbf{R}\mathrm{P}^n)$ of automorphisms of $\mathbf{R}\mathrm{P}^n$ is a group under composition. There is a homomorphism $GL(n+1, \mathbf{R}) \to \mathrm{Aut}(\mathbf{R}\mathrm{P}^n)$, $T \mapsto \widetilde{T}$, with kernel the center of $GL(n+1, \mathbf{R})$, c.f. lemma 27. Hence,

$$\mathrm{PGL}(n, \mathbf{R}) = GL(n+1, \mathbf{R})/Z(GL(n+1, \mathbf{R}))$$

is isomorphic to a subgroup of $\mathrm{Aut}(\mathbf{R}\mathrm{P}^n)$, and is called the *projective linear group.*

Lemma 28. *If $T \in GL(n+1, \mathbf{R})$, then $\widetilde{T}([v]) = [v]$ iff v is an eigenvector of T.*

Proof. Suppose $\widetilde{T}([v]) = [T(v)] = [v]$. Then $T(v) = \lambda v$, $\lambda \neq 0$, that is, v eigenvector associated with the eigenvalue λ.

Conversely, suppose $T(v) = \lambda v$, $\lambda \neq 0$. Then $\widetilde{T}([v]) = [T(v)] = [\lambda v] = [v]$. □

Proof of lemma 25. By lemma 28, it suffices to construct $T_n \in GL(2n, \mathbf{R})$ such that T_n has no real eigenvalues. Let A be the matrix

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

that is, A is a clockwise rotation of $\mathbf{R}^2$ by angle $\pi/2$, and let T_n be the block matrix

$$\begin{pmatrix} A & & & 0 \\ & A & & \\ & & \ddots & \\ 0 & & & A \end{pmatrix}.$$

Since $\det A = 1$ and $\det(T_n) = (\det A)^n = 1$, $T \in GL(2n, \mathbf{R})$. Now,

$$\det(\lambda I - T_n) = (\det \begin{pmatrix} \lambda & -1 \\ -1 & \lambda \end{pmatrix})^n = (\lambda^2 + 1)^n.$$

Hence, T_n has no real eigenvalues. □

Ex. 2.2.4. The loop $f\colon S^1 \to S^1$, $f = \mathrm{id}_{S^1} \cdot \overline{\mathrm{id}_{S^1}}$ where $\cdot$ denotes path composition, is surjective and nullhomotopic, hence f has degree zero. Since the suspension of a surjective map is surjective, $S^n f\colon S^n \to S^n$ is surjective of degree zero, c.f. 2.33.

Ex. 2.2.7. Let $M(n, \mathbf{R})$ denote the $n \times n$ matrices with coefficients in $\mathbf{R}$. We view $M(n, \mathbf{R})$ as the vector space $\mathbf{R}^{n^2}$ with the standard topology. Note that $M(n, \mathbf{R})$ is locally compact. The group $GL(n, \mathbf{R})$ of invertible matrices is topologized as a subspace of $M(n, \mathbf{R}) = \mathbf{R}^{n^2}$. In particular $GL(n, \mathbf{R})$ is locally path connected, so particular the components and path components coincide, and locally compact.

Recall that the determinant is a map $\det\colon M(n, \mathbf{R}) \to \mathbf{R}$, and continuous since it is a polynomial in the n^2 entries in of a $n \times n$ matrix. Note that $GL(n, \mathbf{R}) = \det^{-1}(\mathbf{R} - \{0\})$ is an open subset of $M(n, \mathbf{R})$, and $\det^{-1}(0)$ is a closed subset of $M(n, \mathbf{R})$. Thus, $\det\colon GL(n, \mathbf{R}) \to \mathbf{R}^\times$ is a continuous group homomorphism, where $\mathbf{R}^\times = \mathbf{R} - \{0\}$ denotes the multiplicative group of units in $\mathbf{R}$..

Let I_n denote the $n \times n$ identity matrix, let $I_{i,j}$ be the $n \times n$ matrix with a 1 at entry (i, j) and 0 otherwise. Let

$$\begin{aligned} \text{Æ}_i(c) &= I_n + (c-1)I_{i,i}, \\ \text{Ø}_{i,j} &= I_n - I_{i,i} + I_{i,j} - I_{j,j} + I_{j,i} \text{ and} \\ \text{Å}_{i,j}(c) &= I_n + cI_{i,j}, \end{aligned}$$

where c is a nonzero real number. These matrices are called the elementary matrices.

Let A be a $n \times n$ matrix. The matrices Æ, Ø and Å correspond to the three types of elementary row operations:

Æ: $\text{Æ}_i(c)A$ is obtained from A by multiplying the ith row of A by c.
Ø: $\text{Ø}_{i,j}A$ is obtained from A by interchanging the ith and jth row of A.
Å: $\text{Å}_{i,j}(c)A$ is obtained from A by adding c times the ith row of A to the jth row of A.

Recall that $\det \text{Æ}_i(c) = c$, $\det \text{Ø}_{i,j} = -1$ and $\det \text{Å}_{i,j}(c) = 1$, that is, the elementary matrices are invertible. The elementary column operations correspond to right multiplication by Æ, Ø and Å.

For $1 \le i < j \le n$ consider the rotation matrix,

$$R_{i,j}(\theta) = I_n - I_{i,i} - I_{j,j} + \cos(\theta)(I_{i,i} + I_{j,j}) + \sin(\theta)(I_{j,i} - I_{i,j}),$$

that is, a clockwise rotation by θ radians in the plane spanned by the ith and jth basis vectors of $\mathbf{R}^n$. In particular, $\det R_{i,j}(\theta) = 1$.

Lemma 29. *$GL(n, \mathbf{R})$ is not connected.*

Proof. Suppose γ is a path in $GL(n, \mathbf{R})$ connecting $\text{Æ}_1(-1)$ and I_n. Then $\det \circ \gamma\colon I \to \mathbf{R}$ is a path in $\mathbf{R} - \{0\}$ connecting $\det \text{Æ}_1(-1) = -1$ and $\det I_n = 1$, a contradiction.

Another proof is noting that $\det^{-1}(\mathbf{R}_-)$ and $\det^{-1}(\mathbf{R}_+)$ is a separation of $GL(n, \mathbf{R})$. □

But, $\det^{-1}(\mathbf{R}_-)$ and $\det^{-1}(\mathbf{R}_+)$ are path connected subspaces of $GL(n, \mathbf{R})$:

Lemma 30. *There is a path in $GL(n, \mathbf{R})$ from any $A \in GL(n, \mathbf{R})$ to $\text{Æ}_1(\mathrm{sign}(\det A))$.*

Proof. Let $A \in GL(n, \mathbf{R})$. First, we define three types of paths in $GL(n, \mathbf{R})$:

Type Æ: Suppose $c > 0$ and $c \neq 1$. Then $(1-t) + tc > 0$ for all $t \in I$. Define $Æ(A, i, c)\colon I \to GL(n, \mathbf{R})$ by $Æ(A, i, c)(t) = Æ_i((1-t)+tc)A$. Clearly, $Æ(A, i, c)$ is continuous, i.e., a path from A to $Æ_i(c)A$. Furthermore, $\det Æ(A, i, c)(t) = \det Æ_i((1-t)+tc) \det A = ((1-t)+tc) \det A \neq 0$ for all $t \in I$. Observe that $\operatorname{sign}(\det Æ(A, i, c)) = \operatorname{sign}(\det A)$, that is, the determinant do not change sign along the path. Hence, the path is well-defined.

Type Å: Define $Å(A, i, j, c)\colon I \to GL(n, \mathbf{R})$ by $Å(A, i, j, c)(t) = Å_{i,j}(tc)A$. Clearly, $Å(A, i, j, c)$ is continuous, i.e., a path from A to $Å_{i,j}(c)A$. Furthermore, $\det Å(A, i, j, c) = \det Å_{i,j}(tc) \det A = \det A$, that is, the determinant is constant along the path. Hence, the path is well-defined.

Type R: Define $R(A, i, j, \theta)\colon I \to GL(n, \mathbf{R})$ by $R(A, i, j, \theta)(t) = R_{i,j}(t\theta)A$. Clearly, $R(A, i, j, \theta)$ is continuous, i.e., a path from A to $R_{i,j}(\theta)A$. Furthermore, $\det R(A, i, j, \theta)(t) = \det R_{i,j}(\theta) \det A = \det A$, that is, the determinant is constant along the path. Hence, the path is well-defined.

By composing paths of type Å, there is a path from A to a diagonal matrix $B = \operatorname{diag}(b_1, \ldots, b_n)$, $b_i \neq 0$, $1 \leq i \leq n$.

By composing paths of type Æ, there is a path from B to $C = \operatorname{diag}(\operatorname{sign}(b_1), \ldots, \operatorname{sign}(b_n))$.

Suppose $\operatorname{sign}(b_i) = \operatorname{sign}(b_j) = -1$, $i < j$. Then $R(C, i, j, \pi)$ is a path from C to the matrix $C + 2(I_{i,i} + I_{j,j})$, i.e., the pair of -1's in B is changed to 1's.

Thus, if there is an even number of -1's in C, then there is a path from A to $Æ_1(1) = I_n$. If there is an odd number of -1's in C, then there is a path from C to $Æ_i(-1)$ for some $1 \leq i \leq n$. If $i > 1$, then composing with the path $R(C, 1, i, \pi)$ gives a path from $Æ_i(-1)$ to $Æ_1(-1)$. □

In particular, $GL(n, \mathbf{R})$ has two components, $\det^{-1}(\mathbf{R}_-)$ and $\det^{-1}(\mathbf{R}_+)$. Next, is a topological property of the determinant map:

Lemma 31. *The determinant* $\det\colon M(n, \mathbf{R}) \to \mathbf{R}$ *is an open map.*

Proof. It suffices to show that the determinant maps connected neighborhoods to neighborhoods. Let $A \in M(n, \mathbf{R})$, and let U be a connected neighborhood of A, e.g. an open n^2-ball. The proof is divided into two cases, A invertible and A not invertible:

Suppose $A \in GL(n, \mathbf{R})$. For $t \in \mathbf{R}$, $\det(1-t)A = (1-t)^n \det A$, and $|A - (1-t)A| = |t|\,|A|$. If $0 < t < 1$, then $(1-t)^n < 1 < (1-(-t))^n = (1+t)^n$, i.e. $\det(1-t)A < \det A < \det(1+t)A$. Hence, for $t > 0$ sufficiently small, $(1-t)A, (1+t)A \in U$. Since $\det(U)$ is a connected subspace of $\mathbf{R}$, $\det A \in [\det(1-t)A, \det(1+t)A] \subset \det(U)$, i.e., $\det(U)$ is a neighborhood of $\det A$.

Suppose $\det A = 0$. It suffices to show that U contains matrices of opposite sign. Using row operations of type Å, there exists an invertible matrix B (a product of Å's) such that $C = BA$ is an upper triangular matrix. In particular, $\det B = 1$, i.e., $c_{1,1} \cdots c_{n,n} = \det C = \det A = 0$. Let I_0 be the non-empty subset of $\{1, \ldots, n\}$ such that $c_{i,i} = 0$ iff $i \in I_0$.

Let $i_0 \in I_0$. For $t \in \mathbf{R}$, define matrices $D_+(t)$ and $D_-(t)$ by

$$D_+(t) = C + tI_n$$

and

$$D_-(t) = C + tÆ_{i_0}(-1).$$

Thus,

$$\det D_+(t) = \prod_{i=1}^{n} (t + c_{i,i}) = -\det D_-(t),$$

and $\det D_+(t) = 0$ iff $t = 0$, and $\operatorname{sign}(\det D_+(t)) = -\operatorname{sign}(\det D_-(t))$. Now,

$$\det(B^{-1}D_+(t)) = \det D_+(t) = -\det D_-(t) = -\det(B^{-1}D_+(t)).$$

Furthermore,

$$|A - B^{-1}D_+(t)| = |A - B^{-1}(BA + tI_n)| = |t|\,|B^{-1}|$$

and

$$|A - B^{-1}D_-(t)| = |t||B^{-1}Æ_{i_0}(-1)| = |t|\,|B^{-1}|.$$

It follows that there exists invertible matrices with opposite signs arbitrarily close to A. □

Since the restriction of an open map to an open subspace is an open map, $\det\colon GL(n,\mathbf{R}) \to \mathbf{R}$ is open. Since there exists invertible matrices with opposite signs arbitrarily close to any non-invertible matrix, the boundary of both components of $GL(n,\mathbf{R})$ is $\det^{-1}(0)$. Another immediate consequence:

Corollary 32. *$GL(n,\mathbf{R})$ is dense in $M(n,\mathbf{R})$.*

In other words, a random matrix is likely to be invertible. Recall that the orthogonal group $O(n)$ is the group of $n \times n$ matrices A such that $AA^t = I_n$.

Lemma 33. *$GL(n,\mathbf{R})$ deformation retracts onto $O(n)$.*

Proof. For $B \in M(n,\mathbf{R})$ let B_i denote the ith column of B. Furthermore, let $\langle -,- \rangle\colon \mathbf{R}^n \to \mathbf{R}$ be the usual inner product on $\mathbf{R}^n$. Recall that the projection of a vector u onto a vector v is the vector

$$\mathrm{proj}_v(u) = \frac{\langle u, v \rangle}{\langle v, v \rangle} v,$$

and projection is a linear map. Furthermore, recall that the Gram-Schmidt orthogonalization (GS) process applied to the columns of an invertible matrix gives an orthogonal matrix. In other words, GS is a map $GL(n,\mathbf{R}) \to O(n)$. Concretely, applying GS to A yields the matrix E with columns $E_i = B_i/|B_i|$, where B_i are defined inductively by

$$B_i = \begin{cases} A_1, & i = 1 \\ A_i - \sum_{j=1}^{i-1} \mathrm{proj}_{B_j}(A_i), & i = 2, \ldots, n. \end{cases}$$

Note that $\mathrm{GS}(A) = A$ if $A \in O(n)$. By construction, GS is continuous, that is, a retraction.

Define $H\colon GL(n,\mathbf{R}) \times I \to GL(n,\mathbf{R})$ by $H(A,t) = (1-t)A + t\,\mathrm{GS}(A)$. It is straightforward to verify that H is continuous, e.g. use A.14.

Note that $\det H(A,t) = 0$ iff $H(A,t)$ has a zero column iff $|H(A,t)_i| = 0$ for some $1 \le i \le n$. Since $|A_i| > 0$ and $|\mathrm{GS}(A)_i| = 1$,

$$\begin{aligned}
|H(A,t)_i|^2 &= |(1-t)A_i + t\,\mathrm{GS}(A)_i|^2 \\
&= \langle (1-t)A_i + t\,\mathrm{GS}(A)_i, (1-t)A_i + t\,\mathrm{GS}(A)_i \rangle \\
&= \langle (1-t)A_i, (1-t)A_i \rangle + \langle t\,\mathrm{GS}(A)_i, t\,\mathrm{GS}(A)_i \rangle + 2\langle (1-t)A_i, t\,\mathrm{GS}(A)_i \rangle \\
&= (1-t)^2|A_i|^2 + t^2|\mathrm{GS}(A)_i|^2 + 2(1-t)t\langle A_i, \mathrm{GS}(A)_i \rangle > 0
\end{aligned}$$

for all $t \in \mathbf{R}$. Hence, $H(A,t) \in GL(n,\mathbf{R})$, i.e. H is well-defined. □

Recall that $SL(n,\mathbf{R}) = \mathrm{Ker}\,(\det\colon GL(n,\mathbf{R}) \to \mathbf{R}^\times)$ and $SO(n) = \mathrm{Ker}\,(\det\colon O(n) \to \mathbf{R}^\times)$. In particular, $GL(n,\mathbf{R})/SL(n,\mathbf{R}) \cong \mathbf{R}^\times$. Careful use of the Gram-Schmidt orthogonalization process will show that $SL(n,\mathbf{R})$ deformation retracts onto $SO(n)$.

The orthogonal group is compact: Define $O_{i,j}\colon GL(n,\mathbf{R}) \to \mathbf{R}$ by $O_{i,j}(A) = \langle A_i, A_j \rangle$. Clearly, $O_{i,j}$ is continuous. Now,

$$O(n) = \bigcap_{1 \le i < j \le n} O_{i,j}^{-1}(0) \cap \bigcap_{1 \le i \le n} O_{i,i}^{-1}(1)$$

is an intersection of closed set, i.e., closed. If $A \in O(n)$, then

$$|A|^2 = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{i,j}^2 = \sum_{j=1}^{n} |A_i|^2 = n.$$

Thus, $O(n)$ is also bounded, hence compact.

Exercise 2.2.7. Let $f\colon \mathbf{R}^n \to \mathbf{R}^n$ be an invertible linear transformation. Fix a basis for $\mathbf{R}^n$, e.g. the standard basis, and let $A \in GL(n, \mathbf{R})$ be the matrix of f. Write A instead of f. In particular, A is a map $(\mathbf{R}^n, \mathbf{R}^n - \{0\}) \to (\mathbf{R}^n, \mathbf{R}^n - \{0\})$.

Recall that $\mathbf{R}^n - \{0\}$ deformation retracts on S^{n-1}, c.f. exercise 0.2. Let $r\colon \mathbf{R}^n - \{0\} \to S^{n-1}$, $r(x) = x/|x|$, be the usual retraction, and let $i\colon S^{n-1} \to \mathbf{R}^n - \{0\}$ be the inclusion map. In particular, $ri = \mathrm{id}$ and $ir \simeq \mathrm{id}$, i.e., $r_*i_* = \mathrm{id}$ and $i_*r_* = \mathrm{id}$.

By naturality of the long exact sequence of reduced homology groups of the pair $(\mathbf{R}^n, \mathbf{R}^n - \{0\})$, the diagram

$$\begin{array}{ccccccc}
 & & & & \widetilde{H}_{n-1}(S^{n-1}) & & \\
 & & & & \cong \Big\downarrow i_* & & \\
0 = H_n(\mathbf{R}^n) & \longrightarrow & H_n(\mathbf{R}^n, \mathbf{R}^n - \{0\}) & \xrightarrow[\cong]{\partial} & \widetilde{H}_{n-1}(\mathbf{R}^n - \{0\}) & \longrightarrow & \widetilde{H}_{n-1}(\mathbf{R}^n) = 0 \\
 & & \Big\downarrow A_* & & \Big\downarrow A_* & & \\
0 = H_n(\mathbf{R}^n) & \longrightarrow & H_n(\mathbf{R}^n, \mathbf{R}^n - \{0\}) & \xrightarrow[\cong]{\partial} & \widetilde{H}_{n-1}(\mathbf{R}^n - \{0\}) & \longrightarrow & \widetilde{H}_{n-1}(\mathbf{R}^n) = 0 \\
 & & & & \cong \Big\downarrow r_* & & \\
 & & & & \widetilde{H}_{n-1}(S^{n-1}) & &
\end{array}$$

commutes, and has exact rows, that is, the connecting homomorphisms are isomorphisms. Let $1 \in H_n(\mathbf{R}^n, \mathbf{R}^n - \{0\})$ be a generator, i.e., $1' = r_*(\partial(1)) \in H_{n-1}(S^{n-1})$ is a generator.

By lemma 30, there is a path $\gamma\colon I \to GL(n, \mathbf{R})$ from A to Æ$_1(\mathrm{sign}(\det A))$. Now, $F\colon \mathbf{R}^n \times I \to \mathbf{R}^n$ defined by $F(x,t) = \gamma(t)x$ is continuous by A.14, that is, a homotopy from A to $B =$ Æ$_1(\mathrm{sign}(\det A))$.

Consider S^{n-1} as the vectors of unit length in $\mathbf{R}^n$. Since $B \in O(n)$, the restriction $B|S^{n-1}$ is a map $S^{n-1} \to S^{n-1}$. If $\det A > 0$ then $B = I_n$, and if $\det A < 0$ then $B(x_1, \ldots, x_n)^t = (-x_1, \ldots, x_n)^t$, i.e., a reflection. Hence, $r_*B_*i_* = r_*A_*i_*$ is multiplication by $\mathrm{sign}(\det A)$. Since

$$\begin{aligned}
r_*A_*i_*(1') = \mathrm{sign}(\det A)1' &\Leftrightarrow r_*A_*i_*(r_*(\partial(1))) = \mathrm{sign}(\det A)r_*(\partial(1)) \\
&\Leftrightarrow r_*A_*(\partial(1)) = \mathrm{sign}(\det A)r_*(\partial(1)) \\
&\Leftrightarrow i_*r_*A_*(\partial(1)) = \mathrm{sign}(\det A)i_*r_*(\partial(1)) \\
&\Leftrightarrow A_*(\partial(1)) = \mathrm{sign}(\det A)\partial(1),
\end{aligned}$$

it follows that $A_*\colon H_n(\mathbf{R}^n, \mathbf{R}^n - \{0\}) \to H_n(\mathbf{R}^n, \mathbf{R}^n - \{0\})$ is also multiplication by $\mathrm{sign}(\det A)$.

Ex. 2.2.9. All boundary maps are computed using the cellular boundary formula.

(a). Let X be S^2 with the north and south poles identified to a point.

(1) By 0.8, the space X is homotopy equivalent to $S^2 \vee S^1$.

(2) A CW complex structure consists of one 0-cell x, one 1-cell a, and one 2-cell U. The attachment map is aa^{-1}, which is nullhomotopic. The cellular chain complex is

$$0 \longrightarrow \mathbf{Z}\{U\} \xrightarrow{\partial_2} \mathbf{Z}\{a\} \xrightarrow{\partial_1} \mathbf{Z}\{x\} \longrightarrow 0\,,$$

where $\partial_2 = \partial_1 = 0$. Hence,

$$H_n(X) = \begin{cases} \mathbf{Z}, & n = 0, 1, 2 \\ 0, & n \geq 3, \end{cases}$$

(b). A CW complex structure of S^1 consists of one 0-cell x and one 1-cell a with obvious attachment map. A CW complex structure of $S^1 \vee S^1$ consists of one 0-cell y, two 1-cells b, c with obvious attachment maps.

Using A.6, a CW complex of $S^1 \times (S^1 \vee S^1)$ consists of one 1-cell $x \times y$, three 1-cells $x \times b$, $x \times c$, $a \times y$, and two 2-cells $a \times b$, $a \times c$. The attachment map of the 2-cell $a \times b$ is $[a \times y, x \times b]$, and $[a \times y, x \times c]$ for $a \times c$. The CW complex structure is illustrated in 12

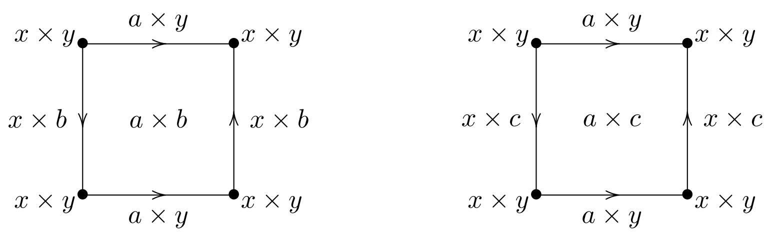


FIGURE 12. The product CW complex structure on $S^1 \times (S^1 \vee S^1)$.

The cellular chain complex is

$$0 \longrightarrow \mathbf{Z}\{a \times b, a \times c\} \xrightarrow{\partial_2=0} \mathbf{Z}\{x \times b, x \times c, a \times y\} \xrightarrow{\partial_1=0} \mathbf{Z}\{x \times y\} \longrightarrow 0.$$

Thus,

$$H_n(S^1 \times (S^1 \vee S^1)) = \begin{cases} \mathbf{Z}, & n = 0 \\ \mathbf{Z}^3, & n = 1 \\ \mathbf{Z}^2, & n = 2 \\ 0, & n \geq 3. \end{cases}$$

(c). Let Y be space under consideration. A CW complex structure consists of one 0-cell x, three 1-cells a, b, c, and one 2-cell U, see figure 13. The attachment map of the 2-cell is $aba^{-1}b^{-1}ca^{-1}c^{-1}$.

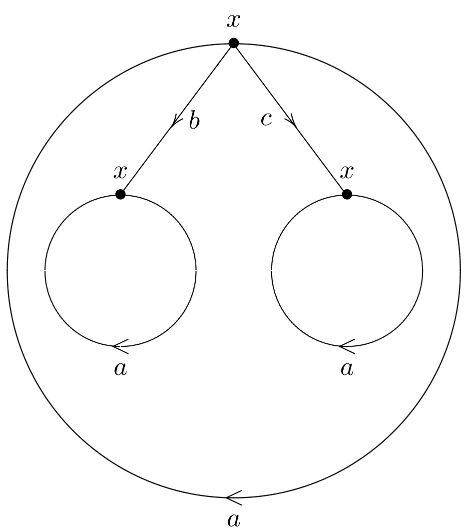


FIGURE 13. A CW complex structure on Y.

The cellular chain complex is

$$0 \longrightarrow \mathbf{Z}\{U\} \xrightarrow{\partial_2} \mathbf{Z}\{a, b, c\} \xrightarrow{\partial_1=0} \mathbf{Z}\{x\} \longrightarrow 0,$$

and

$$\partial_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.$$

Thus,

$$H_n(Y) = \begin{cases} \mathbf{Z}, & n = 0 \\ \mathbf{Z}^2, & n = 1 \\ 0, & n \geq 2. \end{cases}$$

(d). Let Z be the space under consideration. A CW complex structure consists of one 0-cell x, two 1-cells a, b, and one 2-cell U. See figure 14, where there are n horizontal repetitions of a, and m vertical repetitions of b. The attachment map is $a^n b^m a^{-n} b^{-m}$.

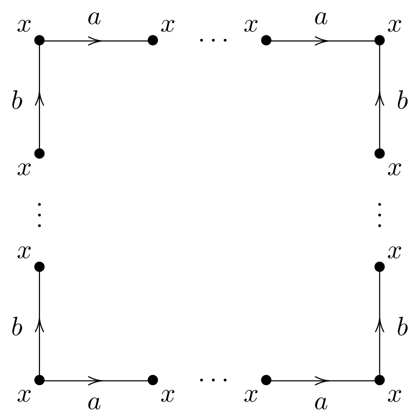


FIGURE 14. A CW complex structure on Z.

The cellular chain complex is

$$0 \longrightarrow \mathbf{Z}\{U\} \xrightarrow{\partial_2=0} \mathbf{Z}\{a,b\} \xrightarrow{\partial_1=0} \mathbf{Z}\{x\} \longrightarrow 0.$$

Thus,

$$H_n(Z) = \begin{cases} \mathbf{Z}, & n=0 \\ \mathbf{Z}^2, & n=1 \\ \mathbf{Z}, & n=2 \\ 0, & n \geq 3. \end{cases}$$

Ex. 2.2.13. Let $2, 3\colon S^1 \to S^1$ denote the attachment maps of degree 2 and 3, respectively, of the 2-cells e_1^2 and e_2^2. Let $S^1 = e^0 \cup e^1$ be the usual CW complex structure consisting of one 0-cell e^0, and one 1-cell e^1.

(a). Since $X = S^1 \cup_2 e_1^2 \cup_3 e_2^2$, the subcomplexes are the e^0, S^1, $S^1 \cup_2 e_1^2$, $S^1 \cup_3 e_2^2$ and X. Recall that $\widetilde{H}_n(e^0) = 0$ for all n, and

$$\widetilde{H}_n(S^1) = \begin{cases} \mathbf{Z}, & n=1 \\ 0, & n \neq 1. \end{cases}$$

The homology of $S^1 \cup_2 e_1^2$: The cellular chain complex is

$$0 \longrightarrow \mathbf{Z}\{e_1^2\} \xrightarrow{d_2} \mathbf{Z}\{e^1\} \xrightarrow{d_1=0} \mathbf{Z}\{e^0\} \longrightarrow 0.$$

Since ∂_2 is multiplication by 2,

$$\widetilde{H}_n(S^1 \cup_2 e_1^2) = \begin{cases} \mathbf{Z}/2\mathbf{Z}, & n=1 \\ 0, & n \neq 1. \end{cases}$$

The homology of $S^1 \cup_3 e_2^2$: The cellular chain complex is

$$0 \longrightarrow \mathbf{Z}\{e_2^2\} \xrightarrow{d_2} \mathbf{Z}\{e^1\} \xrightarrow{d_1=0} \mathbf{Z}\{e^0\} \longrightarrow 0.$$

Since ∂_2 is multiplication by 3,

$$\widetilde{H}_n(S^1 \cup_3 e_2^2) = \begin{cases} \mathbf{Z}/3\mathbf{Z}, & n=1 \\ 0, & n \neq 1. \end{cases}$$

The homology of X: The cellular chain complex is

$$0 \longrightarrow \mathbf{Z}\{e_1^2, e_2^2\} \xrightarrow{d_2} \mathbf{Z}\{e^1\} \xrightarrow{d_1=0} \mathbf{Z}\{e^0\} \longrightarrow 0,$$

where $d_2 = \begin{pmatrix} 2 & 3 \end{pmatrix}$. Note that we write elements as columns. Thus,

$$\operatorname{Ker} d_2 = \mathbf{Z}\{-3e_1^2 + 2e_2^2\} = \mathbf{Z}\left\{\begin{pmatrix} -3 \\ 2 \end{pmatrix}\right\},$$

that is,

$$\widetilde{H}_n(X) = \begin{cases} \mathbf{Z}, & n = 2 \\ 0, & n \neq 2. \end{cases}$$

The quotient complexes are $X/e^0 = X$, $X/S^1 = S^2 \vee S^2$, $X/(S^1 \cup_2 e_1^2) = S^2$, and $X/(S^1 \cup_3 e_2^2) = S^2$, the homology of these spaces are well-known.

(b). Clearly, the quotient map $X \to X/e^0 = X$ is a homotopy equivalence.

The quotient map $X \to X/S^1 = S^2 \vee S^2$ is not a homotopy equivalence, since $H_2(X) = \mathbf{Z} \neq \mathbf{Z}^2 = H_2(S^2 \vee S^2)$.

Consider the quotient map $q\colon X \to X/(S^1 \cup_2 e_1^2) = e^0 \cup e_2^2 = S^2$. Since q is cellular, q induces a cellular chain map, see ex. 2.2.17. Since $q(X - e_2^2) = e^0$,

$$q_\# = q_*\colon H_2(X, S^1) = \widetilde{H}_2(X/S^1) = \mathbf{Z}\{e_1^2, e_2^2\} \to \mathbf{Z}\{e_2^2\} = \widetilde{H}_2(S^2) = H_2(S^2, e^0)$$

is given by the matrix $\begin{pmatrix} 0 & 1 \end{pmatrix}$. Summarizing, the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z}\{e_1^2, e_2^2\} & \xrightarrow{d_2} & \mathbf{Z}\{e^1\} & \xrightarrow{d_1=0} & \mathbf{Z}\{e^0\} & \longrightarrow & 0 \\ & & \downarrow q_\# & & \downarrow q_\# & & \downarrow q_\# & & \\ 0 & \longrightarrow & \mathbf{Z}\{e_2^2\} & \xrightarrow{d_2=0} & 0 & \xrightarrow{d_1=0} & \mathbf{Z}\{e^0\} & \longrightarrow & 0 \end{array}$$

commutes, where the left $q_\#$ is $\begin{pmatrix} 0 & 1 \end{pmatrix}$. Now, $q_*\colon H_2(X) \to H_2(S^2)$ induced by $q_\#$ satisfy

$$q_*(\left[\begin{pmatrix} -3 \\ 2 \end{pmatrix}\right]) = \left[\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} -3 \\ 2 \end{pmatrix}\right] = [2].$$

Hence, q_* is not an isomorphism, i.e., q is not a homotopy equivalence.

A similar argument shows that the quotient map $X \to X/(S^1 \cup_3 e_2^2) = S^2$ is not a homotopy equivalence.

The sphere S^2 and X are homotopy equivalent: Using 1.26, $\pi_1(S^1 \cup_2 e_1^2, e^0) = \langle e^1 \mid (e^1)^2 \rangle$, which is not surprising since $S^1 \cup_2 e_1^2 = \mathbf{R}\mathrm{P}^2$. The attachment map $3\colon S^1 \to S^1 \subset S^1 \cup_2 e_1^2$ is an element in $\pi_1(S^1 \cup_2 e_1^2)$ and $[3] = (e^1)^3 = e^1$. Thus, the attachment map is homotopic to the degree one attachment map $1\colon S^1 \to S^1 \subset S^1 \cup_2 e_1^2$.

Let $0\colon S^1 \to S^1$, $S^1 \mapsto e^0$, be the constant map. Using 0.18,

$$\begin{aligned} X &= S^1 \cup_2 e_1^2 \cup_3 e_2^2 \\ &\simeq S^1 \cup_2 e_1^2 \cup_1 e_2^2 \\ &= S^1 \cup_1 e_2^2 \cup_2 e_1^2 \\ &= D^2 \cup_2 e_1^2 \\ &\simeq D^2 \cup_0 e_1^2 \\ &= D^2 \vee S^2 \\ &\simeq S^2. \end{aligned}$$

Ex. 2.2.17. Let X and Y be CW complexes, and let $f\colon X \to Y$ be a cellular map, that is, $f(X^n) \subset Y^n$ for all n.

By naturality of singular homology, the diagram

$$\begin{array}{ccccc}
 & & \overset{d_n}{\frown} & & \\
H_n(X^n, X^{n-1}) & \xrightarrow{\partial_n} & H_{n-1}(X^{n-1}) & \xrightarrow{j_n} & H_{n-1}(X^{n-1}, X^{n-2}) \\
\big\downarrow f_* & & \big\downarrow f_* & & \big\downarrow f_* \\
H_n(Y^n, Y^{n-1}) & \xrightarrow{\partial_n} & H_{n-1}(Y^{n-1}) & \xrightarrow{j_n} & H_{n-1}(Y^{n-1}, Y^{n-2}) \\
 & & \underset{d_n}{\smile} & &
\end{array}$$

commutes, where f_* is the map induced on singular homology by f. Thus, f induces a cellular chain map $f_\#$ between the cellular chain complexes of X and Y, that is,

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H_{n+1}(X^n, X^{n-1}) & \xrightarrow{d_{n+1}} & H_n(X^n, X^{n-1}) & \xrightarrow{d_n} & H_{n-1}(X^{n-1}, X^{n-2}) & \longrightarrow & \cdots \\
 & & \big\downarrow f_\# & & \big\downarrow f_\# & & \big\downarrow f_\# & & \\
\cdots & \longrightarrow & H_{n+1}(Y^n, Y^{n-1}) & \xrightarrow{d_{n+1}} & H_n(Y^n, Y^{n-1}) & \xrightarrow{d_n} & H_{n-1}(Y^{n-1}, Y^{n-2}) & \longrightarrow & \cdots
\end{array}$$

commutes. Thus, f induces a map on cellular homology which we denote f_*^{CW}.

Using 2.34 and the long exact sequence of the pair (X^{n+1}, X^n), there is a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \big\downarrow & & \\
0 & \longrightarrow & \operatorname{Im}\partial_{n+1} & \longrightarrow & H_n(X^n) & \xrightarrow{i^n} & H_n(X^{n+1}) & \longrightarrow & 0 \\
 & & & & & & \big\downarrow i_{n+1} & & \\
 & & & & & & H_n(X) & & \\
 & & & & & & \big\downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

with exact row and column, where $i^n \colon X^n \to X^{n+1}$ and $i_{n+1} \colon X^{n+1} \to X$ are the inclusion maps. Since $i_{n+1}i^n$ is the inclusion $i_n \colon X^n \to X$, there is a short exact sequence

$$0 \longrightarrow \operatorname{Im}\partial_{n+1} \longrightarrow H_n(X^n) \xrightarrow{i_n} H_n(X) \longrightarrow 0.$$

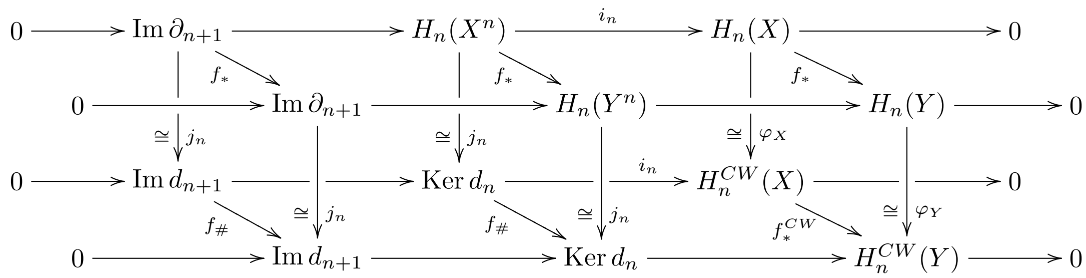
By the definition of cellular homology, there is a short exact sequence

$$0 \longrightarrow \operatorname{Im} d_{n+1} \longrightarrow \operatorname{Ker} d_n \longrightarrow H_n^{CW}(X) \longrightarrow 0.$$

By the proof of 2.35, the isomorphism $\varphi_X \colon H_n(X) \to H_n^{CW}(X)$ between singular and cellular homology is induced by j_n, that is, it fits into the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Im}\partial_{n+1} & \longrightarrow & H_n(X^n) & \xrightarrow{i_n} & H_n(X) & \longrightarrow & 0 \\
 & & \cong \big\downarrow j_n & & \cong \big\downarrow j_n & & \cong \big\downarrow \varphi_X & & \\
0 & \longrightarrow & \operatorname{Im} d_{n+1} & \longrightarrow & \operatorname{Ker} d_n & \longrightarrow & H_n^{CW}(X) & \longrightarrow & 0
\end{array}$$

with exact rows. Consider the diagram



We already know that the front and the back of the diagram commutes. By naturality of singular homology, the left cube, the top, and the bottom of the diagram also commutes. By the construction of the isomorphism between singular and cellular homology, the right cube must also commute, that is,

$$\begin{array}{ccc} H_n(X) & \xrightarrow{f_*} & H_n(Y) \\ \cong \downarrow \varphi_Y & & \cong \downarrow \varphi_Y \\ H_n^{CW}(X) & \xrightarrow{f_*^{CW}} & H_n^{CW} \end{array}$$

commutes. In other words, the isomorphism between singular and cellular homology is natural.

Ex. 2.2.20. For a finite CW complex X let $c_n(X)$ denote the number of n-cells. Recall that the n-cells in $X \times Y$ are the products of an i-cell of X and an j-cell of Y with $i + j = n$, c.f. A.6. Now,

$$\begin{aligned} \chi(X \times Y) &= \sum_n (-1)^n c_n(X \times Y) \\ &= \sum_n \sum_{i+j=n} (-1)^{i+j} c_i(X) c_j(Y) \\ &= \sum_i (-1)^i c_i(X) \sum_j (-1)^j c_j(Y) \\ &= \chi(X)\chi(Y). \end{aligned}$$

Ex. 2.2.21. If X is the union of subcomplexes A and B, then $A \cap B$ is a subcomplex of X consisting of the cells of X that are cells in both A and B. Clearly,

$$c_n(X) = c_n(A) + c_n(B) - c_n(A \cap B).$$

Hence, $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$.

Ex. 2.2.22. Let $p\colon \widetilde{X} \to X$ be an n-sheeted, $n < \infty$, covering space of a finite CW complex X with c_d cells in dimension d. Then, by corollary 41, $\widetilde{X}$ is a finite CW complex with nc_d cells in dimension d. Thus, $\chi(\widetilde{X}) = \sum_d (-1)^d n c_d = n\chi(X)$.

Ex. 2.2.23. Note that, since M_g is compact, any covering space $M_g \to M_h$ is finite sheeted. Let $M_g \to M_h$ be an n-sheeted covering space. Then

$$2 - 2g = \chi(M_g) = n\chi(M_h) = n(2 - 2h),$$

by ex. 2.2.22, which implies that $g = n(h - 1) + 1$. Note that the converse statement also hold, c.f. 1.41.

Ex. 2.2.32. The suspension SX is the union $CX \cup CX$ of two cones on X with $CX \cap CX = X$. The reduced Mayer-Vietoris sequence gives exact sequences

$$0 = \widetilde{H}_n(CX) \oplus \widetilde{H}_n(CX) \longrightarrow \widetilde{H}_n(SX) \longrightarrow \widetilde{H}_{n-1}(X) \longrightarrow \widetilde{H}_{n-1}(CX) \oplus \widetilde{H}_{n-1}(CX) = 0$$

for all n, that is, $\widetilde{H}_n(SX) \cong \widetilde{H}_{n-1}(X)$ for all n.

Ex. 2.2.36. Let $\{x_0, x_1\} = S^0 \subset S^1 \subset \cdots \subset S^{n-1} \subset S^n$.

Lemma 34. *If X retracts onto a subspace A, then the homomorphism $i_*\colon H_n(A) \to H_n(X)$ induced by the inclusion is injective for all n. If X deformation retracts onto A, then i_* is an isomorphism for all n.*

Proof. The first part is ex. 2.1.11.

For the second part, suppose X deformation retracts onto A, that is, id_X is homotopic relative to A to a retraction $r\colon X \to A \subset X$. Since $ri = \mathrm{id}_X$ and $ir \simeq \mathrm{id}_X$, the inclusion $i\colon A \to X$ is a homotopy equivalence. In particular, $i_*\colon H_n(A) \to H_n(X)$ is an isomorphism for all n. □

Lemma 35. $H_i(X \times S^n) = H_i(X) \oplus H_i(X \times S^n, X \times \{x_0\})$.

Proof. Let $A = X \times D^{n+1} \simeq X \times \{x_0\} = X$, $x_0 \in \partial D^{n+1} = S^n$, and let $B = X \times S^n$. Furthermore, let $C = \{x_1\} \times \{x_0\} \subset A$, $x_1 \in X$, and let $D = X \times \{x_0\} \subset B$. We will use the relative Mayer-Vietoris of the pair $(A \cup B, C \cup D) = (X \times D^{n+1}, X \times \{x_0\})$, where $(A, C) = (X \times D^{n+1}, \{x_1\} \times \{x_0\})$, $(B, D) = (X \times S^n, X \times \{x_0\})$ and $(A \cap B, C \cap D) = (X \times D^{n+1}, \{x_1\} \times \{x_0\})$.

Since $X \times D^{n+1}$ deformation retracts onto $X \times \{x_0\}$, the inclusion $X \times \{x_0\} \to X \times D^{n+1}$ induces isomorphisms on homology groups. In particular, $H_i(A \cup B, C \cup D) = H_i(X \times D^{n+1}, X \times \{x_0\}) = 0$ for all i. Thus, the relative Mayer-Vietoris sequence, 2.17 and 2.18 gives that

$$\begin{aligned}
\widetilde{H}_i(X \times S^n) &= H_i(X \times S^n, \{x_1\} \times \{x_0\}) \\
&= H_i(A \cap B, C \cap D) \\
&= H_i(A, C) \oplus H_i(B, D) \\
&= H_i(X \times D^{n+1}, \{x_1\} \times \{x_0\}) \oplus H_i(X \times S^n, X \times \{x_0\}) \\
&= \widetilde{H}_i(X) \oplus H_i(X \times S^n, X \times \{x_0\}).
\end{aligned}$$

for all i. Hence, $H_i(X \times S^n) = H_i(X) \oplus H_i(X \times S^n, X \times \{x_0\})$ for all i. □

Lemma 36. $H_i(X \times S^n, X \times \{x_0\}) = H_{i-1}(X \times S^{n-1}, X \times \{x_0\})$.

Proof. Write $S^n = D^n_u \cup D^n_l$ as the union of the upper and lower hemispheres, and let $S^{n-1} \subset S^n$ denote the equator.

Let $A = X \times D^n_u \simeq X \times \{x_0\}$, $x_0 \in S^{n-1}$, and let $B = X \times D^n_l$. Furthermore, let $C = D = X \times \{x_0\}$.

We will use the relative Mayer-Vietoris of the pair $(A \cup B, C \cup D) = (X \times S^n, X \times \{x_0\})$, where $(A, C) = (X \times D^n_u, X \times \{x_0\})$, $(B, D) = (X \times D^n_l, X \times \{x_0\})$ and $(A \cap B, C \cap D) = (X \times S^{n-1}, X \times \{x_0\})$.

As in the proof of lemma 35, $H_i(A, C) = H_i(B, D) = H_i(X \times D^n, X \times \{x_0\}) = 0$ for all i. Thus, the relative Mayer-Vietoris sequence gives that

$$H_i(X \times S^n, X \times \{x_0\}) = H_i(A \cup B, C \cup D) = H_{i-i}(A \cap B, C \cap D) = H_{i-1}(X \times S^{n-1}, X \times \{x_0\}).$$

□

Now, using the lemmas and induction,

$$\begin{aligned}
H_i(X \times S^n) &= H_i(X) \oplus H_i(X \times S^n, X \times \{x_0\}) \\
&= H_i(X) \oplus H_{i-1}(X \times S^{n-1}, X \times \{x_0\}) \\
&= \cdots \\
&= H_i(X) \oplus H_{i-n}(X \times S^0, X \times \{x_0\}) \\
&= H_i(X) \oplus H_{i-n}(X).
\end{aligned}$$

The latter isomorphism follows since $C_*(X \times S^0) = C_*(X \times \{x_0\}) \oplus C_*(X \times \{x_1\})$, that is, $C_*(X \times S^0, X \times \{x_0\}) = C_*(X \times \{x_1\}) = C_*(X)$.

Ex. 2.B.10. Recall that S^∞ is contractible, c.f. 1B.3. Hence, $H_n(S^\infty; \mathbf{Z}_2) = 0$ for $n > 0$ and $H_0(S^\infty; \mathbf{Z}_2) = \mathbf{Z}_2$. The transfer sequence for the universal covering space $p\colon S^\infty \to \mathbf{R}\mathrm{P}^\infty$ gives that the connecting homomorphism $\partial\colon H_{n+1}(\mathbf{R}\mathrm{P}^\infty; \mathbf{Z}_2) \to H_n(\mathbf{R}\mathrm{P}^\infty; \mathbf{Z}_2)$ is an isomorphism for $n > 0$. Consider the exact sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow & H_1(\mathbf{R}\mathrm{P}^\infty; \mathbf{Z}_2) & \xrightarrow{\partial} & H_0(\mathbf{R}\mathrm{P}^\infty; \mathbf{Z}_2) & \xrightarrow{\tau_*} & H_0(S^\infty; \mathbf{Z}_2) & \xrightarrow{p_*} & H_0(\mathbf{R}\mathrm{P}^\infty; \mathbf{Z}_2) & \longrightarrow 0 \\ & & & \| & & \| & & \| & \\ & & & \mathbf{Z}_2 & & \mathbf{Z}_2 & & \mathbf{Z}_2 & \end{array}$$

from the transfer sequence. Since p_* is surjective, p_* is an isomorphism. Hence, $\tau_* = 0$, i.e., ∂ is surjective. Since ∂ is also injective, $\partial\colon H_1(\mathbf{R}\mathrm{P}^\infty; \mathbf{Z}_2) \to H_0(\mathbf{R}\mathrm{P}^\infty; \mathbf{Z}_2)$ is an isomorphism. Thus, $H_n(\mathbf{R}\mathrm{P}^\infty; \mathbf{Z}_2) = \mathbf{Z}_2$ for all n.

Ex. 2.C.2.

Lemma 37. *If X is path connected and $f\colon X \to X$ is a map, then $f_*\colon H_0(X) \to H_0(X)$ is the identity map.*

Proof. Let $\sigma\colon \Delta^0 \to X$ be a 0-simplex, and let $\gamma\colon \Delta^1 \to X$ be a path from σ to $f\sigma$. Then $\partial_1(\gamma) = f\sigma - \sigma$, where $\partial_1 : C_1(X) \to C_0(X)$ is the singular boundary map. Hence, $[f\sigma] = [\sigma]$ in $H_0(X)$, that is, f_* is the identity map. □

Lemma 38. *Let $f\colon S^n \to S^n$, $n > 0$. Then $\Lambda(f) = 1 + (-1)^n \deg f$.*

Proof. Using lemma 37,

$$\begin{aligned} \Lambda(f) &= \sum_m (-1)^m \operatorname{tr}(f_*\colon H_m(S^n) \to H_m(S^n)) \\ &= \operatorname{tr}(f_*\colon H_0(S^n) \to H_0(S^n)) + (-1)^n \operatorname{tr}(f_*\colon H_n(S^n) \to H_n(S^n)) \\ &= 1 + (-1)^n \deg f. \end{aligned}$$

□

Now consider a map $f\colon S^n \to S^n$. By Lefschetz fixed point theorem and lemma 38, f has a fixed point unless $\Lambda(f) = 1 + (-1)^n \deg f = 0$, i.e. unless $\deg f = (-1)^{n+1} = \deg(x \mapsto -x)$.

Ex. 2.C.8. Suppose X is homotopy equivalent to a finite simplicial complex $K = (V_K, S_K)$ and Y is homotopy equivalent to a countable simplicial complex $L = (V_L, S_L)$. Let $\varphi_X : X \to K$ be a homotopy equivalence with homotopy inverse $\psi_X : K \to X$, and let $\varphi_Y : Y \to L$ be a homotopy equivalence with homotopy inverse $\psi_Y : K \to Y$.

$|[X, Y]| = |[K, L]|$: Define $\Lambda\colon [X, Y] \to [K, L]$ by $\Lambda([f]) = \varphi_Y f \psi_X$ for $[f] \in [X, Y]$, and $\lambda\colon [K, L] \to [X, Y]$ by $\lambda([f]) = \psi_Y f \varphi_X$ for $[f] \in [K, L]$. Now,

$$\Lambda\lambda([f]) = [\varphi_Y \psi_Y f \varphi_X \psi_X] = [\mathrm{id}_L\, f\, \mathrm{id}_K] = [f]$$

and

$$\lambda\Lambda([f]) = [\psi_Y \varphi_Y f \psi_X \varphi_X] = [\mathrm{id}_Y\, f\, \mathrm{id}_X] = [f],$$

i.e., $\Lambda\lambda = \mathrm{id}_{[K,L]}$ and $\lambda\Lambda = \mathrm{id}_{[X,Y]}$. Hence, $|[X, Y]| = |[K, L]|$.

By the simplicial approximation theorem,

$$|[X, Y]| = |[K, L]| \leq |\cup_{n \in \mathbf{N}} V_L^{V_{\mathrm{sd}^n K}}| \leq \aleph_0,$$

since a countable union of countable sets is countable (assuming, as always, the Axiom of Choice), and for each n the set $V_{\mathrm{sd}^n K}$ is finite, i.e., $V_L^{V_{\mathrm{sd}^n K}}$ is countable.

Ex. 2.C.9. By 2C.5, a finite CW complex is homotopy equivalent to a finite simplicial complex. Thus, it suffices to prove there are only countably many finite simplicial complexes. Since

$$|\cup_{n \in \mathbf{N}} \mathcal{P}(\mathcal{P}(\{1, \ldots, n\}))| = |\cup_{n \in \mathbf{N}} \{0, 1\}^{\{0,1\}^{\{1,\ldots,n\}}}| = \aleph_0,$$

there are only countably many finite simplicial complexes.

Appendix

Ex. 1. First a little lemma.

Lemma 39. *A covering map is an open map.*

Proof. Let $p\colon \widetilde{X} \to X$ be a covering map, and let $\widetilde{U}$ be an open set of $\widetilde{X}$. For $x \in p(\widetilde{U})$ choose a neighborhood U of x evenly covered by p, that is, $p^{-1}(U)$ is a set of disjoint open sets $\{\widetilde{U}_\alpha\}$ in $\widetilde{X}$ such that each $\widetilde{U}_\alpha$ is mapped homeomorphically onto V by p.

Choose $\widetilde{x} \in \widetilde{U}$ with $p(\widetilde{x}) = x$, and let $\widetilde{U}_\beta$ be the open set in $\{\widetilde{U}_\alpha\}$ containing $\widetilde{x}$. Now, $\widetilde{U} \cap \widetilde{U}_\beta$ is open in $\widetilde{U}_\beta$. Since the restriction of p to $\widetilde{U}_\beta$ is a homeomorphism onto U, $p(\widetilde{U} \cap \widetilde{U}_\beta)$ is open in U. Since U is open in X, $p(\widetilde{U} \cap \widetilde{U}_\beta)$ is also open in X, that is, it is an open neighborhood of x contained in $p(\widetilde{U})$. □

Theorem 40. *Let $p\colon \widetilde{X} \to X$ be a covering space. If X is a CW complex, then $\widetilde{X}$ is a CW complex with n-skeleton $\widetilde{X}^n = p^{-1}(X^n)$, and p maps open cells homeomorphically onto open cells.*

Proof. We have a CW complex structure on X, that is, a sequence

$$X^0 \subset X^1 \subset \cdots \subset X^{n-1} \subset X^n \subset \cdots \subset X$$

of subspaces of X such that $X = \bigcup_n X^n$, and X has the weak topology with respect to the collection $\{X^n\}$, and X^n is homeomorphic to an n-cellular extension of X^{n-1} for $n > 0$, that is,

$$X^n = X^{n-1} \cup_\varphi \coprod_\alpha D^n,$$

where $\varphi = \coprod_\alpha(\varphi_\alpha\colon \partial D^n \to X^{n-1})$ is the attaching map and $\Phi\colon \coprod_\alpha(\Phi_\alpha\colon D^n \to X)$ is the characteristic map. As usual, let e^n_α denote the (open) n-cell corresponding to Φ_α.

Note that $\widetilde{X}^0 = p^{-1}(X^0)$ is a discrete set of points, and

$$p^{-1}(X^0) \subset p^{-1}(X^1) \subset \cdots \subset p^{-1}(X^{n-1}) \subset p^{-1}(X^n) \subset \cdots \subset \widetilde{X}$$

with

$$\widetilde{X} = p^{-1}(X) = p^{-1}(\bigcup_n X^n) = \bigcup_n p^{-1}(X^n) = \bigcup_n \widetilde{X}^n.$$

To prove that $\widetilde{X}$ is a CW complex, it remains to prove that $\widetilde{X}^n$ is homeomorphic to an n-cellular extension of $\widetilde{X}^{n-1}$ for $n > 0$, and that the topology on $\widetilde{X}$ is the weak topology with respect to the collection $\{\widetilde{X}^n\}$.

$\widetilde{X}^n$ is homeomorphic to an n-cellular extension of $\widetilde{X}^{n-1}$: Let $x_0 \in D^n - \partial D^n$, and let $x_\alpha = \Phi_\alpha(x_0)$. For each $\widetilde{x}_{\alpha,\beta} \in p^{-1}(\widetilde{x}_\alpha)$ let $\widetilde{\Phi}_{\alpha,\beta}\colon D^n \to \widetilde{X}$ be the lift of Φ_α, i.e., $\Phi_\alpha = p \circ \widetilde{\Phi}_{\alpha,\beta}$, with $\widetilde{\Phi}_{\alpha,\beta}(x_0) = \widetilde{x}_{\alpha,\beta}$. These lifts exists since D^n is contractible, by the lifting criterion. Since D^n is connected, there is exactly one lift for each point in $p^{-1}(x_\alpha)$, by the unique lifting property. Note that $\widetilde{\Phi}_{\alpha,\beta}$ restricts to a map $\widetilde{\varphi}_{\alpha,\beta}\colon \partial D^n \to \widetilde{X}^{n-1}$.

Since Φ_α maps $D^n - \partial D^n$ homeomorphically onto e^n_α and $p \circ \widetilde{\Phi}_{\alpha,\beta} = \Phi_\alpha$, p maps $\widetilde{\Phi}_{\alpha,\beta}(D^n - \partial D^n)$ bijectively onto e^n_α. By lemma 39, p is also open, that is, p maps $\widetilde{\Phi}_{\alpha,\beta}(D^n - \partial D^n)$ homeomorphically onto e^n_α. Thus, $\widetilde{\Phi}_{\alpha,\beta}$ maps $D^n - \partial D^n$ homeomorphically onto its image.

Now, the n-cellular extension

$$\widetilde{X}^{n-1} \cup_{\widetilde{\varphi}} \coprod_{\alpha,\beta} D^n$$

where $\widetilde{\varphi} = \coprod_{\alpha,\beta}(\widetilde{\varphi}_{\alpha,\beta}\colon \coprod_{\alpha,\beta} \partial D^n \to \widetilde{X}^{n-1})$, has $\widetilde{\Phi} = \coprod_{\alpha,\beta}(\widetilde{\Phi}_{\alpha,\beta}\colon \coprod_{\alpha,\beta} D^n \to \widetilde{X}^{n-1})$ as characteristic map. Let $\tilde{e}^n_{\alpha,\beta}$ be the n-cell corresponding to $\widetilde{\Phi}_{\alpha,\beta}$.

Since X^n is the disjoint union (as a set) of X^{n-1} and the n-cells e^n_α, $\widetilde{X}^n$ is the disjoint union (as a set) of $\widetilde{X}^{n-1}$ and the n-cells $\tilde{e}^n_{\alpha,\beta}$. Now, it suffices to prove that the topology of $\widetilde{X}^n$ as a subspace of $\widetilde{X}$ corresponds to the quotient topology of the n-cellular extension, that is, a subset $\widetilde{U}$ of $\widetilde{X}^n$ is open iff $\widetilde{U} \cap \widetilde{X}^{n-1}$ is open in $\widetilde{X}^{n-1}$ as a subspace of $\widetilde{X}$, and each of the sets $\widetilde{\Phi}^{-1}_{\alpha,\beta}(\widetilde{U})$

are open: For the non-trivial direction, suppose $\widetilde{U} \cap \widetilde{X}^{n-1}$ is open and $\widetilde{\Phi}^{-1}_{\alpha,\beta}(\widetilde{U})$ is open for all α, β. By ex. 1.3.1, the restriction $p\colon \widetilde{X}^n \to X^n$ is a covering space. Hence, there is an open cover $\{U_\gamma\}$ of X^n such that each U_γ is evenly covered by p. Thus, we may assume that $\widetilde{U}$ is contained in an open set which is mapped homeomorphically to an open set of X^n by p. Hence, it suffices to prove that $U = p(\widetilde{U})$ is open in X^n. It is straightforward to verify that

$$p(\widetilde{U} \cap \widetilde{X}^{n-1}) = U \cap X^{n-1}$$

and

$$\Phi^{-1}_\alpha(U) = \bigcup_\beta \widetilde{\Phi}^{-1}_{\alpha,\beta}(\widetilde{U}).$$

Using the assumptions and lemma 39, $U \cap X^{n-1}$ is open, and $\Phi^{-1}_\alpha(U)$ are open for all α. Thus, U is open in X^n.

The topology on $\widetilde{X}$ is the weak topology with respect to the collection $\{\widetilde{X}^n\}$: Let $\widetilde{U}$ be a subset of $\widetilde{X}$ with $\widetilde{U} \cap \widetilde{X}^n$ open for all n. As above, we may assume $\widetilde{U}$ is contained in an open set of $\widetilde{X}$ which is mapped homeomorphically onto an open set of X by p. Again, it suffices to prove that $U = p(\widetilde{U})$ is open. It is easily verified that $p(\widetilde{U} \cap \widetilde{X}^n) = U \cap X^n$ for all n. Since p is open, $U \cap X^n$ open for all n, that is, U is open in X. □

The construction of the CW complex structure on a covering space in the proof gives the following result.

Corollary 41. *If $p\colon \widetilde{X} \to X$ is an n-sheeted, $n < \infty$, covering space of a finite CW complex X with c_d cells in dimension d, then $\widetilde{X}$ is a finite CW complex with nc_d d-cells.*

Actually, a covering space of a topological group is a topological group, and a covering space of an n-manifold is an n-manifold, the proofs are left to the reader.

Additional Exercises

Ex. 2.1.1. Consider the Δ-complex of S^1 with n 0-simplices $v_1, \dots, v_n$, and n 1-simplices $a_1, \dots, a_n$, where $a_i\colon \Delta^1 \to S^1$ are given by $a_i([v_1]) = v_{i+1}$ and $a_i([v_0]) = v_i$ ($v_{n+1} = v_1$) for $1 \le i \le n$.

The simplicial chain complex is

$$0 \longrightarrow \mathbf{Z}\{a_1, \dots, a_n\} \xrightarrow{\partial_1} \mathbf{Z}\{v_1, \dots, v_n\} \longrightarrow 0.$$

The nontrivial boundary map is $\partial_1(a_i) = v_{i+1} - v_i$. Hence, $[v_1] = \cdots = [v_n]$ in $H^\Delta_0(S^1) = \mathbf{Z}\{v_1, \dots, v_n\}/\operatorname{Im}\partial_1$.

Suppose $z_1a_1 + \cdots z_{n-1}a_{n-1} + z_na_n \in \operatorname{Ker}\partial_1$, that is,

$$\begin{aligned} 0 &= \partial_1(z_1a_1 + \cdots z_{n-1}a_{n-1} + z_na_n) \\ &= z_1(v_2 - v_1) + \cdots + z_{n-1}(v_n - v_{n-1}) + z_n(v_1 - v_n) \\ &= (z_n - z_1)v_1 + (z_1 - z_2)v_2 + \cdots + (z_{n-1} - z_n)v_n. \end{aligned}$$

Since $\{v_1, \dots, v_n\}$ is a basis, $z_n - z_1 = z_1 - z_2 = \cdots = z_{n-1} - z_n = 0$, that is, $z_1 = z_2 = \cdots = z_n$. Hence, $\operatorname{Ker}\partial_1 = \mathbf{Z}\{a_1 + \cdots + a_n\}$.

Summarizing, $H^\Delta_0(S^1) = H^\Delta_1(S^1) = \mathbf{Z}$ and zero otherwise.

Ex. 2.2.4. Consider a simplicial complex structure on a closed surface with v vertices, e edges and f faces, and with Euler characteristic χ. Since each face has 3 edges, and a closed surface is a compact 2-manifold without boundary, each edge in a face is an edge in exactly one other face, i.e., $e = 3f/2$. Substituting $e = 3f/2$ and $f = 2e/3$ into $\chi = v - e + f$ gives that $f = 2(v - \chi)$ and $e = 3(v - \chi)$. The maximal number of edges from each vertex is $v - 1$. Thus, the maximal number of edges is $v(v-1)/2$, i.e., $e \le v(v-1)/2$. Since $e = 3(v - \chi)$, $6(v - \chi) \le v^2 - v$.

Consider a simplicial complex structure on $S^1 \times S^1$ with v vertices, e edges and f faces. Recall that $\chi(S^1 \times S^1) = 0$. Thus, $6v \le v^2 - v$, i.e., $6 \le v - 1$, i.e., $v \ge 7$. Now, $f = 2v \ge 14$ and $e = 3v \ge 21$. Clearly, the diagram is a simplicial complex structure on the torus.

Consider a simplicial complex structure on $\mathbf{R}\mathrm{P}^2$ with v vertices, e edges and f faces. Recall that $\chi(\mathbf{R}\mathrm{P}^2) = 1$. Thus $6(v-1) \leq v^2 - v$, i.e. $v^2 - 7v + 6 \geq 0$. We require that $v > 1$, why $v \geq 6$. Hence, $f = 2(v-1) \geq 10$ and $e = 3(v-1) \geq 15$.

Recall that $\mathbf{R}\mathrm{P}^2$ is S^2 with antipodal points identified. The icosahedron is a simplicial complex structure on S^2 with 12 vertices, 30 edges and 20 faces. Identifying antipodal point on the icosahedron is a simplicial map. Thus, the icosahedron describes a triangulation of $\mathbf{R}\mathrm{P}^2$ with 6 vertices, 15 edges and 10 faces. The simplicial complex structure is illustrated in figure 15.

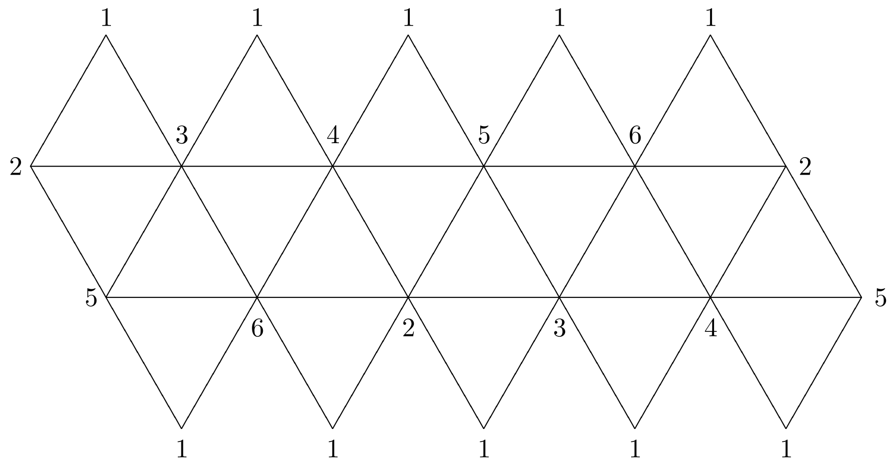


FIGURE 15. A triangulation of $\mathbf{R}\mathrm{P}^2$ using an icosahedron

Ex. 2.2.5. Assume $\mathbf{R}^n = X \times X$. Define

$$f\colon \mathbf{R}^n \times \mathbf{R}^n = X \times X \times X \times X \to X \times X \times X \times X = \mathbf{R}^n \times \mathbf{R}^n$$

by $f(x_1, x_2, x_3, x_4) = (x_2, x_3, x_4, x_1)$, that is, cyclic permutation. In particular, f is a homeomorphism, i.e., $\deg(f) = \pm 1$.

Observe that $f^2(x, y) = (y, x)$, $(x, y) \in \mathbf{R}^n \times \mathbf{R}^n$. Thus, f is an invertible linear map $\mathbf{R}^{2n} \to \mathbf{R}^{2n}$ with matrix

$$F = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

where I_n is the $n \times n$ identity matrix. Since F is obtained from I_{2n} by interchanging n rows, $\det(F) = (-1)^n$. By ex. 2.2.7, $\deg(f^2) = -1$ if n is odd, a contradiction.