April 21, 2006

Munkres §29

Ex. 29.1. Closed intervals $[a, b] \cap \mathbf{Q}$ in $\mathbf{Q}$ are not compact for they are not even sequentially compact [Thm 28.2]. It follows that all compact subsets of $\mathbf{Q}$ have empty interior (are nowhere dense) so $\mathbf{Q}$ can not be locally compact.

To see that compact subsets of $\mathbf{Q}$ are nowhere dense we may argue as follows: If $C \subset \mathbf{Q}$ is compact and C has an interior point then there is a whole open interval $(a, b) \cap \mathbf{Q} \subset C$ and also $[a, b] \cap \mathbf{Q} \subset C$ for C is closed (as a compact subset of a Hausdorff space [Thm 26.3]). The closed subspace $[a, b] \cap \mathbf{Q}$ of C is compact [Thm 26.2]. This contradicts that no closed intervals of $\mathbf{Q}$ are compact.

Ex. 29.2.

(a). Assume that the product $\prod X_\alpha$ is locally compact. Projections are continuous and open [Ex 16.4], so X_α is locally compact for all α [Ex 29.3]. Furthermore, there are subspaces $U \subset C$ such that U is nonempty and open and C is compact. Since $\pi_\alpha(U) = X_\alpha$ for all but finitely many α, also $\pi_\alpha(C) = X_\alpha$ for all but finitely many α. But C is compact so also $\pi_\alpha(C)$ is compact.

(b). We have $\prod X_\alpha = X_1 \times X_2$ where X_1 is a finite product of locally compact spaces and X_2 is a product of compact spaces. It is clear that finite products of locally compact spaces are locally compact for finite products of open sets are open and all products of compact spaces are compact by Tychonoff. So X_1 is locally compact. X_2 is compact, hence locally compact. Thus the product of X_1 and X_2 is locally compact.

Conclusion: $\prod X_\alpha$ is locally compact if and only if X_α is locally compact for all α and compact for all but finitely many α.

Example: $\mathbf{R}^\omega$ and $\mathbf{Z}_+^\omega$ are not locally compact.

Ex. 29.3. Local compactness is not preserved under continuous maps. For an example, let $S \subset \mathbf{R}^2$ be the graph of $\sin(1/x)$, $x \in (0, 1]$. The space $\{(0, 0)\} \cup S$ is not locally compact at $(0, 0)$: Any neighborhood U of $(0, 0)$ contains an infinite subset without limit points, the intersection of S and a horizontal straight line, so U can not [Thm 28.1] be contained in any compact subset of S. On the other hand, $\{(0, 0)\} \cup S$ is the image of a continuous map defined on the locally compact Hausdorff space $\{-1\} \cup (0, 1]$ [Thm 29.2].

Local compactness is clearly preserved under *open* continuous maps as open continuous maps preserve both compactness and openness.

Ex. 29.4 (Morten Poulsen). Let d denote the uniform metric. Suppose $[0, 1]^\omega$ is locally compact at 0. Then $0 \in U \subset C$, where U is open and C is compact. There exists $\varepsilon > 0$ such that $B_d(0, \varepsilon) \subset U$. Note that $A = \{0, \varepsilon/3\}^\omega \subset B_d(0, \varepsilon)$, hence $A \subset C$. By theorem 28.2 A has a limit point in C, contradicting Ex. 28.1.

Ex. 29.5 (Morten Poulsen).

Lemma 1. *A homeomorphism between locally compact Hausdorff spaces extends to a homeomorphism between the one-point compactifications. In other words, homeomorphic locally compact Hausdorff spaces have homeomorphic one-point compactifications.*

Proof. Let $f : X_1 \to X_2$ be a homeomorphism between locally compact Hausdorff spaces. Furthermore let $\omega X_1 = X_1 \cup \{\omega_1\}$ and $\omega X_2 = X_2 \cup \{\omega_2\}$ denote the one-point compactifications. Define $\widetilde{f} : \omega X_1 \to \omega X_2$ by

$$\widetilde{f}(x) = \begin{cases} f(x), & x \in X_1 \\ \omega_2, & x = \omega_1. \end{cases}$$

Note that $\widetilde{f}$ is bijective. Recall that for a locally compact Hausdorff space X the topology on the one-point compactification, ωX, is the collection

$$\{\, U \mid U \subset X \text{ open} \,\} \cup \{\, \omega X - C \mid C \subset X \text{ compact} \,\},$$

c.f. the proof of theorem 29.1.

If $U \subset X_2$ is open then $\widetilde{f}^{-1}(U) = f^{-1}(U)$ is open in ωX_1. If $C \subset X_2$ is compact then $\widetilde{f}^{-1}(\omega X_2 - C) = \widetilde{f}^{-1}(\omega X_2) - \widetilde{f}^{-1}(C) = \omega X_1 - f^{-1}(C)$ is open in ωX_1, since $f^{-1}(C) \subset X_1$ is compact. It follows that $\widetilde{f}$ is continuous, hence a homeomorphism, by theorem 26.6. □

Finally note that the converse statement does not hold: If $X_1 = [0, 1/2) \cup (1/2, 1]$ and $X_2 = [0, 1)$ then $\omega X_1 = [0, 1] = \omega X_2$. But X_1 and X_2 are not homeomorphic, since X_1 is not connected and X_2 is connected.

Ex. 29.6 (Morten Poulsen). Let S^n denote the unit sphere in $\mathbf{R}^{n+1}$. Let p denote the point $(0, \ldots, 0, 1) \in \mathbf{R}^{n+1}$.

Lemma 2. *The punctured sphere $S^n - p$ is homeomorphic to $\mathbf{R}^n$.*

Proof. Define $f : (S^n - p) \to \mathbf{R}^n$ by

$$f(x) = f(x_1, \ldots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \ldots, x_n).$$

The map f is also known as stereographic projection. It is straightforward to check that the map $g : \mathbf{R}^n \to (S^n - p)$ defined by

$$g(y) = g(y_1, \ldots, y_n) = (t(y)\, y_1, \ldots, t(y)\, y_n, 1 - t(y)),$$

where $t(y) = 2/(1 + \|y\|^2)$, is a right and left inverse for f. □

Theorem 3. *The one-point compactification of $\mathbf{R}^n$ is homeomorphic to S^n.*

Proof. By the preceding lemma $\mathbf{R}^n$ is homeomorphic to $S^n - p$. The one-point compactification of $S^n - p$ is clearly S^n. Now the result follows from Ex. 29.5. □

Ex. 29.7. Let X be any linearly ordered space with the least upper bound property. As $[a, b] = [a, b) \cup \{b\}$ is compact Hausdorff [Thm 27.1, Thm 17.11], the right half-open interval $[a, b)$ is locally compact Hausdorff and its Alexandroff compactification is $[a, b]$ [Thm 29.1]. Apply this to $S_\Omega = [1, \Omega) \subset \overline{S}_\Omega = [1, \Omega]$. (Apply also to $\mathbf{Z}_+ = [1, \omega) \subset \mathbf{Z}_+ \times \mathbf{Z}_+$ where $\omega = 2 \times 1$ for (an alternative answer to) [Ex 29.8])

Also X itself is locally compact Hausdorff [Thm 17.11] as all closed and bounded intervals in X are compact [Thm 27.1]. Is the one-point compactification of X a linearly ordered space?

Ex. 29.9. This follows from Ex 29.3 for the quotient map $G \to G/H$ is open [SupplEx 22.5.(c)].

Ex. 29.11. It is not always true that the product of two quotient maps is a quotient map [Example 7, p. 143] but here is a case where it is true.

Lemma 4 (Whitehead Theorem)**.** [1, 3.3.17] *Let $p\colon X \to Y$ be a quotient map and Z a locally compact space. Then*

$$p \times 1\colon X \times Z \to Y \times Z$$

is a quotient map.

Proof. Let $A \subset X \times Z$. We must show: $(p \times 1)^{-1}(A)$ is open $\Rightarrow$ A is open. This means that for any point $(x, y) \in (p \times 1)^{-1}(A)$ we must find a saturated neighborhood U of x and a neighborhood V of y such that $U \times V \subset (p \times 1)^{-1}(A)$.

Since $(p \times 1)^{-1}(A)$ is open in the product topology there is a neighborhood U_1 of x and a neighborhood V of y such that $U_1 \times V \subset (p \times 1)^{-1}(A)$. Since Y is locally compact Hausdorff we may assume [Thm 29.2] that $\overline{V}$ is compact and $U_1 \times \overline{V} \subset (p \times 1)^{-1}(A)$. Note that also $p^{-1}(pU_1) \times \overline{V}$ is contained in $(p \times 1)^{-1}(A)$. The tube lemma [Lemma 26.8] says that each point of $p^{-1}(pU_1)$ has a neighborhood such that the product of this neighborhood with $\overline{V}$ is contained in the open set $(p \times 1)^{-1}(A)$. Let U_2 be the union of these neighborhoods. Then $p^{-1}(pU_1) \subset U_2$ and $U_2 \times \overline{V} \subset (p \times 1)^{-1}(A)$. Continuing inductively we find open sets $U_1 \subset U_2 \subset \cdots \subset U_i \subset U_{i+1} \subset \cdots$ such that $p^{-1}(pU_i) \subset U_{i+1}$ and $U_{i+1} \times \overline{V} \subset (p \times 1)^{-1}(A)$. The open set $U = \bigcup U_i$ is saturated because $U \subset p^{-1}(pU) = \bigcup p^{-1}(pU_i) \subset \bigcup U_{i+1} = U$. Thus also $U \times V$ is saturated and $U \times V \subset \bigcup U_i \times V \subset (p \times 1)^{-1}(A)$. □

Example: If $p\colon X \to Z$ is a quotient map, then also $p \times \mathrm{id}\colon X \times [0,1] \to Z \times [0,1]$ is a quotient map. This fact is important for homotopy theory.

Theorem 5. *Let $p\colon A \to B$ and $q\colon C \to D$ be quotient maps. If B and C are locally compact Hausdorff spaces then $p \times q\colon A \times C \to B \times D$ is a quotient map.*

Proof. The map $p \times q$ is the composition

$$A \times C \xrightarrow{p\times 1} B \times C \xrightarrow{1\times q} B \times D$$

of two quotient maps and therefore itself a quotient map [p. 141]. □

References

[1] Ryszard Engelking, *General topology*, second ed., Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989, Translated from the Polish by the author. MR 91c:54001