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Munkres §22

Ex. 22.2.

(a). The map $p\colon X \to Y$ is continuous. Let U be a subspace of Y such that $p^{-1}(U) \subset X$ is open. Then

$$f^{-1}(p^{-1}(U)) = (pf)^{-1}(U) = \mathrm{id}_Y^{-1}(U) = U$$

is open because f is continuous. Thus $p\colon X \to Y$ is a quotient map.

(b). The map $r\colon X \to A$ is a quotient map by (a) because it has the inclusion map $A \hookrightarrow X$ as right inverse.

Ex. 22.3. Let $g\colon \mathbf{R} \to A$ be the continuous map $f(x) = x \times 0$. Then $q \circ g$ is a quotient map, even a homeomorphism. If the compostion of two maps is quotient, then the last map is quotient; see [1]. Thus q is quotient.

The map q is not closed for $\{x \times \frac{1}{x} \mid x > 0\}$ is closed but $q(A \cap \{x \times \frac{1}{x} \mid x > 0\} = (0, \infty)$ is not closed.

The map q is not open for $\mathbf{R} \times (-1, \infty)$ is open but $q(A \cap (\mathbf{R} \times (-1, \infty))) = [0, \infty)$ is not open.

Ex. 22.5. Let $U \subset A$ be open in A. Since A is open, U is open in X. Since p is open, $p(U) = q(U) \subset p(A)$ is open in Y and also in $p(A)$ because $p(A)$ is open [Lma 16.2].

SupplEx. 22.3. Let G be a topological group and H a subgroup. Let $\varphi_G\colon G \times G \to G$ be the map $\varphi_G(x, y) = xy^{-1}$ and $\varphi_H\colon H \times H \to H$ the corresponding map for the subgroup H. Since these maps are related by the commutative diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\varphi_G} & G \\ \uparrow & & \uparrow \\ H \times H & \xrightarrow{\varphi_H} & H \end{array}$$

and φ_G is continuous, also φ_H is continuous [Thm 18.2]. Moreover, any subspace of a T_1-space is a T_1-space, so H is a T_1-space. Thus H is a topological group by Ex 22.1.

Now, by using [Thm 18.1.(2), Thm 19.5], we get

$$\varphi_G(\overline{H} \times \overline{H}) = \varphi_G(\overline{H \times H}) \subset \overline{\varphi_G(H \times H)} = \overline{H}$$

which shows that $\overline{H}$ is a subgroup of G. But then $\overline{H}$ is a topological group as we have just shown.

SupplEx. 22.5. Let G be a topological group and H a subgroup of G.

(a). Left multiplication with α induces a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{f_\alpha} & G \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ G/H & \xrightarrow[f_\alpha/H]{} & G/H \end{array}$$

which shows [Thm 22.2] that f_α/H is ,

(b). The saturation $gH = f_g(H)$ of the point $g \in G$ is closed because H is closed and left multiplication f_g is a homeomorphism [SupplEx 22.4].

(c). The saturation $UH = \bigcup_{h \in H} Uh = \bigcup g_h(U)$ is open because U is open and right multiplication g_h is a homeomorphism [SupplEx 22.4].

(d). The space G/H is T_1 by point (b). The map $G \times G \xrightarrow{\varphi} G\colon (x,y) \mapsto xy^{-1}$ is continuous [SupplEx 22.1] and it induces a commutative diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\varphi} & G \\ {\scriptstyle p\times p}\downarrow & & \downarrow{\scriptstyle p} \\ G/H \times G/H & \xrightarrow[\varphi/H]{} & G/H \end{array}$$

where $p \times p$ is a quotient map since it is open as we have just shown. (It is not true that the product of two quotient maps is a quotient map [Example 7, p. 143] but is is true that a product of two open maps is an open map.) This shows [Thm 22.2] that φ/H is continuous and hence [SupplEx 22.1] G/H is a topological group.

SupplEx. 22.7. Let G be a topological group.

(a). Any neighborhood of U of e contains a symmetric neighborhood $V \subset U$ such that $VV \subset U$. By continuity of $(x,y) \to xy$, there is a neighborhood W_1 of e such that $W_1W_1 \subset U$. By continuity of $(x,y) \to xy^{-1}$, there is a neighborhood W_2 of e such that $W_2W_2^{-1} \subset W_1$. (Any neighborhood of e contains a symmetric neighborhood of e.) Now $V = W_2W_2^{-1}$ is a symmetric neighborhood of e and $VV \subset W_1W_1 \subset U$.

(b). G is Hausdorff.
Let $x \neq y$. Since the set $\{xy^{-1}\}$ is closed in G, there is a neighborhood U of e not containing xy^{-1}. Find a symmetric neighborhood V of e such that $VV \subset U$. Then $Vx \cap Vy = \emptyset$. (If not, then $gx = hy$ for some $g, h \in V$ and $xy^{-1} = g^{-1}h \in V^{-1}V = VV \subset U$ contradicts $xy^{-1} \notin U$.)

(c). G is regular.
Let $A \subset G$ be a closed subspace and x a point outside A. Since e is outside the closed set xA^{-1}, a whole neighborhood U of e is disjoint from xA^{-1}. Find a symmetric neighborhood V of e such that $VV \subset U$. Then $Vx \cap VA = \emptyset$. (If not, then $gx = ha$ for some $g, h \in V$ and $xA^{-1} \ni xa^{-1} = g^{-1}h \in V^{-1}V = VV \subset U$ contradicts $xA^{-1} \cap U = \emptyset$)

(d). G/H is regular for any closed subgroup $H \subset G$.
The point xH is closed in G/H because the subspace xH is closed in G. Let $x \in G$ be a point ouside a saturated closed subset $A \subset G$. Find a neighborhood V of e such that Vx and VA are disjoint. Then also $VxH \cap VA = \emptyset$ since $A = AH$ is saturated. Since $p\colon G \to G/H$ is open [SupplEx 22.5.(c)], $p(Vx)$ is an open neighborhood of $p(x)$ disjoint from the open neighborhood $p(VA)$ of $p(A)$. This shows that G/H is regular.

References

[1] Jesper M. Møller, *General topology*, `http://www.math.ku.dk/ moller/e03/3gt/notes/gtnotes.dvi`.