1st December 2004

Munkres §35

Ex. 35.3. Let X be a metrizable topological space.

(i) $\Rightarrow$ (ii): (We prove the contrapositive.) Let d be any metric on X and $\varphi\colon X \to \mathbf{R}$ be an unbounded real-valued function on X. Then $\overline{d}(x,y) = d(x,y) + |\varphi(x) - \varphi(y)|$ is an unbounded metric on X that induces the same topology as d since

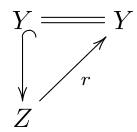
$$B_{\overline{d}}(x,\varepsilon) \subset B_d(x,\varepsilon) \subset B_{\overline{d}}(x,\delta)$$

for any $\varepsilon > 0$ and any $\delta > 0$ such that $\delta < \frac{1}{2}\varepsilon$ and $d(x,y) < \delta \Rightarrow |\varphi(x) - \varphi(y)| < \frac{1}{2}\varepsilon$.

(ii) $\Rightarrow$ (iii): (We prove the contrapositive.) Let X be a normal space that is not limit point compact. Then there exists a closed infinite subset $A \subset X$ [Thm 17.6]. Let $f\colon X \to \mathbf{R}$ be the extension [Thm 35.1] of any surjection $A \to \mathbf{Z}_+$. Then f is unbounded.

(iii) $\Rightarrow$ (i): Any limit point compact metrizable space is compact [Thm 28.2]; any metric on X is continuous [Ex 20.3], hence bounded [Thm 26.5].

Ex. 35.4. Let Z be a topological space and $Y \subset Z$ a subspace. Y is a retract of Z if the identity map on Y extends continuously to Z, i.e. if there exists a continuous map $r\colon Z \to Y$ such that



commutes.

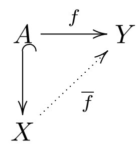
(a). $Y = \{z \in Z \mid r(z) = z\}$ is closed if Z is Hausdorff [Ex. 31.5].

(b). Any retract of $\mathbf{R}^2$ is connected [Thm 23.5] but A is not connected.

(c). The continuous map $r(x) = x/|x|$ is a retraction of the punctured plane $\mathbf{R}^2 - \{0\}$ onto the circle $S^1 \subset \mathbf{R}^2 - \{0\}$.

Ex. 35.5. A space Y has the UEP if the diagram

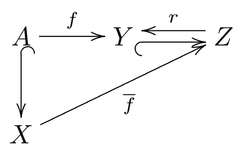
(1)



has a solution for any closed subspace A of a normal space X.

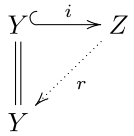
(a). Another way of formulating the Tietze extension theorem [Thm 35.1] is: $[0,1]$, $[0,1)$, and $(0,1) \simeq \mathbf{R}$ have the UEP. By the universal property of product spaces [Thm 19.6], $\operatorname{map}(A, \prod X_\alpha) = \prod \operatorname{map}(A, X_\alpha)$, any product of spaces with the UEP has the UEP.

(b). Any retract Y of a UEP space Z is a UEP space for in the situation



the continuous map $r\overline{f}\colon X \to Y$ extends $f\colon A \to Y$.

0.1. **Ex. 35.6.** Let Y be a normal space. We say that Y is an absolute retract if for any imbedding $Y \xhookrightarrow{i} Z$ of Y into a closed subspace of a normal space Z there is a map $r\colon Z \to Y$ such that



commutes, i.e. such that ri is the identity on Y.

(a). Any space Y with the UEP is an absolute retract: Apply (1) with input $Z \hookleftarrow Y = Y$.

(b). If Y is compact: Y has the UEP $\Leftrightarrow$ Y is an absolute retract. (Cf. [Ex 35.8])
The compact Hausdorff spaces are precisely the spaces that are homeomorphic to a closed subspace of $[0,1]^J$ for some set J [Thm 34.3, Thm 37.3]. Therefore any compact Hausdorff space that is also an absolute retract is a retract of the UEP space $[0,1]^J$, hence is itself a UEP space [Ex 31.5.(b)].

Ex. 35.7.

(a). The space $C \subset \mathbf{R}^2$ is a closed subspace of the normal space $\mathbf{R}^2$ homeomorphic to $[0,\infty)$. The Tietze theorem (small variation of [Thm 35.1.(b)]) says that $[0,\infty)$ has the UEP. Therefore $[0,\infty)$ is an absolute retract [Ex 35.6] and C is retract of $\mathbf{R}^2$. The continuous map $r\colon \mathbf{R}^2 \to C$ given by

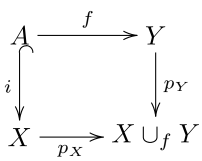
$$r(x) = \begin{cases} |x| \cos\log|x| \times \log|x| \sin\log|x| & x \neq 0 \times 0 \\ 0 \times 0 & x = 0 \times 0 \end{cases}$$

is a retraction of $\mathbf{R}^2$ onto the logarithmic spiral C.

(b). The space $K \subset \mathbf{R}^3$ is a closed subspace of the normal space $\mathbf{R}^3$ homeomorphic to $\mathbf{R}$. The Tietze theorem says that $\mathbf{R}$ has the UEP. Therefore $\mathbf{R}$ is an absolute retract [Ex 35.6] and K is a retract of $\mathbf{R}^3$; I can't find an explicit retraction $\mathbf{R}^3 \to K$, though.

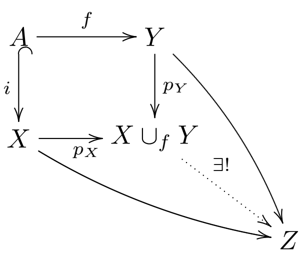
Ex. 35.8. (Adjunction spaces [1, p 93] [2, Chp I, Exercise B, p 56]) Let X and Y be two disjoint topological spaces and $f\colon A \to Y$ a continuous map defined on a closed subspace A of X. Define $X \cup_f Y$ to be the quotient of $X \cup Y$ by the smallest equivalence relation such that $a \in A$ and $f(a) \in Y$ are equivalent for all points $a \in A$. (To picture this, tie an elastic band from each point a of A to its image $f(a)$ in Y and let go!) The equivalence classes, $[y] = f^{-1}(y) \cup \{y\}$ for $y \in Y$ and $[x] = \{x\}$ for $x \in X - A$, are represented by points in Y or in $X - A$. Let $p\colon X \cup Y \to X \cup_f Y$ be the quotient map; p_X the restriction of p to X and p_Y the restriction of p to Y.

The adjunction space $X \cup_f Y$ fits into a commutative diagram



called a *push-out diagram* because of this universal property: If $X \to Z$ and $Y \to Z$ are continuous maps that agree on A then there is [Thm 22.2] a unique continuous map $X \cup_f Y \to Z$ such that the diagram

(2)



commutes. (This is just the universal property for quotient spaces in this particular situation.)

Here are the main properties of adjunction spaces.

Lemma 1. *Let* $p\colon X \cup Y \to X \cup_f Y$ *be the quotient map.*

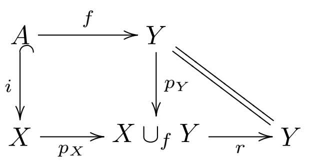
1. *The quotient map* p *embeds* Y *into a closed subspace of* $X \cup_f Y$*. (We therefore identify* Y *with its image* $p_Y(Y)$ *in the adjunction space.)*
2. *The quotient map* p *embeds* $X - A$ *into the open subspace* $(X \cup_f Y) - Y$ *of the adjunction space.*
3. *If* X *and* Y *are normal, also the adjunction space* $X \cup_f Y$ *is normal.*
4. *The projection map* $p\colon X \cup Y \to X \cup_f Y$ *is closed if (and only if* [1, p 93]*)* f *is closed.*

Proof. (1) The map $p_Y = p|Y\colon Y \to X \cup_f Y$ is closed for closed sets $B \subset Y \subset X \amalg Y$ have closed saturations $f^{-1}(B) \amalg B$. Since p_Y is also injective it is an embedding.
(2) The map $p_X|X - A\colon X - A \to (X \cup_f Y$ open because the saturation of any (open) subset U of $X - A$ is $U \cup \emptyset \subset X \cup Y$ itself. Since $p_X|X - A$ is also injective it is an embedding.
(3) Points are closed in the quotient space $X \cup_f Y$ because the equivalence classes are closed in $X \cup Y$. Let C and D be two disjoint closed subspaces of $X \cup_f Y$. We will show that there is a continuous map $X \cup_f Y \to [0, 1]$ with value 0 on C and value 1 on D. Since Y is normal, there exists [Thm 33.1] a Urysohn function $g\colon Y \to [0, 1]$ such that $g(Y \cap C) = \{0\}$ and $g(Y \cap D) = \{1\}$. Since X is normal, by the Tietze extension theorem [Thm 35.1], there is a continuous map $X \to [0, 1]$ which is 0 on $p_X^{-1}(C)$, 1 on $p_X^{-1}(D)$, and is $g \circ f$ on A. By the universal property for adjunction spaces (2), there is a map $X \cup_f Y \to [0, 1]$ that is 0 on C and 1 on D. This shows that C and D can be separated by a continuous function and that $X \cup_f Y$ is normal.
(4) Closed subsets of Y always have closed saturations as we saw in item (1). If f is closed then also the saturation, $B \cup f^{-1}f(A \cap B) \cup f(A \cap B) \subset X \cup Y$, of a closed subset $B \subset X$ is closed. (Since closed quotient maps (surjective closed maps) preserve normality [Ex 31.6, Thm 73.3] this gives an easy proof of (3) under the additional assumption that $f\colon A \to Y$ be a closed map.) □

The adjunction space is the disjoint union of a closed subspace homeomorphic to Y and an open subspace homeomorphic to $X - A$.

Theorem 2. [2, Chp I, Exercise C, p 56] *Let* Y *be a normal space. Then* Y *has the universal extension property if and only if* Y *is an absolute retract.*

Proof. One direction was proved already in Ex 35.6. For the other direction, suppose that the normal space Y is an absolute retract. Let X be any normal space, A a closed subspace of X, and $f\colon A \to Y$ a continuous map. Form the adjunction space $Z = X \cup_f Y$. Then Z is normal (as we have just seen) and Y is (homeomorphic) to a closed subspace of Z. Since Y is an absolute retract, there is a retraction $r\colon Z \to Y$ of Z onto Y. These maps are shown in the commutative diagram



which says that $r \circ p_X\colon X \to Y$ is an extensiom of $f\colon A \to Y$. This shows that Y has the universal extension property. □

References

[1] Ryszard Engelking, *General topology*, second ed., Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989, Translated from the Polish by the author. MR **91c:**54001

[2] Edwin H. Spanier, *Algebraic topology*, Springer-Verlag, New York, 1981, Corrected reprint. MR **83i:**55001