April 2007
JMM

Topology

Books, notes, calculators, and computers are allowed at this three hour written exam. You may write your answers in pencil.

Problem 1 (33%)

For each natural number $n = 1, 2, 3, \dots$ let $C_n = \{(x, y) \in \mathbf{R}^2 \mid x^2 + (y-n)^2 = n^2\}$ be the circle with radius n centered at $(0, n)$. Consider the subspace $C = \bigcup_{n=1}^{\infty} C_n$ of the plane $\mathbf{R}^2$ that is the union of these expanding circles.

(1) What is the interior of C? Is C open?
(2) What is the closure of C? Is C closed?
(3) Is C locally compact?

Problem 2 (33%)

Let X be a topological space and $\mathcal{A}$ an open covering of X by connected subspaces. This means that $\mathcal{A}$ is a set of subsets of X such that

- any set $A \in \mathcal{A}$ is an open and connected subspace of X, and
- any point of X lies in A for some $A \in \mathcal{A}$.

For any two points $x, y \in X$, we shall write $x \sim y$ if and only if there exist finitely many sets $A_0, \dots, A_k \in \mathcal{A}$ such that

$$x \in A_0, \text{ and } A_{i-1} \cap A_i \neq \emptyset \text{ for } 1 \leq i \leq k, \text{ and } y \in A_k$$

for some integer $k \geq 1$.

(1) Show that $\sim$ is an equivalence relation on X.
(2) Show that if $x \sim y$ and $A_0, \dots, A_k$ are as in the definition of the equivalence relation $\sim$, then $A_0 \cup \dots \cup A_k$ is connected.
(3) Show that the equivalence classes for $\sim$ are connected and open.
(4) Show that the equivalence classes for $\sim$ are the connected components of X.

Problem 3 (34%)

(1) Find a 2-dimensional abstract simplicial complex (ASC) T (with 6 vertices) such that $|T|$ is homeomorphic to the cylinder $\partial\Delta^2 \times \Delta^1$.
(2) If K_1 and K_2 are finite ASCs, such that $|K_1|$ and $|K_2|$ are (nonempty) surfaces, describe a new ASC, $K_1 \# K_2$, containing T as a subcomplex, such that $|K_1 \# K_2| = |K_1| \# |K_2|$.
(3) Show that $\chi(K_1 \# K_2) = \chi(K_1) + \chi(K_2) - 2$.

(THE END)