

## ABELIANIZATION & EXERCISE 1.2.9

MORTEN POULSEN

Let  $G$  be a group, let  $A$  be a subset of  $G$ . We write group composition multiplicatively and denote the identity element by 1. Let  $\mathcal{H}$  be the set of subgroups of  $G$  containing  $A$ . Now the subgroup

$$\langle A \rangle = \bigcap_{H \in \mathcal{H}} H$$

of  $G$  is called the subgroup of  $G$  generated by  $A$ . By construction,  $\langle A \rangle$  is the smallest subgroup of  $G$  containing  $A$ . It is easily verified that any element of  $\langle A \rangle$  can be written as a product  $a_1 \cdots a_n$ ,  $a_1, \dots, a_n \in A$ ,  $n \in \mathbf{Z}_+$ , repetitions are allowed.

The commutator of two elements  $a$  and  $b$  in  $G$  is the element  $aba^{-1}b^{-1}$  and is denoted  $[a, b]$ . Clearly,  $[a, b] = 1$  iff  $a$  and  $b$  commute. Let  $G'$  denote the subgroup of  $G$  generated by the set of commutators in  $G$ , i.e.  $G' = \langle \{ [a, b] \mid a, b \in G \} \rangle$ .

For any  $a, b$  and  $g$  in  $G$ ,  $g[a, b]g^{-1} = [gag^{-1}, gbg^{-1}]$ . Let  $[a_1, b_1] \cdots [a_n, b_n]$ ,  $a_1, b_1, \dots, a_n, b_n \in G$ ,  $n \in \mathbf{Z}_+$ , be any element of  $G'$ . Now

$$g[a_1, b_1] \cdots [a_n, b_n]g^{-1} = g[a_1, b_1]g^{-1}g \cdots g^{-1}g[a_n, b_n]g^{-1} = [ga_1g^{-1}, gb_1g^{-1}] \cdots [ga_ng^{-1}, gb_ng^{-1}]$$

for any  $g$  in  $G$ . Hence  $g[a_1, b_1] \cdots [a_n, b_n]g^{-1} \in G'$ , i.e.  $G'$  is a normal subgroup of  $G$ .

**Theorem 1.** *If  $H$  is a normal subgroup of  $G$  then  $G/H$  is abelian iff  $G' \subset H$ .*

*Proof.* Suppose  $G/H$  is abelian. Let  $a, b \in G$ . Now  $[a][b] = [b][a]$  in  $G/H$ , i.e.  $[a][b][a]^{-1}[b]^{-1} = [aba^{-1}b^{-1}] = [1]$ , i.e.  $aba^{-1}b^{-1} \in H$ . Hence  $G' \subset H$ .

Conversely suppose  $G' \subset H$ , i.e.  $aba^{-1}b^{-1} \in H$  for all  $a, b \in G$ . Hence  $[a][b][a]^{-1}[b]^{-1} = [1]$  in  $G/H$ , i.e.  $[a]$  and  $[b]$  commutes. □

In other words,  $G'$  is the smallest normal subgroup of  $G$  with abelian factor group or  $G/G'$  is the largest abelian factor group of  $G$ . Clearly,  $G$  abelian iff  $G'$  is trivial. Let  $\pi: G \rightarrow G/G'$  denote the residue homomorphism, i.e.  $\pi$  maps an element  $g$  in  $G$  to the residue class  $[g]$  in  $G/G'$ .

**Theorem 2.** *If  $f: G \rightarrow H$  be a homomorphism into an abelian group  $H$  then there exists a unique homomorphism  $f': G/G' \rightarrow H$  such that*

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ \pi \downarrow & \nearrow f' & \\ G/G' & & \end{array}$$

*commutes.*

*Proof.* Let  $a, b \in G$ . Since  $H$  abelian,  $f([a, b]) = f(aba^{-1}b^{-1}) = f(a)f(b)f(a)^{-1}f(b)^{-1} = 1$ , i.e.  $G' \subset \text{Ker } f$ . By the homomorphism theorem,  $f$  factors uniquely through the factor group  $G/G'$ . □

For any group  $G$  we define the abelianization  $G_{ab}$  of  $G$  to be  $G/G'$ . Theorem 1 says that  $G_{ab}$  is the largest abelian factor group of  $G$ . Theorem 2 says that any homomorphism from  $G$  into an abelian group factors uniquely through  $G_{ab}$ .

**Exercise 1.2.9.** A CW complex structure on  $M'_h$  consists of one 0-cell,  $x_0$ ,  $2g + 1$  1-cells,  $a_1, b_1, \dots, a_g, b_g, C$ , and one 2-cell. Hence the 1-skeleton is a wedge of  $2g + 1$  circles. The attachment map of the 2-cell is  $[a_1, b_1] \cdots [a_g, b_g]C^{-1}$ . (It might be instructive to draw the situation for  $h = 2$  using the usual visualization of the 1-skeleton of  $M_2$  as a polygon with 8 edges and the circle  $C$  as a circle inside the polygon intersecting the 1-skeleton at one of the points representing  $x_0$ . Remember to choose the orientations in accordance with the attachment map or adjust the attachment map appropriately.)

Now, by 1.26,

$$\pi_1(M'_h, x_0) = \langle a_1, b_1, \dots, a_g, b_g, C \mid [a_1, b_1] \cdots [a_g, b_g]C^{-1} = 1 \rangle.$$

Since

$$(\dagger) \quad [a_1, b_1] \cdots [a_g, b_g] = C,$$

another presentation of the fundamental group is

$$\pi_1(M'_h, x_0) = \langle a_1, b_1, \dots, a_g, b_g \rangle,$$

i.e. the free product of  $2g$  copies of  $\mathbf{Z}$ .

Suppose  $r: M'_h \rightarrow C$  is a retraction. Let  $i: C \rightarrow M'_h$  be the inclusion map. As always,  $ri = \text{id}_C$  and  $r_*i_* = \text{id}_{\pi_1(C, x_0)}$ . Note that  $\pi_1(C, x_0) = \langle C \rangle = \mathbf{Z}$ , and  $i_*(C) = C = [a_1, b_1] \cdots [a_g, b_g]$ , by  $(\dagger)$ . We will derive a contradiction to the existence of the retraction  $r$  in two almost identical ways:

(1) Since  $\pi_1(C, x_0)$  is abelian,

$$r_*i_*(C) = r_*([a_1, b_1] \cdots [a_g, b_g]) = [r_*(a_1), r_*(b_1)] \cdots [r_*(a_g), r_*(b_g)] = 1,$$

i.e.  $r_*i_*$  is the trivial homomorphism, a contradiction.

(2) Since  $\pi_1(C, x_0)$  is abelian, there exists a homomorphism  $r'_*: \pi_1(M'_h, x_0)_{ab} \rightarrow \pi_1(C, x_0)$  such that the diagram

$$\begin{array}{ccccc} & & \text{id}_{\pi_1(C, x_0)} & & \\ & \nearrow & \text{---} & \searrow & \\ \pi_1(C, x_0) & \xrightarrow{i_*} & \pi_1(M'_h, x_0) & \xrightarrow{r_*} & \pi_1(C, x_0) \\ & & \downarrow \pi & \nearrow r'_* & \\ & & \pi_1(M'_h, x_0)_{ab} & & \end{array}$$

commutes, c.f. theorem 2. Since  $C$  is a product of commutators in  $\pi_1(M'_h, x_0)$ , c.f.  $(\dagger)$ ,  $\pi i_*(C) = 1$ , hence  $r_*i_* = r'_*\pi i_*$  is the trivial homomorphism, a contradiction.

In particular, there is no retraction  $M_g \rightarrow C$ , since restriction would give a retraction  $M'_h \rightarrow C$ .

Recall that the usual CW complex structure on  $M_g$  consists of one 0-cell,  $2g$  1-cells,  $a_1, b_1, \dots, a_g, b_g$  and one 2-cell. The 1-skeleton is the wedge

$$\bigvee_{i=1}^g (S_{a_i}^1 \vee S_{b_i}^1).$$

The attachment map of the 2-cell is  $[a_1, b_1] \cdots [a_g, b_g]$ . Collapsing

$$\bigvee_{i=2}^g (S_{a_i}^1 \vee S_{b_i}^1)$$

gives a quotient map  $q: M_g \rightarrow M_1$ . The map  $r: M_1 = S^1 \times S^1 \rightarrow S^1 \times \{s_0\} = C'$ ,  $s_0 \in S^1$ , defined by  $r(x, y) = (x, s_0)$  is a retraction. Now  $rq: M_g \rightarrow C'$  is a retraction.