From singular chains to Alexander Duality

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Contents

CHAPTER 1

Singular homology

In this chapter we construct the singular homology functor from the category of topological spaces to the category of abelian groups. [1]

1. The standard geometric n-simplex Δ^n

Let **VCT** be the category of real vector spaces and

$$\mathbf{R}[\bullet]\colon \Delta_< \to \mathbf{VCT}$$

the functor from $\Delta_<$ (2.26) that takes n_+ to the real vector space $\mathbf{R}[n_+]$ with basis e_i, $i \in n_+$. The coface map $d^i \in \Delta_<((n-1)_+, n_+)$, $i \in n_+$, induces the *geometric coface map* $d^i = \mathbf{R}[d^i]\colon \mathbf{R}[(n-1)_+] \to \mathbf{R}[n_+]$

$$d^i(t_0, \dots, t_{n-1}) = \begin{cases} (0, t_0, \dots, t_{n-1}) & i = 0 \\ (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}) & 0 < i < n \\ (t_0, \dots, t_{n-1}, 0) & i = n \end{cases} \tag{1.1}$$

The coface map d^i takes $\mathbf{R}[(n-1)_+]$ into $\mathbf{R}[n_+]$ as the hyperplane orthogonal to e_i, $d_i(\mathbf{R}^{n-1}) = e_i^\perp$. The geometric coface maps satisfy the cosimplicial identities (2.18).

1.1. Definition. The *standard geometric n-simplex* is the convex hull

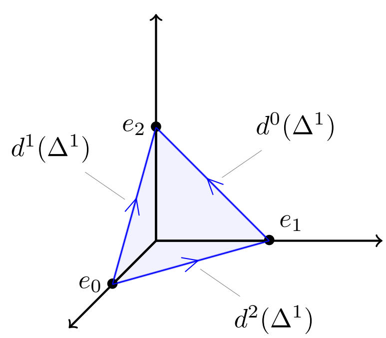
$$\Delta^n = \mathrm{conv}(n_+) = \{(t_0, \dots, t_n) \in \mathbf{R}[n_+] \mid 0 \le t_i \le 1, \sum_{i=0}^{n+1} t_i = 1\}$$

of the basis n_+ of $\mathbf{R}[n_+]$, $n = 0, 1, 2, \dots$.

The geometric coface maps $d^i\colon \mathbf{R}[(n-1)_+] \to \mathbf{R}[n_+]$, $i \in n_+$, of vector spaces restrict to geometric coface maps $d^i\colon \Delta^{n-1} \to \Delta^n$ of standard geometric simplices. The coface map d^i takes Δ^{n-1} into Δ^n as the facet opposite vertex e_i, $d^i(\Delta^{n-1}) = \Delta^n \cap e_i^\perp$. In short, there is a co-Δ-space

$$\Delta[\bullet]\colon \Delta_< \to \mathbf{TOP}$$

taking n_+ to the geometric n-simplex $\Delta[n_+] = \Delta^n$ and the coface map $d^i\colon (n-1)_+ \to n_+$ to the geometric coface map $d^i\colon \Delta^{n-1} \to \Delta^n$.



[1] These notes were modified April 20, 2016

$$\begin{array}{ccccccccc}
\mathbf{R}[0_+] & \rightrightarrows & \mathbf{R}[1_+] & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & \mathbf{R}[2_+] & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} & \mathbf{R}[3_+] & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} & \mathbf{R}[4_+]\cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Delta^0 & \rightrightarrows & \Delta^1 & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & \Delta^2 & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} & \Delta^3 & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} & \Delta^4\cdots \\
 & & & & \big\downarrow \mathbf{TOP}(\bullet, X) & & & & \\
\mathrm{Sing}(X)_0 & \leftleftarrows & \mathrm{Sing}(X)_1 & \substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow} & \mathrm{Sing}(X)_2 & \substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow} & \mathrm{Sing}(X)_3 & \substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow} & \mathrm{Sing}(X)_4\cdots \\
 & & & & \big\downarrow \mathbf{Z}[\bullet] & & & & \\
\mathbf{Z}[\mathrm{Sing}(X)_0] & \xleftarrow{\partial} & \mathbf{Z}[\mathrm{Sing}(X)_1] & \xleftarrow{\partial} & \mathbf{Z}[\mathrm{Sing}(X)_2] & \xleftarrow{\partial} & \mathbf{Z}[\mathrm{Sing}(X)_3] & \xleftarrow{\partial} & \mathbf{Z}[\mathrm{Sing}(X)_4]\cdots \\
\big\downarrow H_0 & & \big\downarrow H_1 & & \big\downarrow H_2 & & \big\downarrow H_3 & & \big\downarrow H_4 \\
H_0(X) & & H_1(X) & & H_2(X) & & H_3(X) & & H_4(X)\cdots
\end{array}$$

FIGURE 1. The singular Δ-set, the singular chain complex, and the singular homology groups of a topological space X

2. The singular Δ-set, chain complex, and homology groups of a topological space

Let X be a topological space. An n-simplex in X is a (continuous) map $\sigma\colon \Delta^n \to X$ of the geometric n-simplex into X.

- The *singular Δ-set* of X is the Δ-set $\mathrm{Sing}(X)_\bullet$ which in degree n is the set $\mathrm{Sing}(X)_n = \mathbf{TOP}(\Delta^n, X)$ of all singular n-simplices $\sigma\colon \Delta^n \to X$ in X; the face maps $d_i\colon \mathrm{Sing}(X)_n \to \mathrm{Sing}(X)_{n-1}$, $i \in n_+$, are induced by the coface maps $d^i\colon \Delta^{n-1} \to \Delta^n$: $d_i\sigma = \sigma d^i$.
- The *singular chain complex* $(C_*(X), \partial)$ of X is the chain complex $(\mathbf{Z}[\mathrm{Sing}(X)], \partial)$ of the singular Δ-set of X.
- The *singular homology groups* $H_*(X)$ of X are the homology groups of the singular Δ-set $\mathrm{Sing}(X)$ of X: $H_n(X) = H_n^\Delta(\mathrm{Sing}(X))$.

The nth chain group of X is the free abelian group $C_n(X)$ generated by the set of all n-simplices $\sigma\colon \Delta^n \to X$ in X. The elements of $C_n(X)$ are linear combinations $\sum n_\sigma \sigma$, where $n_\sigma \in \mathbf{Z}$ and $n_\sigma = 0$ for all but finitely many σ. (By convention $C_n(X) = 0$ for $n < 0$.) The n-th boundary map $\partial_n\colon C_n(X) \to C_{n-1}(X)$ is the linear map with value

$$\partial_n(\sigma) = \sum_{i \in n_+} (-1)^i d_i\sigma = \sum_{i \in n_+} (-1)^i \sigma d^i \tag{1.2}$$

on the n-simplex $\sigma\colon \Delta^n \to X$.

The nth singular homology group of X is the quotient

$$H_n(X) = \ker(\partial_n)/\operatorname{im}(\partial_{n+1})$$

of the n-cycles $Z_n(X) = \ker(\partial_n)$ by the n-boundaries $B_n(X) = \operatorname{im}(\partial_{n+1})$. We let $[z] \in H_n(X)$ be the homology class represented by the n-cycle $z \in Z_n(X)$.

Any map $f\colon X \to Y$ induces a homomorphism $H_n(f)\colon H_n(X) \to H_n(Y)$ given by $H_n(f)[z] = [C_n(f)z]$ for any n-cycle $z \in Z_n(X)$. In fact, $H_n(\bullet)$ is a (composite) functor from topological spaces to abelian groups.

1.3. PROPOSITION (Additivity Axiom). Let $\{X_\alpha\}$ be the path-components of X. There is an isomorphism

$$\bigoplus H_k(X_\alpha) \cong H_k(X)$$

induced by the inclusion maps.

PROOF. Any simplex $\sigma\colon \Delta^k \to X$ must factor through one of the path-components X_α as Δ^k is path connected. Therefore $\bigoplus C_k(X_\alpha) = C_k(X)$ and $\bigoplus H_k(X_\alpha) \cong H_k(X)$ for all integers k. $\square$

1.4. PROPOSITION. The group $H_0(X)$ is the free abelian group on the set of path-components of X.

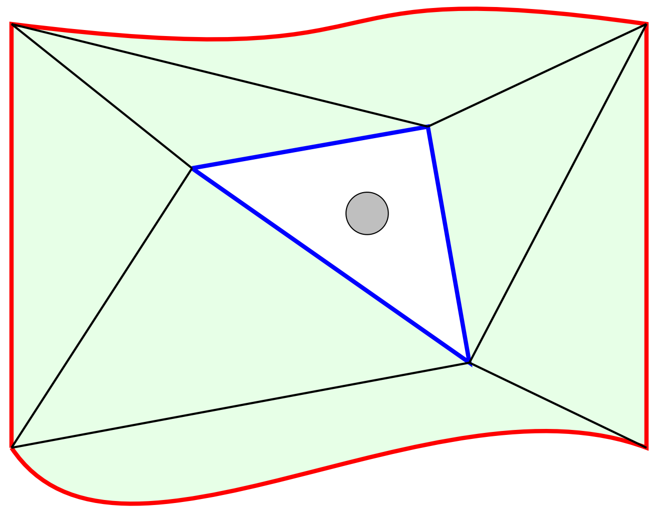


FIGURE 2. The blue and the red 1-cycles are homologous in $\mathbf{R}^2 - D^2$ by the green 2-chain

PROOF. The proposition is true if $X = \emptyset$ is empty. Suppose that X is nonempty. By the Additivity Axiom we can also assume that X is path-connected. Define $\varepsilon\colon C_0(X) \to \mathbf{Z}$ to be the group homomorphism with value 1 on any point of X. The sequence

$$C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbf{Z} \to 0$$

is exact: Clearly, $\operatorname{im}\partial_1 \subset \ker\varepsilon$ for $\varepsilon\partial_1 = 0$. Fix a point, x_0 in X. Since X is path-connected, for every $x \in X$ there is a 1-simplex $\sigma_x\colon \Delta^1 \to X$ with $\sigma_x(0) = x_0$ and $\sigma_x(1) = x$. Let $\sum_{x\in X}\lambda_x x$ be a 0-chain in X with $\sum_{x\in X}\lambda_x = 0$. Then

$$\partial_1(\sum_{x\in X}\lambda_x\sigma_x) = \sum_{x\in X}\lambda_x(x - x_0) = \sum_{x\in X}\lambda_x x - (\sum_{x\in X}\lambda_x)x_0 = \sum_{x\in X}\lambda_x x$$

This shows that $\ker\varepsilon \subset \operatorname{im}\partial_1$. Thus $\mathbf{Z} = \operatorname{im}\varepsilon \cong C_0(X)/\ker\varepsilon = C_0(X)/\operatorname{im}\partial_1 = H_0(X)$ by exactness of the sequence above. □

1.5. PROPOSITION (Dimension Axiom). The homology groups of the space $\{*\}$ consisting of one point are $H_0(\{*\}) \cong \mathbf{Z}$ and $H_k(\{*\}) = 0$ for $k \neq 0$.

1.6. Homology with coefficients. Let G be any abelian group. The nth chain group of X with coefficients in G is the abelian group $C_n(X;G)$ consisting of all linear combinations $\sum_\sigma g_\sigma\sigma$, where $g_\sigma \in G$ and $g_\sigma = 0$ for all but finitely many n-simplices $\sigma\colon \Delta^n \to X$. The boundary map $\partial_n\colon C_n(X;G) \to C_{n-1}(X;G)$ is defined as

$$\partial_n\left(\sum_\sigma g_\sigma\sigma\right) = \sum_\sigma\sum_{i=0}^n (-1)^i g_\sigma \sigma d^i$$

where $\pm 1 g_\sigma$ means $\pm g_\sigma$. Again, the composition of two boundary maps is zero so that $(C_n(X;G), \partial_n)$ is a chain complex. We define the nth homology group of X with coefficients in G, $H_n(X;G)$, to be the nth homology group of this chain complex.

In particular, $H_n(X;\mathbf{Z}) = H_n(X)$. Most of the following results are true for $H_n(X;G)$ even though they will only be stated for $H_n(X)$.

1.7. Reduced homology. The *reduced homology groups* of X with coefficients in the abelian group G are the homology groups $\widetilde{H}_n(X;G)$ of the *augmented chain complex*

$$\cdots \to C_1(X;G) \xrightarrow{\partial_1} C_0(X;G) \xrightarrow{\varepsilon} G \to 0$$

where $\varepsilon(\sum_{x\in X} g_x x) = \sum_{x\in X} g_x$. This is again a chain complex as $\varepsilon\partial_1 = 0$. There is no difference between reduced and unreduced homology in positive degrees while $\widetilde{H}_0(X;G) = \ker\varepsilon/\operatorname{im}\partial_1$.

When $X = \emptyset$ is empty, $H_0(X) = 0$ and $H_{-1}(X) = \mathbf{Z}$. Suppose now that $X \neq \emptyset$ is nonempty.

$\widetilde{H}_0(X)$ and path-connectedness: We noted in the proof of Proposition 1.4 that X is path-connected if and only if $\operatorname{im}\partial_1 = \ker\varepsilon$. This means that

$$\widetilde{H}_0(X;G) = 0 \iff X \text{ is path-connected}$$

A short split-exact sequence for $\widetilde{H}_0(X)$: In degree 0 there are natural short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \ker(\varepsilon)/\mathrm{im}(\partial) \arrow[r] \arrow[d, equal] & C_0(X;G)/\mathrm{im}(\partial) \arrow[r] \arrow[d, equal] & C_0(X;G)/\ker(\varepsilon) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \widetilde{H}_0(X;G) \arrow[r] & H_0(X;G) \arrow[r, "\varepsilon"] & G \arrow[r] & 0
\end{tikzcd}$$

The right vertical map is an isomorphism and the homomorphism $G \ni g \to gx_0$, where x_0 is some fixed point in X, is a right inverse to $\varepsilon\colon H_0(X) \to G$. Since the short exact sequence splits there is a non-natural isomorphism

$$H_0(X;G) \cong \widetilde{H}_0(X;G) \oplus G$$

Another short split-exact sequence for $\widetilde{H}_0(X)$: Since $\widetilde{H}_*(\{x_0\};G) = 0$ for the space consisting of a single point, the long exact sequence in reduced homology (1.11) for the pair (X, x_0) gives that $\widetilde{H}_n(X;G) \cong H_n(X, x_0; G)$. The long exact sequence in (unreduced) homology (1.10) breaks into short split exact sequences because the point is a retract of the space and it ends with

$$0 \longrightarrow H_0(x_0;G) \longrightarrow H_0(X;G) \longrightarrow H_0(X, x_0; G) \longrightarrow 0$$

so that $\widetilde{H}_0(X;G) \cong H_0(X, x_0; G) \cong H_0(X;G)/H_0(x_0;G)$.

1.8. Proposition. Let X be any topological space.

$$\widetilde{H}_{-1}(X;\mathbf{Z}) = 0 \iff X \neq \emptyset$$

$$\widetilde{H}_{-1}(X;\mathbf{Z}) = 0 \text{ and } \widetilde{H}_0(X;\mathbf{Z}) = 0 \iff X \text{ is nonempty and path-connected}$$

3. The long exact sequence of a pair

Let (X, A) be a pair of spaces consisting of a topological space X with a subspace $A \subset X$. Define $C_n(X, A)$ to be the quotient of $C_n(X)$ by its subgroup $C_n(A)$. Then we have the situation

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C_{n+1}(A) \arrow[r, "i"] \arrow[d, "\partial_{n+1}"'] & C_{n+1}(X) \arrow[r, "j"] \arrow[d, "\partial_{n+1}"'] & C_{n+1}(X,A) \arrow[r] \arrow[d, dashed, "\overline{\partial}_{n+1}"] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r, "i"] \arrow[d, "\partial_n"'] & C_n(X) \arrow[r, "j"] \arrow[d, "\partial_n"'] & C_n(X,A) \arrow[r] \arrow[d, dashed, "\overline{\partial}_n"] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r, "i"] \arrow[d] & C_{n-1}(X) \arrow[r, "j"] \arrow[d] & C_{n-1}(X,A) \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

where the rows are short exact sequences. Since the morphism $j \circ \partial_n \colon C_n(X) \to C_{n-1}(X,A)$ vanishes on the subgroup $C_n(A)$ of $C_n(X)$ there is a unique morphism $\overline{\partial}_n \colon C_n(X,A) \to C_{n-1}(X,A)$ such that the diagram commutes (Chapter 1, Section 10.1.1). Then $(C_*(X,A), \overline{\partial}_*)$ is a chain complex for $\overline{\partial}_{n-1} \circ \overline{\partial}_n = 0$ since it is induced from $\partial_{n-1} \circ \partial_n = 0$. Define the *relative homology*

$$H_n(X,A) = \frac{Z_n(X,A)}{B_n(X,A)} = \frac{j^{-1}Z_n(X,A)}{j^{-1}B_n(X,A)} = \frac{\partial_n^{-1}C_{n-1}(A)}{B_n(X)+C_n(A)} \tag{1.9}$$

to be the degree n homology of this *relative chain complex*. The above commutative diagram can now be enlarged to a short exact sequence of chain complexes. Lemma 1.100, the *Fundamental Lemma of Homological Algebra*, tells us that there is an associated long exact sequence[2]

$$\cdots \longrightarrow H_n(A) \xrightarrow{i} H_n(X) \xrightarrow{j} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \cdots \tag{1.10}$$

of homology groups [**21**, 1.3]. The *connecting homomorphism*

$$Z_n(X,A)/B_n(X,A) = H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) = Z_{n-1}(A)/B_{n-1}(A)$$

is induced by the zig-zag $i^{-1}\partial \colon \partial^{-1}C_n(A) \to Z_{n-1}(A)$.

Two more long exact sequences in homology can be obtained in a similar way:

- The long exact sequence in *reduced* homology

$$\cdots \longrightarrow \widetilde{H}_n(A) \xrightarrow{i} \widetilde{H}_n(X) \xrightarrow{j} H_n(X,A) \xrightarrow{\partial} \widetilde{H}_{n-1}(A) \longrightarrow \cdots \tag{1.11}$$

  is obtained by using the augmented chain complexes.
- The long exact sequence for a triple (X,A,B)

$$\cdots \longrightarrow H_n(A,B) \longrightarrow H_n(X,B) \longrightarrow H_n(X,A) \xrightarrow{\partial} H_{n-1}(A,B) \longrightarrow \cdots \tag{1.12}$$

  comes from the short exact sequence

$$0 \to C_*(A)/C_*(B) \to C_*(X)/C_*(B) \to C_*(X)/C_*(A) \to 0$$

  of relative chain complexes.

4. The Eilenberg–Steenrod Axioms

The Eilenberg–Steenrod Axioms are theorems. They say that singular homomology is a functor from the category of pairs (X,A) of topological spaces to the category of abelian groups that satisfies the following list of axioms (theorems):

Exactness Axiom: For any pair (X,A) of topological spaces there is a natural long excact sequence

$$\cdots \leftarrow H_{n-1}(A) \leftarrow H_n(X,A) \leftarrow H_n(X) \leftarrow H_n(A) \leftarrow \cdots$$

Homotopy Axiom: $f \simeq g : (X,A) \to (Y,B) \implies f_* = g_* \colon H_n(X,A) \to H_n(Y,B)$
Excision Axiom: $\mathrm{cl}(U) \subset \mathrm{int}(A) \implies H_n(X-U, A-U) \cong H_n(X,A)$
Dimension Axiom: $H_0(\{*\}) = \mathbf{Z}$ and $H_j((\{*\}) = 0$ for $i \neq 0$
Additivity Axiom: $\bigoplus_{\alpha\in A} H_n(X_\alpha) \xrightarrow{\cong} H_n(\coprod_{\alpha\in A} X_\alpha)$

5. Homotopy invariance

Homology does not distinguish between homotopic maps.

1.13. Theorem (Homotopy Axiom). *Homotopic maps induce identical maps on homology: If $f_0 \simeq f_1 \colon X \to Y$, then $H_n(f_0) = H_n(f_1) \colon H_n(X) \to H_n(Y)$.*

The $(n+1)$-simplex $P^i \in \mathbf{POSI}((n+1)_+, n_+ \times 1_+)$ from (2.16) induces a linear map between vector spaces $P^i \colon \mathbf{R}[(n+1)_+] \to \mathbf{R}[n_+] \times \mathbf{R}[1_+]$ that restricts to an injective map from $\Delta^{n+1} = \mathrm{conv}((n+1)_+) \subset \mathbf{R}[(n+1)_+]$ to $\mathrm{conv}(n_+ \times 1_+) = \mathrm{conv}(n_+) \times \mathrm{conv}(1_+) = \Delta^n \times \Delta^1 \subset \mathbf{R}[n_+] \times \mathbf{R}[1_+]$ (see Lemma 1.14). (Alternatively, the geometric prism map $P^i \colon \Delta^{n+1} = B(n+1)_+ \to B(n_+ \times 1_+) = \Delta^n \times \Delta^1$ is induced from the prism map $P^i \colon (n+1)_+ \to n_+ \times 1_+$ of Definition 2.16.)

[2]The long exact sequence of a pair adorns the facade of the university library in Warsaw.

1.14. LEMMA. The convex hull of $n_+ \times 1_+$ in $\mathbf{R}[n_+] \times \mathbf{R}[1_+]$ is $\mathrm{conv}(n_+ \times 1_+) = \mathrm{conv}(n_+) \times \mathrm{conv}(1_+) = \Delta^n \times \Delta^1$

PROOF. The inclusion $\subseteq$ is clear. To prove the opposite inclusion, write the basis vectors of $\mathbf{R}[n_+]$ as $n_+ = \{e_i \mid i \in n_+\}$ and the basis vectors of $\mathbf{R}[1_+]$ as $1_+ = \{f_0, f_1\}$. An arbitrary element of $\mathrm{conv}(n_+) \times \mathrm{conv}(1_+)$ has the form $(\sum s_i(e_i, 0), t_0(0, f_0) + t_1(0, f_1)) = t_0 \sum s_i(e_i, f_0) + t_1 \sum s_i(e_i, f_1)$ which is a convex combination of vectors from $n_+ \times 1_+ = \{(e_i, f_j) \mid i \in n_+, j \in 1_+\}$. $\square$

In coordinates, P^i_n has the form

$$P^i \colon \Delta^{n+1} \to \Delta^n \times \Delta^1, \qquad \sum_{h=0}^{n+1} t_h e_h \to \Big(\sum_{h \le i} t_h e_h + \sum_{h > i} t_h e_{h-1}, \sum_{h \le i} t_h e_0 + \sum_{h > i} t_h e_1\Big)$$

as a map from $\Delta^{n+1} = \mathrm{conv}((n+1)_+) \subset \mathbf{R}[(n+1)_+]$ to $\mathrm{conv}(n_+ \times 1_+) = \mathrm{conv}(n_+) \times \mathrm{conv}(1_+) = \Delta^n \times \Delta^1 \subset \mathbf{R}[n_+] \times \mathbf{R}[1_+]$.

For any topological space X, let $i_0, i_1 \colon X \to X \times \Delta^1$ be the inclusions of X as the bottom and top of the cylinder on X.

1.15. LEMMA. $(i_0)_* = (i_1)_* \colon H_n(X) \to H_n(X \times \Delta^1)$

PROOF. Let $P \colon C_n(X) \to C_{n+1}(X \times \Delta^1)$ be the *prism operator* given by

$$P(\Delta^n \xrightarrow{\sigma} X) = \sum_{i=0}^{n} (-1)^i (\Delta^{n+1} \xrightarrow{P^i} \Delta^n \times \Delta^1 \xrightarrow{\sigma \times 1} X \times \Delta^1)$$

The prism operator is natural and, in particular, $P\sigma = (\sigma \times 1)_* P\delta_n$ where $P\delta_n = \sum_{i \in n_+} (-1)^i P^i$ is the prism on the identity map $\delta_n \in C_n(\Delta^n)$. Corollary 2.24 shows that P is a chain homotopy (Definition 1.97): The relation $\partial P = i_1 - P\partial - i_0$ holds between the abelian group homomorphisms of the diagram

$$\begin{tikzcd}
C_{n-1}(X) \arrow[dr, "P"'] & & C_n(X) \arrow[ll, "\partial"'] \arrow[d, shift right, "i_1"'] \arrow[d, shift left, "i_0"] \arrow[dr, "P"] & \\
 & & C_n(X \times \Delta^1) & C_{n+1}(X \times \Delta^1) \arrow[l, "\partial"']
\end{tikzcd}$$

But then $H_n(i_0) = H_n(i_1) \colon H_n(X) \to H_n(X \times \Delta^1)$ as chain homotopic chain maps are identical in homology (Lemma 1.98). $\square$

PROOF OF THEOREM 1.13. Suppose that f_0 and f_1 are homotopic maps of X into Y. Let $F \colon X \times \Delta^1 \to Y$ be a homotopy. The diagrams

$$\begin{tikzcd}
X \arrow[r, shift left, "i_1"] \arrow[r, shift right, "i_0"'] & X \times \Delta^1 \arrow[r, "F"] & Y \arrow[r, squiggly, "H_n"] & H_n(X) \arrow[r, shift left, "(i_1)_*"] \arrow[r, shift right, "(i_0)_*"'] & H_n(X \times \Delta^1) \arrow[r, "F_*"] & H_n(Y)
\end{tikzcd}$$

tell us that $(f_0)_* = (Fi_0)_* = F_*(i_0)_* = F_*(i_1)_* = (Fi_1)_* = (f_1)_*$. $\square$

1.16. COROLLARY. If $f \simeq g \colon (X, A) \to (Y, B)$ then $f_* = g_* \colon H_n(X, A) \to H_n(Y, B)$.

PROOF. The prism operator on the chain complexes for X and A will induce a prism operator on the quotient chain complex. $\square$

1.17. COROLLARY. Any homotopy equivalence $f \colon X \to Y$ induces an isomorphism $f_* \colon H_*(X) \to H_*(Y)$ on homology. Any homotopy equivalence $f \colon (X, A) \to (Y, B)$ induces an isomorphism $f_* \colon H_*(X, A) \to H_*(Y, B)$ on homology.

6. Excision

Let X be a topological space and $\mathcal{U} = \{U_\alpha\}$ a covering of $X = \bigcup U_\alpha$. Define the $\mathcal{U}$-small n-chains,

$$C_n^{\mathcal{U}}(X) = \sum_\alpha C_n(U_\alpha) = \mathrm{im}\left(\bigoplus_\alpha C_n(U_\alpha) \xrightarrow{+} C_n(X)\right) \subset C_n(X)$$

to be the image of the addition homomorphism from the direct sum of the chain groups $C_n(U_\alpha)$. A singular chain in X is thus $\mathcal{U}$-small if it is a sum of singular simplices with support in one of the subspaces U_α. The boundary map on $C_n(X)$ restricts to a boundary map on $C_n^{\mathcal{U}}(X)$ since the boundary operator is natural. This means that the $C_n^{\mathcal{U}}(X)$, $n \geq 0$, constitute a sub-chain complex of the singular chain complex. Let $H_n^{\mathcal{U}}(X)$ be the homology groups of $C_n^{\mathcal{U}}(X)$.

1.18. Theorem (Excision 1). *Suppose that $X = \bigcup \mathrm{int}\, U_\alpha$. The inclusion chain map $C_n^{\mathcal{U}}(X) \to C_n(X)$ induces an isomorphism of homology groups, $H_n^{\mathcal{U}}(X) \xrightarrow{\cong} H_n(X)$.*

There is also a relative version of excision. Suppose that A is a subspace of X. Let $\mathcal{U} \cap A$ denote the covering $\{U_\alpha \cap A\}$ of A. Define $C_n^{\mathcal{U}}(X, A)$ to be the quotient of $C_n^{\mathcal{U}}(X)$ by $C_n^{\mathcal{U}\cap A}(A)$. Then there is an induced chain map as in the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & C_n^{\mathcal{U}\cap A}(A) \arrow[r] \arrow[d] & C_n^{\mathcal{U}}(X) \arrow[r] \arrow[d] & C_n^{\mathcal{U}}(X,A) \arrow[r] \arrow[d, dashed] & 0 \\ 0 \arrow[r] & C_n(A) \arrow[r] & C_n(X) \arrow[r] & C_n(X,A) \arrow[r] & 0 \end{tikzcd}$$

with exact rows. Let $H_n^{\mathcal{U}}(X, A)$ be the homology groups of the chain complex $C_n^{\mathcal{U}}(X, A)$.

1.19. Corollary (Excision 2). Suppose that $X = \bigcup \mathrm{int}\, U_\alpha$. The inclusion chain map $C_n^{\mathcal{U}}(X, A) \to C_n(X, A)$ induces an isomorphism of relative homology groups, $H_n^{\mathcal{U}}(X, A) \cong H_n(X, A)$.

Proof of Corollary 1.19 assuming Theorem 1.18. The above morphism between short exact sequences of chain complexes induces a morphism

$$\begin{tikzcd} \cdots \arrow[r] & H_{n+1}^{\mathcal{U}}(X,A) \arrow[r] \arrow[d] & H_n^{\mathcal{U}\cap A}(A) \arrow[r] \arrow[d, "\cong"] & H_n^{\mathcal{U}}(X) \arrow[r] \arrow[d, "\cong"] & H_n^{\mathcal{U}}(X,A) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H_{n+1}(X,A) \arrow[r] & H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X,A) \arrow[r] & \cdots \end{tikzcd}$$

between induced long exact sequences in homology. Since two out of three of the vertical homomorphisms, $H_n^{\mathcal{U}}(X) \to H_n(X)$ and $H_n^{\mathcal{U}}(A) \to H_n(A)$, are isomorphisms by Theorem 1.18, the 5-lemma says that also the third vertical maps is an isomorphism. Note here that $A \cap \mathrm{int}\, U_\alpha \subset \mathrm{int}_A(A \cap U_\alpha)$ (General topology, below 2.35) so that we have $A = A \cap X = A \cap \bigcup \mathrm{int}\, U_\alpha = \bigcup (A \cap \mathrm{int}\, U_\alpha) \subset \bigcup \mathrm{int}_A(A \cap U_\alpha) \subset A$ so that, in fact, $A = \bigcup \mathrm{int}_A(A \cap U_\alpha)$. □

We now specialize to the case where the covering $\mathcal{U} = \{A, B\}$ consists of just two subspaces.

1.20. Theorem (Excision 3). *Let X be a topological space.*

(1) *Suppose that $X = A \cup B$ and that $X = \mathrm{int}(A) \cup \mathrm{int}(B)$. The inclusion map $(B, A \cap B) \to (A \cup B, A)$ induces an isomorphism $H_n(B, A \cap B) \xrightarrow{\cong} H_n(X, A)$ for all $n \geq 0$.*
(2) *Suppose that $U \subset A \subset X$ and that $\mathrm{cl}(U) \subset \mathrm{int}(A)$. The inclusion map $(X - U, A - U) \to (X, A)$ induces an isomorphism $H_n(X - U, A - U) \xrightarrow{\cong} H_n(X, A)$ for all $n \geq 0$.*

Proof of Theorem 1.20 assuming Corollary 1.19. (1). Let $\mathcal{U} = \{A, B\}$ be the covering consisting of the two subsets A and B. The $\mathcal{U}$-small chains are $C_n^{\mathcal{U}}(X) = C_n(A) + C_n(B) \subset C_n(X)$ and $C_n^{\mathcal{U}\cap A}(A) = C_n(A)$ since A itself is a member of the covering $\mathcal{U} \cap A = \{A, A \cap B\}$. Using Noether's isomorphism theorem

$$C_n^{\mathcal{U}}(X, A) = \frac{C_n^{\mathcal{U}}(X)}{C_n^{\mathcal{U}}(A)} = \frac{C_n(A) + C_n(B)}{C_n(A)} \xleftarrow{\cong} \frac{C_n(B)}{C_n(A) \cap C_n(B)} = \frac{C_n(B)}{C_n(A \cap B)}$$

we can identify the $\mathcal{U}$-small chain complex with the relative chain complex for the pair $(B, A \cap B)$. Thus $H_n(B, A \cap B) \cong H_n^{\mathcal{U}}(X, A)$ where $H_n^{\mathcal{U}}(X, A) \cong H_n(X, A)$ by Corollary 1.19.

(2). Let $B = X - U$ be the complement to U. Then $X = \mathrm{int}(A) \cup (X - \mathrm{int}(A)) = \mathrm{int}(A) \cup (X - \mathrm{cl}(U)) = \mathrm{int}(A) \cup \mathrm{int}(B) = A \cup B$, and $(B, A \cap B) = (X - U, A - U)$. Thus (1) $\implies$ (2). $\square$

The proof of Theorem 1.18 uses k-fold iterated subdivision in the form of a natural chain map chain homotopic to the identity

$$\mathrm{sd}^k \colon C_*(X) \to C_*(X)$$

that decomposes any simplex in X into a chain that is a sum of smaller simplices and such that the subdivision of a cycle is a homologous cycle. We shall first define subdivision in the special case where $X = \Delta^n$ is the standard geometric n-simplex and then extend the definition by naturality. The plan of the proof is

(1) Define subdivision of a linear simplex
(2) Show that linear subdivision $\mathrm{sd}\colon L_*(\Delta^n) \to L_*(\Delta^n)$ is a chain map chain homotopic to the identity
(3) Deduce that iterated linear subdivision $\mathrm{sd}^k\colon L_*(\Delta^n) \to L_*(\Delta^n)$ is a chain map chain homotopic to the identity
(4) Use naturality to define a general itertaed subdivision operator $\mathrm{sd}^k\colon C_*(X) \to C_*(X)$ chain homotopic to the identity
(5) Prove that $H_n^{\mathcal{U}}(X) \to H_n(X)$ is injective and surjective

1.21. Linear chains. A p-simplex in Δ^n is a map $\Delta^p \to \Delta^n$. A *linear* p-simplex in Δ^n is a linear map of the form $[v_0 v_1 \cdots v_p]\colon \Delta^p \to \Delta^n$ given by

$$[v_0 v_1 \cdots v_p]\left(\sum t_i e_i\right) = \sum t_i v_i, \quad t_i \geq 0, \sum t_i = 1,$$

where $v_0, v_1, \ldots, v_p$ are $p+1$ points in Δ^n. A linear chain is any finite linear combination with $\mathbf{Z}$-coefficients of linear simplices. Let $L_p(\Delta^n)$ be the abelian group of all linear chains. This is a subgroup of the abelian group $C_p(\Delta^n)$ of all singular chains. Since the boundary of a linear p-simplex

$$\partial[v_0 v_1 \cdots v_p] = \sum_{i=0}^{p} (-1)^i [v_0 \cdots \widehat{v_i} \cdots v_p]$$

is a linear $(p-1)$-chain, we have a (sub)chain complex

$$\cdots \xrightarrow{\partial} L_p(\Delta^n) \xrightarrow{\partial} L_{p-1}(\Delta^n) \xrightarrow{\partial} \cdots \xrightarrow{\partial} L_0(\Delta^n) \xrightarrow{\partial} 0$$

of linear chains. As usual the boundary of any 0-simplex is defined to 0. It will be more convenient for us to work with the *augmented* linear chain complex

$$\cdots \xrightarrow{\partial} L_p(\Delta^n) \xrightarrow{\partial} L_{p-1}(\Delta^n) \xrightarrow{\partial} \cdots \xrightarrow{\partial} L_0(\Delta^n) \xrightarrow{\partial} L_{-1}(\Delta^n) \tag{1.22}$$

where $L_{-1}(\Delta^n) \cong \mathbf{Z}$ is generated by the (-1)-simplex $[\emptyset]$ with 0 vertices and the boundary of any 0-simplex $\partial[v_0] = [\emptyset]$.

1.23. Cones on linear chains. Fix a point $b \in \Delta^n$. Define the cone on b to be the linear map

$$b\colon L_p(\Delta^n) \to L_{p+1}(\Delta^n), \quad p \geq -1,$$
$$b[v_0 v_1 \cdots v_p] = [b v_0 v_1 \cdots v_p]$$

that adds b as an extra vertex on any linear simplex. The cone operator is a chain homotopy (Definition 1.97) between the identity map and the zero map: We compute the boundary of a cone.

1.24. Lemma. For all $p \geq -1$

$$\begin{tikzcd} L_{p-1}(\Delta^n) \arrow[dr, "b"'] & L_p(\Delta^n) \arrow[l, "\partial"'] \arrow[d, shift right, "1"'] \arrow[d, shift left, "0"] \arrow[dr, "b"] & \\ & L_p(\Delta^n) & L_{p+1}(\Delta^n) \arrow[l, "\partial"] \end{tikzcd}$$

$$\partial b = 1 - b\partial - 0\colon L_p(\Delta^n) \to L_p(\Delta^n)$$

The boundary of a cone is the base and the cone on the boundary of the base

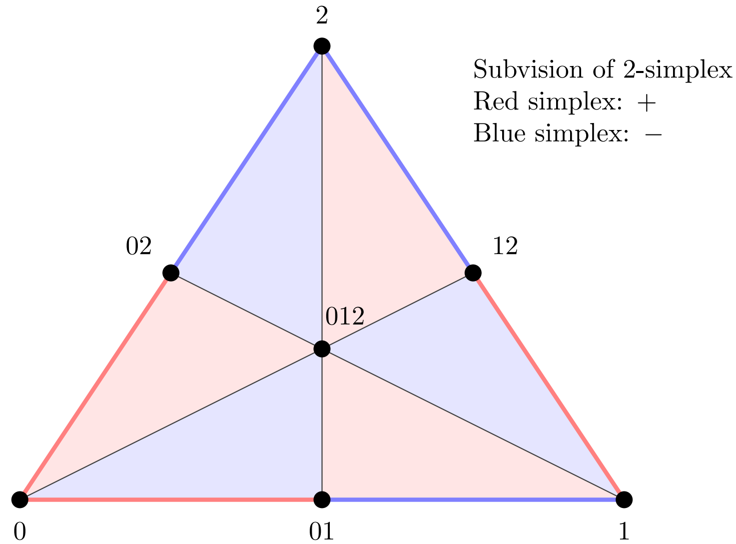


FIGURE 3. Barycentric subdivision of a 2-simplex

PROOF. It is enough to compute the boundary on the cone $\partial b[v_0 \cdots v_p]$ of a linear simplex. For the (-1)-simplex, $\partial b[\emptyset] = \partial[b] = [\emptyset] = (1 - b\partial)[\emptyset]$. For a 0-simplex, $\partial b[v_0] = \partial[bv_0] = [v_0] - [b] = [v_0] - b[\emptyset] = [v_0] - b\partial[v_0] = (\mathrm{id} - b\partial)[v_0]$. For $p > 0$ we take any linear p-simplex $\sigma = [v_0 \cdots v_p] \in L_p(\Delta^n)$ and clearly

$$\partial\partial[bv_0 \cdots v_p] = [v_0 \cdots v_p] - b\partial[v_0 \cdots v_p] = (1 - b\partial - 0)[v_0 \cdots v_p]$$

as asserted. □

Thus the augmented linear chain complex of Δ^n is exactness (the identity map induces the zero map in homology).

1.25. Subdivision of linear chains. The *barycentre* of the linear simplex $\sigma = [v_0 \cdots v_p]$ is the point

$$b(\sigma) = \frac{1}{p+1}(v_0 + \cdots + v_p) = v_{01\cdots p},$$

the center of gravity.

The *subdivision* operator is the linear map $\mathrm{sd}\colon L_p(\Delta^n) \to L_p(\Delta^n)$ defined recursively by

$$\mathrm{sd}(\sigma) = \begin{cases} \sigma & p = -1, 0 \\ b(\sigma)\,\mathrm{sd}(\partial\sigma) & p > 0 \end{cases} \tag{1.26}$$

This means that the subdivision of a point is a point and the subdivision of a p-simplex for $p > 0$ is the cone with vertex in the barycentre on the subdivision of the boundary. For instance

$$\begin{aligned} \mathrm{sd}[v_0, v_1] &= v_{01}([v_1] - [v_0]) = [v_{01}v_1] - [v_{01}v_0] \\ \mathrm{sd}[v_0, v_1, v_2] &= v_{012}\,\mathrm{sd}([v_1v_2] - [v_0v_2] + [v_0v_1]) \\ &= [v_{012}v_{12}v_2] - [v_{012}v_{12}v_1] - [v_{012}v_{02}v_2] + [v_{012}v_{02}v_0] + [v_{012}v_{01}v_1] - [v_{012}v_{01}v_0] \end{aligned}$$

1.27. LEMMA. sd is a chain map: $\partial\,\mathrm{sd} = \mathrm{sd}\,\partial$.

PROOF. The claim is that the boundary of the subdivision is the subdivision of the boundary. In degree -1 and 0, this is clear as sd is the identity map there. Assume that $\mathrm{sd}\,\partial = \partial\,\mathrm{sd}$ in all degrees $< p$ and let σ be a linear p-simplex in Δ^n. Then

$$\begin{aligned} \partial\,\mathrm{sd}\,\sigma \overset{\text{def}}{=} \partial\,(b(\sigma)\,\mathrm{sd}(\partial\sigma)) &\overset{1.24}{=} (\mathrm{id} - b(\sigma)\partial)\,\mathrm{sd}(\partial\sigma) = \mathrm{sd}(\partial\sigma) - b(\sigma)\partial\,\mathrm{sd}(\partial\sigma) \\ &\overset{\text{induction}}{=} \mathrm{sd}(\partial\sigma) - b(\sigma)\,\mathrm{sd}(\partial\partial\sigma) \overset{\partial\partial=0}{=} \mathrm{sd}(\partial\sigma) \end{aligned}$$

since the induction hypothesis applies to $\partial\sigma$ which has degree $p - 1$. □

We now show that subdivision is chain homotopic to the identity map. We need to find morphisms $T\colon L_p(\Delta^n) \to L_{p+1}(\Delta^n)$, $p \geq 0$, such that $\partial T = 1 - T\partial - \mathrm{sd}$. We may let $T = 0$ in degree $p = -1$ and $T(v_0) = (v_0 v_0)$ in degree $p = 0$. Then the formula holds in degree -1 and 0.

Define $T\colon L_p(\Delta^n) \to L_{p+1}(\Delta^n)$ recursively by

$$T(\sigma) = \begin{cases} 0 & p = -1 \\ b(\sigma)(\sigma - T\partial\sigma) & p \geq 0 \end{cases} \tag{1.28}$$

for all $\sigma \in L_p(\Delta^n)$

1.29. Lemma. For all $p \geq -1$

$$\begin{tikzcd} L_{p-1}(\Delta^n) \arrow[dr, "T"'] & L_p(\Delta^n) \arrow[l, "\partial"'] \arrow[d, shift right, "1"'] \arrow[d, shift left, "\mathrm{sd}"] \arrow[dr, "T"] & \\ & L_p(\Delta^n) & L_{p+1}(\Delta^n) \arrow[l, "\partial"] \end{tikzcd}$$

$$\partial T = 1 - T\partial - \mathrm{sd}\colon L_p(\Delta^n) \to L_p(\Delta^n)$$

T is a chain homotopy between subdivision and the identity

Proof. The formula holds in degree $p = -1$ where both sides are 0. The formula also holds in degree $p = 0$, since $T[v_0] = v_0[v_0] = [v_0 v_0]$ so that $(T\partial + \partial T)[v_0] = T[\emptyset] + \partial[v_0 v_0] = 0 + 0 = 0 = (1 - \mathrm{sd})[v_0]$. Assume now inductively that the formula holds in degrees $< p$ and let σ be a linear p-simplex in Δ^n. Then

$$\begin{aligned} \partial T\sigma = \partial(b(\sigma - T\partial\sigma)) = \sigma - T\partial\sigma - b\partial(\sigma - T\partial\sigma) = \sigma - T\partial\sigma - b(\partial\sigma - \partial T\partial\sigma) &\overset{1-\partial T = T\partial + \mathrm{sd}}{=} \\ \sigma - T\partial\sigma - b(T\partial\partial\sigma + \mathrm{sd}\,\partial\sigma) = \sigma - T\partial\sigma - b\,\mathrm{sd}\,\partial\sigma &= \sigma - T\partial\sigma - \mathrm{sd}\,\sigma \end{aligned}$$

where we use Lemma 1.24 and the induction hypothesis. □

1.30. Iterated subdivision of linear chains. For any $k \geq 0$ let

$$\mathrm{sd}^k = \overbrace{\mathrm{sd} \circ \cdots \circ \mathrm{sd}}^{k}\colon L_p(\Delta^n) \to L_p(\Delta^n)$$

be the kth fold iterate of the subdivision operator sd. Since sd is a chain map chain homotopic to the identity map, its k-fold iteration, sd^k, is also a chain map chain homotopic to the identity (Lemma 1.99). Let T_k be a chain homotopy so that

$$\partial T_k = 1 - T_k\partial - \mathrm{sd}^k\colon L_p(\Delta^n) \to L_p(\Delta^n) \tag{1.31}$$

for all $p \geq -1$.

The *diameter* of a compact subspace of a metric space is the maximum distance between any two points of the subspace.

1.32. Lemma. Let $\sigma = [v_0 \cdots v_p]$ be a linear simplex in Δ^n. Any simplex in the chain $\mathrm{sd}^k(\sigma)$ has diameter $\leq \left(\frac{n}{n+1}\right)^k \mathrm{diam}(\sigma)$.

I omit the proof. The important thing to notice is that iterated subdivision will produce simplices of arbitrarily small diameter because the fraction $n/(n+1) < 1$.

1.33. Subdivision for general spaces. Let now X be an arbitrary topological space. For any singular n-simplex $\sigma\colon \Delta^n \to X$ in X we define

$$\mathrm{sd}^k(\sigma) = \sigma_*(\mathrm{sd}^k\,\delta_n) \in C_n(X), \quad T_k(\sigma) = \sigma_*(T_k\delta_n) \in C_{n+1}(X)$$

where, as usual, $\sigma_*\colon L_p(\Delta^n) \to C_p(X)$ is the chain map induced by σ and $\delta_n \in L_n(\Delta^n)$ is the linear n-simplex on the standard n-simplex Δ^n that is identity map $\delta_n = [e_0 \cdots e_n]\colon \Delta^n \to \Delta^n$.

1.34. Lemma. $\mathrm{sd}^k\colon C_n(X) \to C_n(X)$ is a natural chain map, $\partial\,\mathrm{sd}^k = \mathrm{sd}^k\,\partial$, and $T_k\colon C_n(X) \to C_{n+1}(X)$ is a natural chain homotopy, $\partial T_k = 1 - T_k\partial - \mathrm{sd}^k$.

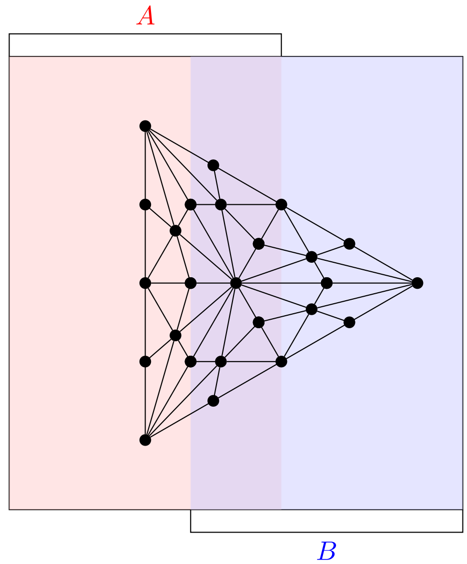


FIGURE 4. Subdivided simplices in $X = \operatorname{int}(A) \cup \operatorname{int}(B)$ are $\mathcal{U}$-small

PROOF. It is a completely formal matter to check this. It is best first to verify that subdivision is natural. Let $\sigma\colon \Delta^n \to X$ be a singular n-simplex in X and let $f\colon X \to Y$ be a map. The computation

$$f_* \operatorname{sd}^k(\sigma) \overset{\text{def}}{=} f_*\sigma_* \operatorname{sd}^k(\delta^n) = (f\sigma)_* \operatorname{sd}^k(\delta^n) \overset{\text{def}}{=} \operatorname{sd}^k(f\sigma) = \operatorname{sd}^k(f_*\sigma)$$

shows that subdivision is natural. Next we show that subdivision is a chain map:

$$\begin{aligned}
\partial(\operatorname{sd}^k \sigma) &= \partial\sigma_* \operatorname{sd}^k(\delta^n) && \text{by definition}\\
&= \sigma_*\partial \operatorname{sd}^k(\delta_n) && \sigma_* \text{ is a chain map}\\
&= \sigma_* \operatorname{sd}^k(\partial\delta_n) && \operatorname{sd}^k \text{ is a chain map on linear chains}\\
&= \operatorname{sd}^k \sigma_*(\partial\delta_n) && \operatorname{sd}^k \text{ is natural}\\
&= \operatorname{sd}^k \partial\sigma_*(\delta_n) && \sigma_* \text{ is a chain map}\\
&= \operatorname{sd}^k(\partial\sigma)
\end{aligned}$$

Similarly, we see that T_k is natural because

$$f_*T_k\sigma \overset{\text{def}}{=} f_*\sigma_*T_k\delta_n = (f\sigma)_*T_k\delta_n \overset{\text{def}}{=} T_k(f\sigma) = T_k(f_*\sigma)$$

It then follows that

$$T_k\partial\sigma = T_k\partial\sigma_*\delta_n = T_k\sigma_*\partial\delta_n \overset{T_k \text{ natural}}{=} \sigma_*T_k\partial\delta_n$$

and

$$\partial T_k\sigma \overset{\text{def}}{=} \partial\sigma_*T_k\delta_n \overset{\sigma_* \text{ chain map}}{=} \sigma_*\partial T_k\delta_n$$

We conclude that

$$(\partial T_k + T_k\partial)\sigma = \sigma_*(\partial T_k + T_k\partial)\delta_n \overset{(1.98)}{=} \sigma_*(\mathrm{id} - \operatorname{sd}^k)\delta_n = (\mathrm{id} - \operatorname{sd}^k)\sigma$$

and this finishes the proof. □

1.35. COROLLARY. The subdivision of an n-cycle is a homologous n-cycle: If $z \in C_n(X)$ is an n-cycle in X, then $\operatorname{sd}^k z = z - \partial T_k z$.

PROOF. Let $z \in C_n(X)$ be an n-cycle, $\partial z = 0$. Then $z - \operatorname{sd}^k z = (1 - \operatorname{sd}^k)z = (\partial T_k + T_k\partial)z = \partial T_k z$. □

1.36. Proof of Theorem 1.18. Let X be a topological space and $\mathcal{U} = \{U_\alpha\}$ a covering of X.

1.37. Lemma. Suppose that $X = \bigcup \operatorname{int} U_\alpha$. Let $c \in C_n(X)$ be any singular chain in X. There exists a $k \gg 0$ (depending on c) such that the k-fold subdivided chain $\operatorname{sd}^k(c)$ is $\mathcal{U}$-small.

Proof. It is enough to show that $\operatorname{sd}^k(\sigma) = \sigma_*(\operatorname{sd}^k(\delta_n))$ is $\mathcal{U}$-small when k is big enough for any singular simplex $\sigma \colon \Delta^n \to X$. Choose k so big that the diameter of each simplex in the chain $\operatorname{sd}^k(\delta_n)$ is smaller than the Lebesgue number (General topology, 2.158) of the open covering $\{\sigma^{-1}(\operatorname{int} U_\alpha)\}$ of the compact metric space Δ^n. $\square$

Proof of Theorem 1.18. We show that the induced homomorphism $H_n^{\mathcal{U}}(X) \to H_n(X)$ is surjective and injective.
Surjective: Let z be any n-cycle. For $k \gg 0$, the subdivided chain $\operatorname{sd}^k(z)$ is a $\mathcal{U}$-small cycle in the same homology class as z (Lemma 1.37, Corollary 1.35).
Injective: Let $b \in C_n^{\mathcal{U}}(X)$ be a $\mathcal{U}$-small n-cycle, $\partial b = 0$, which is a boundary in $C_n(X)$, $b = \partial c$ for some $c \in C_{n+1}(X)$. The situation is sketched in the commutative diagram

$$\begin{tikzcd} C_{n-1}^{\mathcal{U}}(X) \arrow[d, hook] & C_n^{\mathcal{U}}(X) \arrow[l, "\partial"'] \arrow[d, hook] & C_{n+1}^{\mathcal{U}}(X) \arrow[l, "\partial"'] \arrow[d, hook] \\ C_{n-1}(X) & C_n(X) \arrow[l, "\partial"'] & C_{n+1}(X) \arrow[l, "\partial"'] \end{tikzcd}$$

where the rows are chain complexes and the down arrows are inclusions.

The subdivided chain $\operatorname{sd}^k c$ is $\mathcal{U}$-small for $k \gg 0$ (Lemma 1.37) and its boundary is $\partial \operatorname{sd}^k c = \operatorname{sd}^k \partial c = \operatorname{sd}^k b = b - \partial T_k b$ (Corollary 1.35). Thus $b = \partial \operatorname{sd}^k c + \partial T_k b$ is a boundary in $C_*^{\mathcal{U}}(X)$ since $\operatorname{sd}^k c$ is $\mathcal{U}$-small and also $T_k b$ is $\mathcal{U}$-small since b is $\mathcal{U}$-small and T_k natural (Corollary 1.35). $\square$

1.38. The Mayer–Vietoris sequence. Our first application of the excision axiom is the Mayer–Vietoris sequence.

1.39. Corollary (The Mayer–Vietoris sequence). Suppose that $X = A \cup B = \operatorname{int} A \cup \operatorname{int} B$. Then there is a long exact sequence

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(A \cup B) \xrightarrow{\partial} H_{n-1}(A \cap B) \to \cdots$$

for the homology of $X = A \cup B$. There is a similar sequence for reduced homology.

Proof. Note that there is short exact sequence of chain complexes

$$0 \to C_n(A \cap B) \xrightarrow{\sigma \to (\sigma, -\sigma)} C_n(A) \oplus C_n(B) \xrightarrow{(\sigma,\tau) \to \sigma + \tau} C_n(A) + C_n(B) \to 0$$

producing a long exact sequence in homology. The homology of the chain complex to the right is isomorphic to $H_n(X)$ by excision (1.18) applied to the covering $\{A, B\}$ where $C_n^{\{A,B\}}(X) = C_n(A) \oplus C_n(B)$. In degree $n = 0$ we may use the short exact sequences $0 \to 0 \oplus 0 \to 0 \to 0$ or $0 \to \mathbf{Z} \xrightarrow{(1,-1)} \mathbf{Z} \oplus \mathbf{Z} \xrightarrow{+} \mathbf{Z} \to 0$. In the second case, we get the Mayer–Vietoris sequence in reduced homology. $\square$

1.40. The long exact sequence for a quotient space. For good pairs the relative homology groups actually are the homology groups of the quotient space.

1.41. Definition. (X, A) is a *good pair* if the subspace A is closed and is the deformation retract of an open neighborhood $V \supset A$.

Suppose that (X, A) is a good pair and the open subspace V deformation retracts onto the closed subspace A. The quotient map $q \colon X \to X/A$ is a map of triples $q \colon (X, V, A) \to (X/A, V/A, A/A)$ where

- A/A is a point
- V/A is an open subspace that deformation retracts onto the closed point A/A
- the restriction of q to the complement of A is a homeomorphism between $X - A$ and $X/A - A/A$ and between $V - A$ and $V/A - A/A$

Consult (General topology, 2.84.(4)) to verify these facts.

1.42. Proposition (Relative homology as homology of a quotient space). Let (X, A) be a good pair and $A \neq \emptyset$. Then the quotient map $(X, A) \to (X/A, A/A)$ induces an isomorphism $H_n(X, A) \to H_n(X/A, A/A) = \widetilde{H}_n(X/A)$ on homology.

Proof. Look at the commutative diagram

A a deformation retract of V excision of A

$$\begin{tikzcd}
H_*(X,A) \arrow[r, "\cong"'] \arrow[d] & H_*(X,V) \arrow[d] & H_*(X-A, V-A) \arrow[l, "\cong"'] \arrow[d, "\cong"', "q \text{ a homeomorphism } X-A \to X/A - A/A"] \\
H_*(X/A, A/A) \arrow[r, "\cong"] & H_*(X/A, V/A) & H_*(X/A - A/A, V/A - A/A) \arrow[l, "\cong"]
\end{tikzcd}$$

A/A a deformation retract of V/A excision of A/A

where the horizontal maps and the right vertical map are isomorphisms. □

The long exact sequence for the quotient space of a nice pair (X, A),

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} \widetilde{H}_n(X/A) \to H_{n-1}(A) \to \cdots$$

is obtained from the pair sequence (1.10) by replacing $H_n(X, A)$ by $\widetilde{H}_n(X/A)$. (There is a similar long exact sequence in reduced homology.) This exact sequence is in fact a special case of the following slightly more general long exact sequence. (See Homotopy theory for beginners for mapping cones).

1.43. Corollary (The long exact sequence of a map). For any map $f \colon X \to Y$ there is a long exact sequence

$$\cdots \to H_n(X) \xrightarrow{f_*} H_n(Y) \xrightarrow{q_*} \widetilde{H}_n(C_f) \to H_{n-1}(X) \to \cdots$$

where C_f is the mapping cone of f.

1.44. Homology of spheres. The homology groups of the spheres are the most important of all homology groups.

1.45. Corollary (Homology of spheres). The homology groups of the n-sphere S^n, $n \geq 0$, are

$$\widetilde{H}_i(S^n) = \begin{cases} \mathbf{Z} & i = n \\ 0 & i \neq n \end{cases}$$

Proof. The long exact sequence in reduced homology for the good pair (D^n, S^{n-1}) gives that $\widetilde{H}_i(S^n) = \widetilde{H}_i(D^n/S^{n-1}) \cong \widetilde{H}_{i-1}(S^{n-1})$ because $\widetilde{H}_i(D^n) = 0$ for all i. Use this equation n times to get $\widetilde{H}_i(S^n) \cong \widetilde{H}_{i-n}(S^0)$. This homology group is nontrivial if and only if $i = n$. □

1.46. Corollary (Homology of a wedge). Let (X_α, x_α), $\alpha \in A$, be a set of based spaces. For all $n \geq 0$,

$$\bigoplus \widetilde{H}_n(X_\alpha) \cong \widetilde{H}_n(\bigvee X_\alpha)$$

provided that each pair $(X_\alpha, \{x_\alpha\})$ is a good pair.

Proof. $\bigoplus \widetilde{H}_n(X_\alpha) \overset{1.7}{\cong} \bigoplus H_n(X_\alpha, x_\alpha) \cong H_n(\coprod X_\alpha, \coprod \{x_\alpha\}) = H_n(\coprod X_\alpha / \coprod \{x_\alpha\}) = \widetilde{H}_n(\bigvee X_\alpha)$. □

1.47. Corollary (Homology of a suspension). $\widetilde{H}_{n+1}(SX) \underset{\cong}{\xrightarrow{\partial}} \widetilde{H}_n(X)$ for any space $X \neq \emptyset$.

Proof. The suspension, $SX = (X \times [-1, +1])/(X \times \{-1\}, X \times \{+1\}) = C_-X \cup C_+X$ is the union of two cones, $C_-X = X \times [-1, 1/2]/X \times \{-1\}$ and $C_+X = X \times [-1/2, +1]/X \times \{+1\}$, whose intersection $C_-X \cap C_+X = X \times [-1/2, 1/2]$ deformation retracts onto X. The Mayer–Vietoris sequence (1.39) in reduced homology gives the isomorphism. □

We may also use 1.47 to compute the homology groups of spheres as $SS^{n-1} = C_+S^{n-1} \cup_{S^{n-1}} C_+S^{n-1} = D^n \cup_{S^{n-1}} D^n = S^n$.

The Mayer–Vietoris sequence is natural with respect to maps of triples (X, A, B).

1.48. Lemma. The reflection map $R\colon SX \to SX$ given by $R([x,t]) = [x,-t]$ induces multiplication by $R_* = -1$ on the reduced homology groups of SX.

Proof. R is a map of the triple (SX, C_-X, C_+X) to the triple (SX, C_+X, C_-X) that is the identity on $X \subset C_-X \cap C_+X$. Let $\partial\colon \widetilde{H}_{n+1}(SX) \to \widetilde{H}_n(X)$ be the isomorphism associated to the first triple. From the construction of the Mayer–Vietoris sequence we see that the isomorphism associated to the second triple is $-\partial$. (Look at the zig-zag relations defining the connecting homomorphisms.) Thus R induces a commutative diagram

$$\begin{tikzcd} \widetilde{H}_{n+1}(SX) \arrow[r, "\partial", "\cong"'] \arrow[d, "R_*"'] & \widetilde{H}_n(X) \arrow[d, equal] \\ \widetilde{H}_{n+1}(SX) \arrow[r, "\cong", "-\partial"'] & \widetilde{H}_n(X) \end{tikzcd}$$

and we conclude that $R_* = -1$. $\square$

1.49. Corollary. A reflection $R\colon S^n \to S^n$ of an n-sphere, $n \geq 0$, induces multiplication by -1 on the nth reduced homology group $\widetilde{H}_n(S^n)$.

1.50. Proposition (Generating homology classes). The connecting isomorphism $\partial\colon H_n(\Delta^n, \partial\Delta^n) \to \widetilde{H}_{n-1}(\partial\Delta^n)$ is an isomorphism and both homology groups are isomorphic to $\mathbf{Z}$ when $n \geq 1$.

(1) The relative homology group $H_n(\Delta^n, \partial\Delta^n)$ is generated by the homology class $[\delta^n]$
(2) The reduced homology group $\widetilde{H}_{n-1}(\partial\Delta^n)$ is generated by the homology class $[\sum_{i\in n_+}(-1)^i d^i]$

Proof. The identity n-simplex δ^n is indeed a relative n-cycle (1.9) because its boundary $\partial\delta^n = \sum_{i\in n_+}(-1)^i d^i$ is an $(n-1)$-chain in the boundary $\partial\Delta^n$ of Δ^n. By definition of the connecting homomorphism we have $\partial[\delta^n] = [\sum_{i\in n_+}(-1)^i d^i]$ on homology. When $n = 1$, $\partial[\delta^1] = [d^0 - d^1]$ is manifestly a generator of $\widetilde{H}_0(S^0)$. To proceed, for $n \geq 2$, let $\Lambda_0 \subseteq \partial\Delta^n$ be the union of all faces of Δ^n but $d^0\Delta^{n-1}$. The coface map d^0 is a map of (good) pairs $d^0\colon (\Delta^{n-1}, \partial\Delta^{n-1}) \to (\partial\Delta^n, \Lambda_0)$ and the induced map on homology $d^0_*\colon H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}) \to H_{n-1}(\partial\Delta^n, \Lambda_0)$ is an isomorphism since the induced map $\Delta^{n-1}/\partial\Delta^{n-1} \to \partial\Delta^n/\Lambda_0$ is a homeomorphism. The isomorphisms

Long exact sequence $\quad$ Λ_0 is contractible $\quad$ $\partial\Delta^n/\Lambda_0 = \Delta^{n-1}/\partial\Delta^{n-1}$

$$\begin{tikzcd} H_n(\Delta^n, \partial\Delta^n) \arrow[r, "\partial", "\cong"'] & H_{n-1}(\partial\Delta^n) \arrow[r, "\cong"'] & H_{n-1}(\partial\Delta^n, \Lambda_0) & H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}) \arrow[l, "d^0_*"', "\cong"] \\ {[\delta^n]} \arrow[r] & {[\sum_{i\in n_+}(-1)^i d^i]} \arrow[r] & {[d^0] = d^0[\delta^{n-1}]} & {[\delta^{n-1}]} \arrow[l] \end{tikzcd}$$

Definition of connecting homomorphism ∂ in long exact sequence $\quad$ $\sum_{i>0}(-1)^i d^i \in C_{n-1}(\Lambda_0)$

show that $[\delta^{n-1}]$ generates $H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$ if and only $[\delta^n]$ generates $H_n(\Delta^n, \partial\Delta^n)$. An induction argument now completes the proof. $\square$

1.45. Local homology groups. Let X be a topological space where points are closed, for instance a Hausdorff space. Let x be a point of X. The *local homology group of X at x* is the relative homology group $H_i(X, X - x)$.

1.51. Proposition. If U is any open neighborhood of x then $H_i(U, U - x) \cong H_i(X, X - x)$.

Proof. Excise the closed set $X - U$ from the open set $X - x$. $\square$

1.52. Proposition (Local homology groups of manifolds). Let M be an m-manifold and x a point of M. Then

$$H_i(M, M - x) = \begin{cases} \mathbf{Z} & i = m \\ 0 & i \neq m \end{cases}$$

Proof. $H_i(U, U - x) \cong H_i(\mathbf{R}^m, \mathbf{R}^m - x) \cong \widetilde{H}_{i-1}(\mathbf{R}^m - x) \cong \widetilde{H}_{i-1}(S^{m-1})$ when U is an open neighborhood of x homeomorphic to $\mathbf{R}^m$. □

7. Easy applications of singular homology

1.53. Proposition. S^{n-1} is not a retract of D^n, $n \geq 1$.

Proof. Assume that $r \colon D^n \to S^{n-1}$ is a retraction.

$$\begin{tikzcd} D^n \arrow[r, "r"] & S^{n-1} \\ S^{n-1} \arrow[u, hook] \arrow[ur, equal] & \end{tikzcd} \qquad \begin{tikzcd} \widetilde{H}_{n-1}(D^n) \arrow[r, "r_*"] & \widetilde{H}_{n-1}(S^{n-1}) \\ \widetilde{H}_{n-1}(S^{n-1}) \arrow[u] \arrow[ur, equal] & \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & \mathbf{Z} \\ \mathbf{Z} \arrow[u] \arrow[ur, equal] & \end{tikzcd}$$

□

1.54. Corollary (Brouwer's fixed point theorem). Any self-map of D^n, $n \geq 0$, has a fixed point.

Proof. Suppose that f is a self-map of D^n with no fixed points. For any $x \in D^n$ let $r(x) \in S^{n-1}$ be the point on the boundary on the ray from $f(x)$ to x. Then $r \colon D^n \to S^{n-1}$ is a retraction of D^n onto S^{n-1}. □

1.55. Corollary (Homeomorphism type of Euclidean spaces). Let $U \subset \mathbf{R}^m$ and $V \subset \mathbf{R}^n$ be two nonempty open subsets of $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively. Then

$$U \text{ and } V \text{ are homeomorphic} \implies m = n$$

In particular: $\mathbf{R}^m$ and $\mathbf{R}^n$ are homeomorphic $\iff$ $m = n$.

Proof. If $U \subset R^m$ and $V \subset \mathbf{R}^n$ are homeomorphic, then their local homology groups are isomorphic so $m = n$ by Proposition 1.52. □

8. The degree of a self-map of the sphere

Let $f \colon S^n \to S^n$ be a map of an n-sphere, $n \geq 1$, into itself and let $f_* \colon H_n(S^n) \to H_n(S^n)$ be the induced map on the nth homology group. Choose a generator $[S^n]$ of $H_n(S^n) \cong \mathbf{Z}$. Then

$$f_*([S^n]) = (\deg f) \cdot [S^n]$$

for a unique integer $\deg f \in \mathbf{Z}$ called the *degree* of f.

1.56. Lemma. We note these properties of the degree:

(1) The degree of a map $f \colon S^n \to S^n$ only depends on the homotopy class of f.
(2) The degree is a map $\deg \colon [S^n, S^n] \to \mathbf{Z}$.
(3) The degree is multiplicative in the sense that $\deg(\mathrm{id}) = 1$ and $\deg(g \circ f) = \deg(g)\deg(f)$.
(4) The degree of a reflection is -1.
(5) The degree of a homotopy equivalence is ± 1.
(6) The degree of the antipodal map -1 is $(-1)^{n+1}$.
(7) Any map $f \colon S^n \to S^n$ without fixed points is homotopic to -1 and has degree $(-1)^{n+1}$.
(8) Any map $f \colon S^n \to S^n$ of degree not equal to $(-1)^{n+1}$ has a fixed point.
(9) Any map $f \colon S^n \to S^n$ of nonzero degree is surjective.
(10) For any integer d there is a self-map of S^n, $n > 0$, of degree d.

Proof. The degree is obviously multiplicative because $\deg(g \circ f)[S^n] = (g \circ f)_*([S^n]) = g_*(f_*([S^n])) = g_*(\deg(f)[S^n]) = \deg(f)g_*([S^n]) = \deg(f)\deg(g)[S^n]$. We showed in Corollary 1.49 that reflections have degree -1. The linear map $-\mathrm{id}$ on $S^n \subset \mathbf{R}^{n+1}$ has degree $(-1)^{n+1}$ since it is the composition of $n+1$ reflections. If f has no fixed points then the line segment in $\mathbf{R}^{n+1}$ connecting $f(x)$ and $-x$ does not pass through 0 (for $f(x)$ is not opposite $-x$) so that we may construct a linear homotopy between f and $-\mathrm{id}$ considered as maps into $\mathbf{R}^{n+1} - \{0\}$. The normalization

$$H(x,t) = \frac{(1-t)f(x) - tx}{|(1-t)f(x) - tx|}$$

of such a homotopy is a homotopy between f and $-\mathrm{id}$ (as maps $S^n \to S^n$). If a self map f of the sphere S^n is not surjective, say x_0 is not in the image of f, then f factors through the contractible space $S^n - \{x_0\} = \mathbf{R}^n$, so that f is nullhomotopic. A self-map of positive degree d is

$$S^n \xrightarrow{\nabla} \underbrace{S^n \vee \ldots \vee S^n}_{d} \xrightarrow{\nu} S^n$$

where ∇ is a pinch and ν a folding map. Compose with a reflection to get maps of negative degrees. $\square$

1.57. Local degree. Suppose that the map $f\colon S^n \to S^n$ has the (ubiquitous) property that $f^{-1}(y)$ is finite for some point $y \in S^n$, say $f^{-1}(y) = \{x_1, \ldots, x_m\}$. Let $V \subset S^n$ be an open neighborhood of y (eg $V = S^n$) and let $U_i \subset S^n$ be disjoint open neighborhoods of x_i (eg small discs) such that $f(U_i) \subset V$ for all $i = 1, \ldots, m$. Then f maps $U_i - x_i$ into $V - y$ for x_i is the only point in U_i that hits y. Define $[S^n]|x_i$ to be the generator of the local homology group $H_n(U_i, U_i - x_i)$ corresponding to $[S^n] \in H_n(S^n)$ under the isomorphism of the left column and $[S^n]|y$ to be the generator of the local homology group $H_n(V, V-y)$ corresponding to $[S^n] \in H_n(S^n)$ under the isomorphism of the right column of the diagram

$$\begin{tikzcd}
\boxed{[S^n]|x_i} \arrow[rrr, "(f|U_i)_*"] \arrow[ddd, leftrightarrow] & & & \boxed{(\deg f|x_i)([S^n]|y)} \arrow[ddd, leftrightarrow] \\
& H_n(U_i, U_i - x_i) \arrow[r, "(f|U_i)_*"] \arrow[d, "\cong"', "\mathrm{exc}"] & H_n(V, V-y) \arrow[d, "\mathrm{exc}"', "\cong"] & \\
& H_n(S^n, S^n - x_i) & H_n(S^n, S^n - y) & \\
\boxed{[S^n]} & H_n(S^n) \arrow[u, "\cong", "j_*"'] & H_n(S^n) \arrow[u, "j_*"'] & \boxed{(\deg f|x_i)[S^n]}
\end{tikzcd}$$

The *local degree* of f at x_i, $\deg f|x_i$, the integer such that

$$(f|U_i)_*([S^n]|x_i) = (\deg f|x_i) \cdot ([S^n]|y),$$

only depends on f near the point x_i.

1.58. Theorem (Computation of degree). $\deg f = \sum \deg f|x_i$

Proof. The commutative diagram of maps between topological spaces

$$\begin{tikzcd}
(\coprod U_i, \coprod U_i - x_i) \arrow[r, "f"] \arrow[d, hook] & (V, V-y) \arrow[d, hook] \\
(S^n, S^n - f^{-1}y) \arrow[r, "f"] & (S^n, S^n - y) \\
(S^n, \emptyset) \arrow[r, "f"] \arrow[u, hook] & (S^n, \emptyset) \arrow[u, hook]
\end{tikzcd}$$

induces a commutative diagram of homology groups

$$\begin{tikzcd}
\boxed{\bigoplus [S^n]|x_i} \arrow[rrr] & & & \boxed{\bigoplus (\deg f|x_i)[S^n]|y} \arrow[dddd, leftrightarrow] \\
& \bigoplus H_n(U_i, U_i - x_i) \arrow[r, "\sum (f \mid U_i)_*"] \arrow[d, "\sum \mathrm{incl}_*"', "\cong"] & \bigoplus H_n(V, V-y) \arrow[d, equal] & \\
& H_n(\coprod U_i, \coprod U_i - x_i) \arrow[r, "f_*"] \arrow[d, "\text{excision}"', "\cong"] & H_n(V, V-y) \arrow[d, "\cong"', "\text{excision}"] & \\
& H_n(S^n, S^n - f^{-1}y) \arrow[r, "f_*"] & H_n(S^n, S^n - y) & \\
\boxed{[S^n]} \arrow[uuuu] & H_n(S^n) \arrow[r, "f_*"] \arrow[u] & H_n(S^n) \arrow[u] & \boxed{\sum (\deg f|x_i)[S^n]}
\end{tikzcd}$$

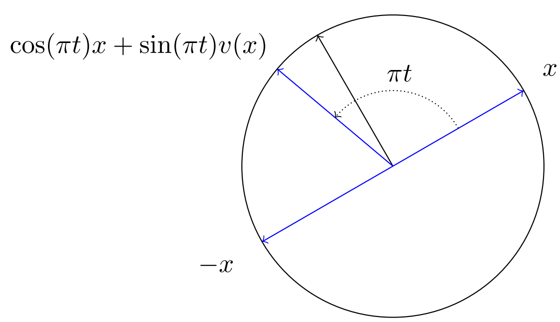
showing that the degree of f is the sum of the local degrees. $\square$

1.59. COROLLARY. The degree of $z \to z^d \colon S^1 \to S^1$ is d.

1.58. Vector fields on spheres.

1.60. THEOREM (Hairy Ball Theorem). *There exists a function $v \colon S^n \to S^n$ such that $v(x) \perp x$ for all $x \in S^n$ if and only if n is odd. (Only odd spheres admit nonzero vector fields.)*

PROOF. If n is odd, let $v(x_0, x_1, \dots, x_{n-1}, x_n) = (-x_1, x_0, \dots, -x_n, x_{n-1})$. Conversely, if v exists, we can rotate x into $-x$ in the plane spanned by x and $v(x)$ and obtain the homotopy $(t, x) \to \cos(\pi t)x + \sin(\pi t)v(x)$ between id and $-$id. Then $1 = \deg(\mathrm{id}) = \deg(-\mathrm{id}) = (-1)^{n+1}$ so n is odd. $\square$



The Hurwitz–Radon number of $n = (2a+1)2^b$ is

$$\rho(n) = 2^c + 8d$$

if $b = c + 4d$ with $0 \le c \le 3$. It is a classical result that $S^{n-1} \subset \mathbf{R}^n$ admits $\rho(n) - 1$ linearly independent vector fields.

1.61. THEOREM (Adams' Vector Fields on Spheres Theorem). [**1**] *S^{n-1} does not admit $\rho(n)$ linearly independent vector fields.*

9. Cellular homology of CW-complexes

It is in principle easy to compute the homology of a Δ-complex from its simplicial chain complex (1.84). In practice, however, one often runs into the problem that Δ-complexes have many simplices. (Consider for instance the compact surfaces of Eamples 2.40–2.42.) CW-complexes are more flexible than Δ-sets, but how can we compute the homology of a CW-complex?

1.62. Homology of an n-cellular extension. Let X be a space and $Y = X \cup_\phi \coprod D^n_\alpha$ an n-cellular extension of X where $n \ge 0$. The characteristic map $\Phi \colon \coprod D^n_\alpha \to Y$ and its restriction, the attaching map,

$\phi = \Phi|\coprod S_\alpha^{n-1} \colon \coprod S_\alpha^{n-1} \to X$ are shown in the commutative diagram

$$\begin{tikzcd}
\coprod S_\alpha^{n-1} \arrow[r, hook] \arrow[d, "\phi"] & \coprod D_\alpha^n \arrow[r, "q"'] \arrow[d, "\Phi"] & \coprod D_\alpha^n / \coprod S_\alpha^{n-1} \arrow[d, "\overline{\Phi}", "\simeq"'] \\
X \arrow[r, hook] & Y \arrow[r, "q"] & Y/X
\end{tikzcd}$$

where the maps labeled q are quotient maps. The map $\overline{\Phi}$ between the quotients induced by the characteristic map is a homeomorphism which makes it very easy to compute relative homology.

Since $(\coprod D_\alpha^n, \coprod S_\alpha^{n-1})$ and (Y, X) are good pairs with homeomorphic quotient spaces there is an isomorphism

$$H_k(Y,X) \xrightarrow[\cong]{\Phi_*} H_k\left(\coprod D_\alpha^n, \coprod S_\alpha^{n-1}\right) = \begin{cases} \bigoplus H_n(D_\alpha^n, S_\alpha^{n-1}) = \bigoplus \mathbf{Z} & k = n \\ 0 & k \neq n \end{cases}$$

by 1.42 and 1.46. The unique nontrivial relative homology group of the pair (Y, X) sits in this 5-term segment

$$0 \to H_n(X) \to H_n(Y) \to H_n(Y,X) \xrightarrow{\partial_n} H_{n-1}(X) \to H_{n-1}(Y) \to 0 \tag{1.63}$$

of the long exact sequence. The 0 to the left is $H_{n+1}(Y,X)$ and the 0 to the right is $H_{n-1}(Y,X)$. The group in the middle, $H_n(X,Y)$, is free abelian on the cells attached.

1.64. Lemma (The effect on homology of an n-cellular extension). Let $Y = X \cup_\phi \coprod D_\alpha^n$ be an n-cellular extension of X where $n \geq 1$. Then

$$H_k(X,Y) \cong \begin{cases} \bigoplus H_n(D_\alpha^n, S_\alpha^{n-1}) \cong \bigoplus \mathbf{Z} & k = n \\ 0 & k \neq n \end{cases}$$

and there are short exact sequences

$$0 \longrightarrow \operatorname{im} \partial_n \longrightarrow H_{n-1}(X) \longrightarrow H_{n-1}(Y) \longrightarrow 0 \qquad 0 \longrightarrow H_n(X) \longrightarrow H_n(Y) \longrightarrow \ker \partial_n \longrightarrow 0$$

so that

$$H_{n-1}(Y) \cong H_{n-1}(X)/\operatorname{im}\partial_n, \qquad H_n(Y) \cong H_n(X) \oplus \ker \partial_n$$

while $H_k(X) \cong H_k(X)$ when $k \neq n-1, n$.

Attaching n-cells to a space introduces extra free generators in degree n, relations in degree $n-1$, and has no effect in other degrees. The isomorphisms of the above lemma are not natural.

1.65. The cellular chain complex. Let X be a CW-complex with skeletal filtration $\emptyset = X^{-1} \subset X^0 \subset \cdots \subset X^n \subset X^{n+1} \subset \cdots \subset X$.

1.66. Lemma. Let X be a CW-complex and $X^n = X^{n-1} \cup_\phi \coprod D_\alpha^n$ the n-skeleton where $n \geq 0$. Then

(1) $H_k(X^n, X^{n-1}) = \begin{cases} \bigoplus H_n(D_\alpha^n, S_\alpha^{n-1}) & k = n \\ 0 & k \neq n \end{cases}$

(2) $H_{>n}(X^n) = 0$

(3) $H_{<n}(X^n) \cong H_{<n}(X)$

Proof. (1) This is obvious when $n = 0$ and is just Lemma 1.64 when $n \geq 1$.
(2) The extensions $X^0 \subset \cdots \subset X^n$ affect homology in degrees $0, 1, \dots, n$ but not in degrees $> n$ and therefore $0 = H_{>n}(X^0) = H_{>n}(X^1) = \cdots = H_{>n}(X^n)$.
(3) The extensions $X^n \subset \cdots \subset X^N$ for $N > n$ affect homology in degrees $n, \dots, N$ but not in degrees $< n$ and therefore $H_{<n}(X^n) = \cdots = H_{<n}(X^N)$. The support of any singular chain is compact and therefore we know from Homotopy theory for beginners that it is contained in a skeleton. This implies that $H_{<n}(X^n) = H_{<n}(X)$: Assume that $k < n$. Let z be a k-cycle in X, representing a homology class $[z] \in H_k(X)$. The support of z lies in a finite skeleton X^N for some $N > n$. Thus $[z]$ lies in the image of $H_k(X^N) \to H_k(X) \ni [z]$. But $H_k(X^n) \to H_k(X^N)$ is an isomorphism, so $[z]$ lies in the image of $H_k(X^n) \to H_k(X)$. Thus this map is surjective. Let next z be a k-cycle in X^n and suppose that the homology class $[z]$ lies in the kernel of $H_k(X^n) \to H_k(X)$. Then $z = \partial u$ is the boundary of some $(k+1)$-chain u in X. The support of u lies in some finite skeleton X^N for some $N > n$. Thus $[z]$ lies in the kernel of the map $H_k(X^n) \to H_k(X^N)$. But this map is an isomorphism so that $[z] = 0$ in $H_k(X^n)$. $\square$

The long exact sequence for the pair (X^n, X^{n-1}) contains the 4-term segment (1.63)

$$0 \longrightarrow H_n(X^n) \xrightarrow{j_n} H_n(X^n, X^{n-1}) \xrightarrow{\partial_n} H_{n-1}(X^{n-1}) \longrightarrow H_{n-1}(X) \longrightarrow 0 \tag{1.67}$$

with $0 = H_n(X^{n-1})$ to the left and $0 = H_{n-1}(X^n, X^{n-1})$ to the right. We have also used that $H_{n-1}(X^n) = H_{n-1}(X)$. Combine the 4-term exact sequences (1.67) for the three pairs that can be formed from $X^{n+1} \supset X^n \supset X^{n-1} \supset X^{n-2}$

$$\boxed{H_n^{\mathrm{CW}}(X) = \frac{\ker(d_n)}{\operatorname{im}(d_{n+1})} \cong \frac{H_n(X^n)}{\ker(i_{n+1})} \cong \operatorname{im}(i_{n+1}) = H_n(X)}$$

$$\begin{tikzcd}
H_{n+1}(X^{n+1}, X^n) \arrow[r, "\partial_{n+1}"] & H_n(X^n) \arrow[r, "i_{n+1}"] \arrow[d, equal] & H_n(X) \arrow[r] & 0 & \\
0 \arrow[r] & H_n(X^n) \arrow[r, "j_n"] \arrow[rr, bend left, "\partial_n j_n = 0"] & H_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] & H_{n-1}(X^{n-1}) \arrow[d, equal] & \\
& & 0 \arrow[r] & H_{n-1}(X^{n-1}) \arrow[r, "j_{n-1}"] & H_{n-1}(X^{n-1}, X^{n-2})
\end{tikzcd}$$

$\boxed{\operatorname{im}(d_{n+1}) \cong \operatorname{im}(\partial_{n+1}) = \ker(i_{n+1})}$ $d_{n+1} = j_n \partial_{n+1}$

$\boxed{\ker(d_n) = \ker(\partial_n) = \operatorname{im}(j_n) = H_n(X^n)}$ $d_n = j_{n-1}\partial_n$

and extract the *cellular chain complex*

$$\cdots \to H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \to \cdots \tag{1.68}$$

where $d_n = j_{n-1} \circ \partial_n$. This is really a chain complex because $d_n \circ d_{n+1} = j_{n-1} \circ \partial_n \circ j_n \circ \partial_{n+1} = 0$ since $\partial_n \circ j_n = 0$ by exactness of (1.67). Define *cellular homology*

$$H_n^{\mathrm{CW}}(X) = \ker d_n / \operatorname{im} d_{n+1}$$

to be the homology of this cellular chain complex.

1.69. Theorem (Cellular and singular homology are isomorphic). $H_n^{\mathrm{CW}}(X) \cong H_n(X)$.

Choose generators (orientations) $[D^n_\alpha] \in H_n(D^n_\alpha, S^{n-1}_\alpha)$ for all n-cells for all $n \geq 0$. As generators for $\widetilde{H}_{n-1}(S^{n-1}_\alpha)$ and $H_n(D^n_\alpha/S^{n-1}_\alpha)$, for $n \geq 1$, we use the images of $[D^n_\alpha]$ under the isomorphisms

$$\begin{tikzcd}
\widetilde{H}_{n-1}(S^{n-1}_\alpha) & H_n(D^n_\alpha, S^{n-1}_\alpha) \arrow[l, "\partial"', "\cong"] \arrow[r, "q_*", "\cong"'] & H_n(D^n_\alpha/S^{n-1}_\alpha) \\
\partial[D^n_\alpha] & {[D^n_\alpha]} \arrow[l] \arrow[r] & q_*[D^n_\alpha]
\end{tikzcd} \tag{1.70}$$

of homology groups.

The elements $e^n_\alpha = \Phi_*([D^n_\alpha]) \in H_n(X^n, X^{n-1})$ form a basis for the free abelian group $H_n(X_n, X^{n-1}) = \mathbf{Z}\{e^n_\alpha\}$. We want to compute the matrix, $(d_{\alpha\beta})$, for the cellular boundary map

$$d_n \colon H_n(X^n, X^{n-1}) = \mathbf{Z}\{e^n_\alpha\} \to H_{n-1}(X^{n-1}, X^{n-2}) = \mathbf{Z}\{e^{n-1}_\beta\}. \tag{1.71}$$

where $\{e^n_\alpha\} = \Phi^n_*[D^n_\alpha]$ are the n-cells and $\{e^{n-1}_\beta\} = \Phi^{n-1}_*[D^{n-1}_\beta]$ the $(n-1)$-cells of X.

Consider first the case $n = 1$. The 0-skeleton of X is the set of 0-cells $X^0 = \{e^0_\beta\}$. The attaching maps for the 1-cells are maps $\varphi_\alpha \colon S^0_\alpha \to X^0 = \{e^0_\beta\}$ given by their values $\varphi_\alpha(\pm 1)$ on the two points of $S^0_\alpha = \{\pm 1\}$. The cellular boundary map fits into the commutative diagram

$$\begin{tikzcd}
\bigoplus H_1(D^1_\alpha, S^0_\alpha) \arrow[r, "\partial"] \arrow[d, "\Phi_*"', "\cong"] & \bigoplus H_0(S^0_\alpha) \arrow[d, "\phi_*"] \\
\mathbf{Z}\{e^1_\alpha\} = H_1(X^1, X^0) \arrow[r, "d_1 = \partial_1"] & H_0(X^0) = \mathbf{Z}\{e^0_\beta\}
\end{tikzcd}$$

and it is given by the differences

$$d_1 e^1_\alpha = \phi_*(\partial[D^1_\alpha]) = \phi_*((+1)_\alpha - (-1)_\alpha) = \phi_\alpha(+1) - \phi_\alpha(-1)$$

between the terminal and the initial values of the attaching maps for the 1-cells. (In case the 0-skeleton X^0 is a single point, the boundary map $d_1 = 0$ is trivial since all attaching maps are constant.)

1.72. Theorem (Cellular boundary formula). *When $n \geq 2$, the cellular boundary map (1.71) is given by*

$$d_n(e^n_\alpha) = \sum d_{\alpha\beta} e^{n-1}_\beta$$

where the integer $d_{\alpha\beta}$ is the degree, relative to the chosen generators $\partial[D^n_\alpha] \in H_{n-1}(S^{n-1}_\alpha)$ and $q_*[D^{n-1}_\beta] \in H_{n-1}(D^{n-1}_\beta/S^{n-2}_\beta)$ (1.70), of the map

$$\begin{tikzcd}
S^{n-1}_\alpha \arrow[r, "i_\alpha"] \arrow[d, dotted, "d_{\alpha\beta}"'] & \coprod S^{n-1}_\alpha \arrow[r, "\phi^n"] & X^{n-1} \arrow[d, "q"] \\
D^{n-1}_\beta/S^{n-2}_\beta & \bigvee D^{n-1}_\beta/S^{n-2}_\beta \arrow[l, "q_\beta"'] \arrow[r, "\overline{\Phi}^{n-1}", "\simeq"'] & X^{n-1}/X^{n-2}
\end{tikzcd}$$

where i_α is an inclusion map and q_β a quotient map.

The cellular boundary formula for $n \geq 2$ follows by inspection of the commutative diagram

$$\begin{tikzcd}
\boxed{[D^n_\alpha]} \arrow[rrr] \arrow[ddd] & & & \boxed{[\partial D^n_\alpha]} \arrow[dd] & & \\
& H_{n-1}(D^n_\alpha, S^{n-1}_\alpha) \arrow[r, "\partial", "\cong"'] \arrow[d, "(i_\alpha)_*"'] & H_{n-1}(S^{n-1}_\alpha) \arrow[d, "(i_\alpha)_*"] & & & \\
& H_n(\coprod D^n_\alpha, \coprod S^{n-1}_\alpha) \arrow[r, "\partial", "\cong"'] \arrow[d, "\Phi^n_*"', "\cong"] & H_{n-1}(\coprod S^{n-1}_\alpha) \arrow[d, "\phi^n_*"] & \boxed{\sum_\beta d_{\alpha\beta} e^{n-1}_\beta} \arrow[rrddd] & & \\
\boxed{e^n_\alpha} & H_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] \arrow[rr, bend right, "d_n"'] & H_{n-1}(X^{n-1}) \arrow[r, "j_{n-1}"] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r, "q_*", "\cong"'] & H_{n-1}(X^{n-1}/X^{n-2}) & \\
& \boxed{d_n e^n_\alpha = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta} & & H_{n-1}(\coprod D^{n-1}_\beta, \coprod S^{n-2}_\beta) \arrow[u, "\Phi^{n-1}_*", "\cong"'] \arrow[r, "q_*", "\cong"'] \arrow[d, "(q_\beta)_*"'] & H_{n-1}(\bigvee D^{n-1}_\beta/S^{n-2}_\beta) \arrow[u, "\Phi^{n-1}_*", "\cong"'] \arrow[d, "(q_\beta)_*"] & \\
& & & H_{n-1}(D^{n-1}_\beta, S^{n-2}_\beta) \arrow[r, "q_*", "\cong"'] & H_{n-1}(D^{n-1}_\beta/S^{n-2}_\beta) & \boxed{d_{\alpha\beta}(q_\beta)_*[D^{n-1}_\beta]}
\end{tikzcd}$$

of homology groups.

1.73. Definition. A map $f \colon X \to Y$ between CW-complexes is *cellular* if it respects the skeletal filtrations in the sense that $f(X^k) \subset Y^k$ for all k.

The cellular chain complex (1.68) and the isomorphism between cellular and singular homology (1.69) are natural with respect to cellular maps.

1.74. Compact surfaces. We compute the homology groups of the compact surfaces.

1.75. Definition. The compact orientable surface of genus $g \geq 1$ is the 2-dimensional CW-complex

$$M_g = \bigvee_{i=1}^{g} (S^1_{a_i} \vee S^1_{b_i}) \cup_{\prod[a_i,b_i]} D^2$$

where the attaching map for the 2-cell is $\prod[a_i, b_i] = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$.

In particular, $M_1 = T$ is a torus. In general $M_g = T \# \cdots \# T$ is the connected sum of g copies of a torus [**17**, Figure 74.8] (M_2) The cellular chain complex (1.68) of M_g has the form

$$0 \longleftarrow \mathbf{Z}\{e^0\} \xleftarrow{d_1} \mathbf{Z}\{a_1, b_1, \ldots, a_g, b_g\} \xleftarrow{d_2} \mathbf{Z}\{e^2\} \longleftarrow 0$$

The only problem is d_2 for the boundary map $d_1 = 0$ as $H_0(M_g) = \mathbf{Z}$. The coefficient in $d_2 e^2$ of, for instance, a_1 is (1.72) the degree of the map

$$S^1 \xrightarrow{\varphi=\prod[a_i,b_i]} \bigvee_{i=1}^g S^1_{a_i} \vee S^1_{b_i} \xrightarrow{q_{a_1}} S^1_{a_1}$$

which is homotopic to the map $a_1 a_1^{-1}\colon S^1 \to S^1$ of degree 0. In this way we see that also $d_2 = 0$. We conclude that

$$H_k(M_g) = H_k^{\mathrm{CW}}(M_g) = \begin{cases} \mathbf{Z} & k=0 \\ \mathbf{Z}^{2g} & k=1 \\ \mathbf{Z} & k=2 \\ 0 & k>2 \end{cases}$$

Note the symmetry in the homology groups.

1.76. Definition. The compact nonorientable surface of genus $g \geq 1$ is the CW-complex

$$N_g = \bigvee_{i=1}^g S^1_{a_i} \cup_{\prod a_i^2} D^2$$

where the attaching map for the 2-cell is $\prod a_i^2 = a_1^2 \cdots a_g^2$.

In particular, $N_1 = S^1 \cup_2 D^2 = \mathbf{R}P^2$ is the real projective plane. In general $N_g = \mathbf{R}P^2 \# \cdots \# \mathbf{R}P^2$ is the connected sum of g copies of $\mathbf{R}P^2$ [**17**, Figure 74.10]. The cellular chain complex of N_g has the form

$$0 \longleftarrow \mathbf{Z}\{e^0\} \xleftarrow{d_1} \mathbf{Z}\{a_1,\dots,a_g\} \xleftarrow{d_2} \mathbf{Z}\{e^2\} \longleftarrow 0$$

The only problem is d_2 for the boundary map $d_1 = 0$ as $H_0(N_g) = \mathbf{Z}$. The coefficient of, for instance, a_1 in $d_2 e^2$ is the degree of the map

$$S^1 \xrightarrow{\varphi=\prod a_i^2} \bigvee_{i=1}^g S^1_{a_i} \xrightarrow{q_{a_1}} S^1_{a_1}$$

which is homotopic to the map $a_1 a_1\colon S^1 \to S^1$ of degree 2. In this way we see that $d_2(e^2) = 2(a_1 + \cdots + a_g)$. We conclude that

$$H_k(N_g) = H_k^{\mathrm{CW}}(N_g) = \begin{cases} \mathbf{Z} & k=0 \\ \mathbf{Z}^{g-1} \oplus \mathbf{Z}/2 & k=1 \\ 0 & k \geq 2 \end{cases}$$

The Classification theorem for compact surfaces [**17**, Thm 77.5] says that any compact surface is homeomorphic to precisely one of the model surfaces M_g, $g \geq 0$, or N_g, $g \geq 1$. Thus two compact surfaces are homeomorphic iff they have isomorphic first homology groups.

1.77. Real projective space. [**6**, V.§6.Exmp 6.13, Ex 4] The 0-sphere $S^0 = \{-1,+\}$ is the (topological) group of real numbers of unit norm. Let $S^n = \{(x_0,\dots,x_n) \in \mathbf{R}^{n+1} \mid |x_0|^2 + \cdots + |x_0|^2 = 1\}$ be the unit sphere in $\mathbf{R}^{n+1}$ and $D^n_\pm = \{(x_0,\dots,x_n) \in S^n \mid \pm x_n \geq 0\}$ its two hemispheres. *Real projective n-space* is the quotient space and ϕ the quotient map

$$\mathbf{R}P^n = S^0 \backslash S^n, \qquad \phi\colon S^n \to \mathbf{R}P^n$$

obtained by identifying two real $(n+1)$-tuples if they are proportional by a real number (necessarily of absolute value 1). The orbit, $\pm(x_0,\dots,x_n)$, of the point $(x_0,\dots,x_n) \in S^n$ is traditionally denoted $[x_0 : \cdots : x_n] \in \mathbf{R}P^n$.

We first give S^n a CW-structure so that the antipodal map $\tau(x) = -x$ becomes cellular and induces an automorphism of the cellular chain complex. We will next consider the quotient CW-structure on $\mathbf{R}P^n$.

Observe that $S^k = S^{k-1} \cup_{\mathrm{act}} (S^0 \times D^k)$ is the pushout

$$\begin{tikzcd} S^0 \times S^{k-1} \arrow[r, "\text{action}"] \arrow[d, hook] & S^{k-1} \arrow[d] \\ S^0 \times D^k \arrow[r, "\Phi"] & S^k \end{tikzcd}$$

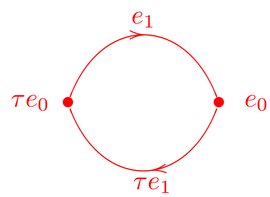
of S^0-maps. Thus

$$S^n = (S^0 \times D^0) \cup (S^0 \times D^1) \cdots \cup (S^0 \times D^n)$$

is a free S^0-CW-complex with one free S^0-cell in each dimension with characteristic map $\Phi^k \colon S^0 \times D^k \to S^k$ given by $\Phi^0(z,u) = z$ and $\Phi^k(z,u) = z(u, \sqrt{1-|u|^2})$ for $k > 0$. Write $D^k_\pm$ for $\pm 1 \times D^k$ and $\Phi^k_\pm$ for the restriction of Φ^k to $D^k_\pm$. Then $S^0 \times D^k = D^k_- \amalg D^k_+$ where $D^k_+ = D^k = D^k_-$ and $\Phi^k_+(u) = (u, \sqrt{1-|u|^2})$, $\Phi^k_- = \tau\Phi^k_+$. The cellular chain complex of this S^0-CW-complex is a chain complex of $\mathbf{Z}S^0$-modules $H_k(S^k, S^{k-1}) \cong H_k(D^k, S^{k-1}) \oplus H_k(D^k, S^{k-1})$.

1.78. Lemma. When $n \geq k \geq 0$ there are generators $[D^k_+] \in H_k(D^k, S^{k-1})$, $k \geq 0$, so that $H_k(S^k, S^{k-1}) = \mathbf{Z}\{e^k, \tau e^k\}$ and $d_k e^k = (1 + (-1)^k \tau) e^{k-1}$ $(k \geq 1)$ where $e^k = (\Phi^k_+)_*[D^k_+]$.

Proof. To start the induction, consider the 1-skeleton of S^n, S^1, with CW-structure



Then $d_1 e^1 = e^0 - \tau e^0 = (1-\tau)e^0$.

Suppose, inductively, that e^k has been found for some k where $n > k \geq 1$. Then $d_k(e^k - (-1)^k \tau e^k) = (1-(-1)^k\tau) d_k e^d = (1-(-1)^k\tau)(1+(-1)^k\tau)e^{k-1} = (1-\tau^2)e^{k-1} = 0$. Consider the commutative diagram

$$\begin{tikzcd}
 & & \boxed{(1+(-1)^{k+1}\tau)e^k} & & & \boxed{e^{k+1}} \arrow[lll, "d_{k+1}"'] \\
H_{k-1}(S^{k-1}, S^{k-2}) & H_{k-1}(S^{k-1}) \arrow[l, "j_{k-1}"'] & H_k(S^k, S^{k-1}) \arrow[l, "\partial_k"'] \arrow[ll, bend right, "d_k"'] & H_k(S^k) \arrow[l, "j_k"'] \arrow[d, equal] & H_{k+1}(S^{k+1}, S^k) \arrow[l, "\partial_{k+1}"'] \arrow[ll, bend right, "d_{k+1}"'] & \\
 & & \text{General fact: } \boxed{\ker(d_k) = \operatorname{im}(j_k)} & H_k(S^k) & H_{k+1}(D^{k+1}, S^k) \arrow[l, "\partial"', "\cong"] \arrow[u, "(\Phi^{k+1}_+)_*"'] & \boxed{[D^{k+1}_+]} \arrow[uu, "(\Phi^{k+1}_+)_*"']
\end{tikzcd}$$

As $e^k - (-1)^k \tau e^k \in \ker(d_k) = \operatorname{im}(j_k) = \operatorname{im}(j_k \circ \partial)$ there is a (unique) generator $[D^{k+1}_+] \in H_{k+1}(D^{k+1}, S^k)$ that hits $(1-(-1)^k\tau)e^k$ under $j_k \circ \partial$. In other words, $d_{k+1}e^{k+1} = (1+(-1)^{k+1}\tau)e^k$, where $e^{k+1} = (\Phi^{k+1}_+)_*[D^{k+1}_+]$, since the diagram commutes. □

The quotient space $\mathbf{R}P^n = S^0 \backslash S^n$ is a CW-complex with one cell in each dimension from 0 through n and the projection map $p \colon \mathbf{R}P^n \to S^n$ is cellular so that there is an induced map

$$\begin{array}{ccccccccccccc}
H_j(S^\infty) & & \mathbf{Z} & & 0 & & 0 & & 0 & & 0 & & \\
S^\infty & 0 \longleftarrow & \mathbf{Z}\oplus\mathbf{Z} & \xleftarrow{1-\tau} & \mathbf{Z}\oplus\mathbf{Z} & \xleftarrow{1+\tau} & \mathbf{Z}\oplus\mathbf{Z} & \xleftarrow{1-\tau} & \mathbf{Z}\oplus\mathbf{Z} & \xleftarrow{1+\tau} & \mathbf{Z}\oplus\mathbf{Z} & \longleftarrow & \cdots \\
\downarrow & & \downarrow{\scriptstyle +} & & \downarrow{\scriptstyle +} & & \downarrow{\scriptstyle +} & & \downarrow{\scriptstyle +} & & \downarrow{\scriptstyle +} & & \\
\mathbf{R}P^\infty & 0 \longleftarrow & \mathbf{Z} & \xleftarrow{0} & \mathbf{Z} & \xleftarrow{2} & \mathbf{Z} & \xleftarrow{0} & \mathbf{Z} & \xleftarrow{2} & \mathbf{Z} & \longleftarrow & \cdots \\
H_j(\mathbf{R}P^\infty) & & \mathbf{Z} & & 0 & & \mathbf{Z}/2\mathbf{Z} & & 0 & & \mathbf{Z}/2\mathbf{Z} & &
\end{array}$$

between the cellular chain complexes. We conclude that the cellular boundary map d_k for $\mathbf{R}P^n$, $d_{2k+1} = 0$ and $d_{2k} = \cdot 2$, alternates between the 0-map and multiplication by 2. In particular, the top homology groups are $H_{2n+1}(\mathbf{R}P^{2n+1}) = \mathbf{Z}$ and $H_{2n}(\mathbf{R}P^{2n}) = 0$. It follows that for instance

$$H_k(\mathbf{R}P^3) = \begin{cases} \mathbf{Z} & k = 0 \\ \mathbf{Z}/2 & k = 1 \\ 0 & k = 2 \\ \mathbf{Z} & k = 3 \\ 0 & k > 3 \end{cases} \qquad H_k(\mathbf{R}P^4) = \begin{cases} \mathbf{Z} & k = 0 \\ \mathbf{Z}/2 & k = 1 \\ 0 & k = 2 \\ \mathbf{Z}/2 & k = 3 \\ 0 & k \geq 4 \end{cases}$$

and in general, $H_k(\mathbf{R}P^n)$ is either 0, $\mathbf{Z}$, or $\mathbf{Z}/2$.

1.79. Complex and quaternion projective space. Let $S^{2n+1} = \{(u, v) \in \mathbf{C}^n \times \mathbf{C} \mid |u|^2 + |v|^2 = 1\}$ be the unit sphere in $\mathbf{C}^{n+1} = \mathbf{R}^{2n+2}$. *Complex projective n-space* is the quotient space and ϕ the quotient map

$$\mathbf{C}P^n = S^1 \backslash S^{2n+1}, \qquad \phi \colon S^{2n+1} \to \mathbf{C}P^n$$

obtained by identifying two points of S^{2n+1} if they are proportional by a complex number (necessarily of absolute value 1). The quotient map $\phi \colon S^{2n+1} \to \mathbf{C}P^n$ is called the Hopf map (in particular for $n = 1$ where we have a map $S^3 \to S^2$ with fibre S^1).

Points of $S^{2k+1} \subset \mathbf{C}^k \times \mathbf{C}$ have the form $z(u, \sqrt{1 - |u|^2})$ for some $u \in \mathbf{C}^k$ with $|u| \leq 1$ and some $z \in \mathbf{C}$ with $|z| = 1$. (If (x, y) lies on S^{2k+1}, then $|y| = \sqrt{1 - |x|^2}$ so that $y = z\sqrt{1 - |x|^2}$ for some $z \in S^1$. Put $u = z^{-1}x$. Then $|u| = |x|$ and $z(u, \sqrt{1 - |u|^2}) = (zu, z\sqrt{1 - |x|^2}) = (x, y)$. If $y \neq 0$, ie $(x, y) \in S^{2k+1} - S^{2k-1}$ then z and u are uniquely determined.) In fact,

(1.80)
$$\begin{tikzcd} S^1 \times S^{2k-1} \arrow[r, "\text{action}"] \arrow[d, hook] & S^{2k-1} \arrow[d] \\ S^1 \times D^{2k} \arrow[r, "\Phi"] & S^{2k+1} \end{tikzcd}$$

is a pushout diagram meaning that $S^{2k+1} = S^{2k-1} \cup_{\text{action}} (S^1 \times D^{2k})$. Thus S^{2n+1} is a free S^1-CW-complex with one free S^1-cell in each even degree up to $2n$. The characteristic map for the $2k$-cell is $\Phi \colon S^1 \times D^{2k} \to S^{2k+1} \subset S^{2n+1}$ given by $\Phi(z, u) = z(u, \sqrt{1 - |u|^2})$.

Consequently, $\mathbf{C}P^n$ is a CW-complex with $2k$-skeleton $\mathbf{C}P^k$ and with one cell in each even dimension $\leq 2n$. The cellular chain complex immediately shows that the homology

$$H_k(\mathbf{C}P^n) = \begin{cases} \mathbf{Z} & k \text{ even and } 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases}$$

is concentrated in even degrees.

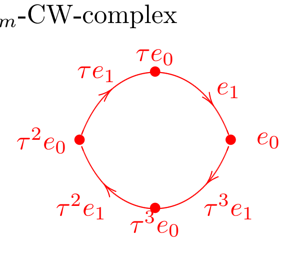
Similarly, the quaternion projective space $\mathbf{H}P^n = S^3 \backslash S^{4n+3}$ is a CW-complex with one cell in dimensions $4k$, $0 \leq k \leq n$, and with homology concentrated in these dimensions. In particular, $\mathbf{H}P^1 = S^4$ and there is a Hopf map $S^7 \to S^4$ with fibre S^3.

1.81. Lens spaces. [6, V.§3.Ex 3, V.§7.Ex 5] The *lens space* is the quotient space

$$L^{2n+1}(m) = C_m \backslash S^{2n+1}$$

for the action on S^{2n+1} of the group $\sqrt[m]{1} = C_m = \langle \tau \rangle \subset S^1$ generated by the primitive mth root of unity $\tau = e^{2\pi i/m}$. (If $m = 2$, $L^{2n+1}(2) = \mathbf{R}P^{2n+1}$.)

In order to obtain a free C_m-CW-structure on S^{2n+1} from the free S^1-CW-structure, first note that $S^1 = (C_m \times D^0) \cup (C_m \times D^1)$ is a free C_m-CW-complex



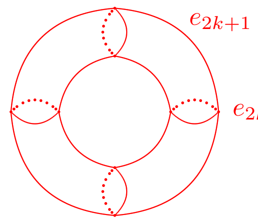
with cellular chain complex of the form

$$0 \longrightarrow \mathbf{Z}C_m \xrightarrow{d_1=\cdot(1-\tau)} \mathbf{Z}C_m \longrightarrow 0$$

concentrated in degree 1 and 0. This implies that

$$S^1 \times D^{2k} = ((C_m \times D^0) \cup (C_m \times D^1)) \times D^{2k} = (C_m \times D^{2k}) \cup (C_m \times D^{2k+1})$$

as in the picture



so that

$$S^{2k+1} \overset{(1.80)}{=} S^{2k-1} \cup (S^1 \times D^{2k}) = S^{2k-1} \cup (C_m \times D^{2k}) \cup (C_m \times D^{2k+1})$$

where we first attach one C_m–cell of dimension $2k$ by the action map $C_m \times S^{2k-1} \to S^{2k-1}$, to obtain the $2k$-skeleton $S^{2k-1} \cup (C_m \times D^{2k})$, and then a C_m–cell of dimension $2k+1$ by the attaching map $\partial(D^1 \times D^{2k}) = \{0,1\} \times D^{2k} \cup D^1 \times S^{2k-1} \to S^{2k-1} \cup (C_m \times D^{2k})$ that takes $\{0\} \times D^{2k}$ to D^{2k} and $\{1\} \times D^{2k}$ to τD^{2k}. Using this construction recursively, we give S^{2n+1} a free C_m-CW-structure with one free C_m-cell in each degree 0 through $2n+1$.

The cellular chain complex for the free C_m–CW-complex S^{2k+1}/S^{2k-1} has the form

$$0 \longrightarrow \mathbf{Z}C_m \xrightarrow{d_{2k+1}=\cdot(1-\tau)} \mathbf{Z}C_m \longrightarrow 0 \tag{1.82}$$

because of the way that the $2k+1$-cell is attached. Observe that the kernel of the boundary map d_{2k+1} is the free abelian subgroup generated by $(1+\tau+\cdots+\tau^{m-1})e_{2k}$. The cellular chain complex for the C_m–CW-complex S^{2n+1} is

$$0 \to \mathbf{Z}C_m\{e^{2n+1}\} \xrightarrow{d_n} \cdots \to \mathbf{Z}C_m\{e^k\} \xrightarrow{d_k} \mathbf{Z}C_m\{e^{k-1}\} \to \cdots \to \mathbf{Z}C_m\{e_1\} \xrightarrow{\cdot(1-\tau)} \mathbf{Z}C_m\{e_0\} \to 0$$

where $d_{2k+1} = \cdot(1-\tau)$ and $d_{2k} = \cdot(1+\tau+\cdots+\tau^{m-1})$ because of exactness as $\ker(d_{2k+1})$ is the $\mathbf{Z}C_m$-module generated by $(1+\tau+\cdots+\tau^{m-1})e_{2k+1}$. The argument is much the same as for $\mathbf{R}P^n$.

This C_m-CW-structure on S^{2n+1} induces a CW-structure on the quotient space $L^{2n+1}(m) = C_m \backslash S^{2n+1}$ such that quotient map $q\colon S^{2n+1} \to L^{2n+1}(m)$ is cellular. The lense space has one cell in each dimension k from 0 through $2n+1$ and the cellular boundary map, $d_{2k+1} = 0$, $d_{2k} = \cdot m$, alternates between 0 and multiplication by m. We conclude that the cellular chain complex of the infinite lense space $L^\infty(m)$ is

$$0 \longleftarrow \mathbf{Z} \xleftarrow{0} \mathbf{Z} \xleftarrow{m} \mathbf{Z} \xleftarrow{0} \mathbf{Z} \xleftarrow{m} \mathbf{Z} \longleftarrow \cdots$$

so that the reduced homology groups

$$\widetilde{H}_k(L^\infty(m)) = \begin{cases} \mathbf{Z}/m & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

alternate between $\mathbf{Z}/m$ and 0.

A homological algebraist will say that this cellular chain complex $C(S^\infty)$ of S^∞ is a resolution of the trivial $\mathbf{Z}C_m$-module $\mathbf{Z}$ by free $\mathbf{Z}C_m$-modules, that $C(C_m \backslash S^\infty) = \mathbf{Z} \otimes_{\mathbf{Z}C_m} C(S^\infty)$ is the cellular chain complex of $L^\infty(m) = C_m \backslash S^\infty$, and that $H_k L^\infty(m) = H_k(C_m \backslash S^\infty) = \mathrm{Tor}_k^{\mathbf{Z}C_m}(\mathbf{Z}, \mathbf{Z})$.

In conclusion we may say that since

$$S^k = S^{k-1} \cup_{\mathrm{act}} (S^0 \times D^k), \quad S^{2k+1} = S^{2k-1} \cup_{\mathrm{act}} (S^1 \times D^{2k}), \quad S^{4k+3} = S^{4k-1} \cup_{\mathrm{act}} (S^3 \times D^{4k})$$

for all $k \geq 0$ we have that

$$\begin{aligned} S^n &= (S^0 \times D^0) \cup (S^0 \times D^1) \cup \cdots \cup (S^0 \times D^n) \\ S^{2n+1} &= (S^1 \times D^0) \cup (S^1 \times D^2) \cup \cdots \cup (S^1 \times D^{2n}) \\ S^{4n+3} &= (S^3 \times D^0) \cup (S^3 \times D^4) \cup \cdots \cup (S^3 \times D^{4n}) \end{aligned}$$

is a free S^0–CW-complex, free S^1–CW-complex, and free S^3–CW-complex, respectively. The quotient CW-structures are

$$\begin{aligned}
\mathbf{R}P^n &= S^0\backslash S^n = D^0 \cup D^1 \cup \cdots \cup D^n \\
\mathbf{C}P^n &= S^1\backslash S^{2n+1} = D^0 \cup D^2 \cup \cdots \cup D^{2n} \\
\mathbf{H}P^n &= S^3\backslash S^{4n+3} = D^0 \cup D^4 \cup \cdots \cup D^{4n}
\end{aligned}$$

Similarly, as $S^1 = (C_m \times D^0) \cup (C_m \times D^1)$ is a free C_m-CW-complex we get that

$$S^{2k+1} = S^{2k-1} \cup (S^1 \times D^{2k}) = S^{2n-1} \cup (C_m \times D^{2n}) \cup (C_m \times D^{2n+1})$$

so that

$$\begin{aligned}
S^{2n+1} &= (C_m \times D^0) \cup (C_m \times D^1) \cup \cdots \cup (C_m \times D^{2n}) \cup (C_m \times D^{2n+1}) \\
L^{2n+1}(m) &= C_m\backslash S^{2n+1} = D^0 \cup D^1 \cup \cdots \cup D^{2n} \cup D^{2n+1}
\end{aligned}$$

In case of $\mathbf{C}P^n$ and $\mathbf{H}P^n$ the cellular boundary maps are trivial for dimensional reasons. For $\mathbf{R}P^n$ and $L^{2n+1}(m)$, look at the chain complex for the spheres and note that it has no homology except at the extreme ends. Since $d_1e_1 = (\tau - 1)e_0$, exactness implies that $d_2e_2 = (1 + \tau + \cdots + \tau^{m-1})e_1$, that $d_3e_3 = (\tau - 1)e_2$ etc. So exactness and d_1 determined the entire C_m-cellular chain complex. Now the cellular chain complex for the real projective or lense space is the quotient of the chain complex for the sphere.

1.83. The equivalence of simplicial and singular homology. Let S be a Δ-set (2.27), $|S|$ its realization (2.37), and let X be a topological space, $\mathrm{Sing}(X)$ its Δ-set. Then $H_*^\Delta(\mathrm{Sing}(|S|) = H_*(|S|)$ and $H_*^\Delta(\mathrm{Sing}(X)) = H_*(X)$.

1.84. Theorem. [10, 2.27] *The unit $\eta_S\colon S \to \mathrm{Sing}(|S|)$ and the counit $X \leftarrow |\mathrm{Sing}(X)|\colon \varepsilon_X$ induce isomorphisms*

$$H_*^\Delta(S) \xrightarrow[\cong]{(\eta_S)_*} H_*^\Delta(\mathrm{Sing}(|S|) = H_*(|S|), \qquad H_*(X) \xleftarrow[\cong]{(\varepsilon_X)_*} H_*(|\mathrm{Sing}(X)|)$$

on homology. These isomorphisms are natural.

Proof. The topological realization $|S|$ is a CW-complex with skeleta $|S|^n = |S_{\le n}|$. The set of n-cells is indexed by S_n. The characteristic map for the n-cells is $\Phi^n(a,x) = (a,x) \in |S_{\le n}|$ and the attaching map is $\varphi^n(a, d^iy) = (d_ia, y)$, $a \in S_n$, $x \in \Delta^n$, $y \in \Delta^{n-1}$, $i \in n_+$. These maps are shown in the commutative diagram

$$\begin{tikzcd}
S_n \times \partial\Delta^n \arrow[r, hook] \arrow[d, "\varphi^n"] & S_n \times \Delta^n \arrow[r, two heads] \arrow[d, "\Phi^n"] & \bigvee_{S_n} \Delta^n/\partial\Delta^n \arrow[d, "\bar{\Phi}^n", "\simeq"'] \\
|S_{\le(n-1)}| \arrow[r, hook] & |S_{\le n}| \arrow[r, two heads] & |S_{\le n}|/|S_{\le(n-1)}|
\end{tikzcd}$$

The cellular chain complex of the CW-complex $|S|$ is a chain complex with the free abelian group $\mathbf{Z}[S_n]$ in degree n. The cellular boundary map $d_n\colon \mathbf{Z}[S_n] \to \mathbf{Z}[S_{n-1}]$ of Theorem 1.72 takes $a \in S_n$ to $d_na = \sum_{b \in S_{n-1}} d_{ab}b$ where the integer d_{ab} is the degree of the left vertical map of the commutative diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, hook, "a"] \arrow[d, dotted, "d_{ab}"'] & S_n \times \partial\Delta^n \arrow[r, "\varphi^n"] & {|S_{\le(n-1)}|} \arrow[d] \\
\Delta^{n-1}/\partial\Delta^{n-1} & \bigvee_{b \in S_{n-1}} \Delta^{n-1}/\partial\Delta^{n-1} \arrow[l, "q_b"'] \arrow[r, "\bar{\Phi}^{n-1}", "\simeq"'] & {|S_{\le(n-1)}|/|S_{\le(n-2)}|}
\end{tikzcd}$$

This map is given by

$$d^iy \to \begin{cases} y & d_ia = b \\ * & d_ia \neq b \end{cases}$$

where $y \in \Delta^{n-1}$ and $i \in n_+$. According to Lemma 1.85 the degree is $d_{ab} = \sum_{\substack{i\in n_+\\ d_i a = b}} (-1)^i$. We conclude that $d_n a = \sum_{b\in S_{n-1}} \sum_{\substack{i\in n_+\\ d_i a=b}} (-1)^i b = \sum_{i\in n_+} (-1)^i d_i a$. This is exactly the boundary map in the chain complex $\mathbf{Z}[S]$ of the Δ-set S (2.29). We have now identified the chain complex $\mathbf{Z}[S]$ of the Δ-set S as the cellular chain complex of its realization $|S|$ and so we may use Theorem 1.69 to conclude that $H^{\Delta}_*(S) = H_*(\mathbf{Z}[S]) \cong H^{\mathrm{CW}}_*(|S|) \cong H_*(|S|)$. $\square$

We will be computing degrees with respect to the generator $[\delta^n] \in H_n(\Delta^n, \partial\Delta^n)$ and its image $[\sum_{i\in n_+}(-1)^i d^i]$ in $H_{n-1}(\partial\Delta^n)$ (Proposition 1.50).

1.85. Lemma. Let $n \geq 2$ and I a subset of the set n_+ of facets of Δ^n. The map $\partial\Delta^n \to \Delta^{n-1}/\partial\Delta^{n-1}$ given by

$$d^i y \to \begin{cases} y & i \in I \\ * & i \notin I \end{cases}$$

for $y \in \Delta^{n-1}$ has degree $\sum_{i\in I}(-1)^i$.

Proof. We consider first the special case where $I = \{i\}$ consists of just one element. Let $\Lambda_i = \bigcup_{j\neq i} d^j \Delta^{n-1}$. The ith coface map $d^i\colon \Delta^{n-1} \to \Delta^n$ induces a homeomorphism $d^i\colon \Delta^{n-1}/\partial\Delta^{n-1} \to \Delta^n/\Lambda_i$. The map of the lemma, in this special case, is the composition $\partial\Delta^n \to \partial\Delta^n/\Lambda_i \xleftarrow[\simeq]{d^i} \Delta^{n-1}/\partial\Delta^{n-1}$. The effect in homology

$$\begin{tikzcd} H_n(\Delta^n,\partial\Delta^n) \arrow[r, "\cong"] & H_{n-1}(\partial\Delta^n) \arrow[r, "\cong"] & H_{n-1}(\partial\Delta^n,\Lambda_i) & H_{n-1}(\Delta^{n-1},\partial\Delta^{n-1}) \arrow[l, "d^*_i"'] \\ {[\delta^n]} \arrow[r] & {[\sum(-1)^i d^i]} \arrow[r] & {[(-1)^i d^i]} & {[(-1)^i\delta^{n-1}]} \arrow[l] \end{tikzcd}$$

shows that this map has degree $(-1)^i$.

Next we consider the other extreme case where $I = n_+$. Let Λ be the $(n-2)$-skeleton of the $(n-1)$-dimensional CW-complex $\partial\Delta^n$. The quotient space $\partial\Delta^n/\Lambda$ is a wedge of $(n-1)$-spheres. The commutative diagram

$$\begin{tikzcd} & \partial\Delta^n/\Lambda & \bigvee_{i\in n_+}\Delta^{n-1}/\partial\Delta^{n-1} \arrow[l, "\bigvee_{i\in n_+} d^i"', "\simeq"] \arrow[d, "q_i"] \\ \partial\Delta^n \arrow[ur] \arrow[dr] & & \\ & \partial\Delta^n/\Lambda_i & \Delta^{n-1}/\partial\Delta^{n-1} \arrow[l, "d^i"', "\simeq"] \end{tikzcd}$$

shows that the homomorphism $H_{n-1}(\partial\Delta^n) \to H_{n-1}(\partial\Delta^n/\Lambda) \xleftarrow[\cong]{(\bigvee d^i)_*} \bigoplus_{i\in n_+} H_{n-1}(\Delta^{n-1}/\partial\Delta^{n-1})$ sends the generator $[\sum_{i\in n_+}(-1)^i d^i]$ of $H_{n-1}(\partial\Delta^n)$ to $\bigoplus_{i\in n_+}[(-1)^i\delta^{n-1}]$.

In the general case the map of the lemma

$$\partial\Delta^n \longrightarrow \partial\Delta^n/\Lambda \xleftarrow[\simeq]{\bigvee_{i\in n_+} d^i} \bigvee_{i\in n_+}\Delta^{n-1}/\partial\Delta^{n-1} \xrightarrow{q_I} \bigvee_{i\in I}\Delta^{n-1}/\partial\Delta^{n-1} \xrightarrow{\bigvee_{i\in I}\mathrm{id}} \Delta^{n-1}/\partial\Delta^{n-1}$$

sends the generator $[\sum_{i\in n_+}(-1)^i d^i]$ of $H_{n-1}(\partial\Delta^n)$ to $\sum_{i\in I}(-1)^i\delta^{n-1}$. This shows that this map has degree $\sum_{i\in I}(-1)^i$. $\square$

1.86. Euler characteristic. According to the Fundamental Theorem for finitely generated Abelian Groups 1.104, any finitely generated abelian group H is isomorphic to $\mathbf{Z}^r \times \mathbf{Z}/q_1 \times \cdots \times \mathbf{Z}/q_t$, where the integer r is an invariant, called the rank of the group, and the numbers $q_1, \ldots, q_t$ are prime powers. We write $r = \text{rank}(H) = \dim_{\mathbf{Q}}(H \otimes_{\mathbf{Z}} \mathbf{Q})$ for the rank of H.

The (integral) Euler characteristic of a space X is

$$\chi(X) = \sum_{j=0}^{\infty} (-1)^j \text{rank} H_j(X)$$

when this sum has a meaning (X has only finitely many nonzero homology groups and they are finitely generated). In particular, any finite CW-complex has an Euler characteristic.

1.87. Theorem. *The Euler characteristic of a finite CW-complex X is*

$$\chi(X) = \sum_{j=0}^{\dim X} (-1)^j n_j$$

where n_j is the number of cells in dimension j.

This is an immediate consequence of a purely algebraic result applied to the cellular chain complex of X.

1.88. Theorem. *Suppose that $C = (0 \leftarrow C_0 \leftarrow C_1 \leftarrow \cdots \leftarrow C_n \leftarrow 0)$ is a finite chain complex of finitely generated abelian groups. Then*

$$\sum_{j=0}^{\infty} (-1)^j \text{rank}(C_j) = \sum_{j=0}^{\infty} (-1)^j \text{rank}(H_j(C))$$

Proof. We assume as known that $\text{rank}(A) - \text{rank}(B) + \text{rank}(C)$ in a short exact sequence $0 \to A \to B \to C \to 0$ of finitely generated abelian groups. (This is the Dimension Formula of Linear Algebra.)

The short exact sequences

$$0 \to Z_k \to C_k \xrightarrow{d} B_{k-1} \to 0, \qquad 0 \to B_k \to Z_k \to H_k \to 0$$

show that $\text{rank}(C_k) = \text{rank}(Z_k) + \text{rank}(B_{k-1})$ and $\text{rank}(H_k) = \text{rank}(Z_k) - \text{rank}(B_k)$. Therefore,

$$\begin{aligned}
&\text{rank}(H_0) - \text{rank}(H_1) + \text{rank}(H_2) - \cdots = \\
&(\text{rank}(Z_0) - \text{rank}(B_0)) - (\text{rank}(Z_1) - \text{rank}(B_1)) + (\text{rank}(Z_2) - \text{rank}(B_2)) - \cdots = \\
&(\text{rank}(Z_0) + \text{rank}(B_{-1}) - (\text{rank}(Z_1) + \text{rank}(B_0)) + (\text{rank}(Z_2) + \text{rank}(B_1)) - \cdots = \\
&\text{rank}(C_0) - \text{rank}(C_1) + \text{rank}(C_2) - \cdots
\end{aligned}$$

which is what we wanted to prove. □

1.89. Corollary (Euler's formula). The alternating sum $\sum_{j=0}^{\dim X} (-1)^j n_j$ of the number of cells is the same for all finite CW-decompositions of the same space X (indeed, for all spaces in the homotopy type of X).

For instance, $F - E + V = 2$ for any finite CW-decomposition of S^2 with F faces, E edges, and V vertices.

1.90. Proposition. Suppose that $X = A \cup B = \text{int}\, A \cup \text{int}\, B$ so that there is a long exact Mayer–Vietoris sequence. If all three spaces, X, A, and B, have Euler characteristics then $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$.

1.91. Example. The Euler characteristic of S^n is 0 if n is odd and 2 if n is even.

The Euler characteristic of $\mathbf{R}P^n$ is 0 if n is odd and 1 if n is even.

The Euler characteristic of $\mathbf{C}P^n$ and $\mathbf{H}P^n$ is $n+1$.

The Euler characteristic of the orientable surface M_g is $2-2g$ and the Euler characteristic of the nonorientable surface N_g is $2-g$. Two orientable (or nonorientable) compact surfaces are homeomorphic if and only if they have the same Euler characteristic.

Note also that $\chi(X) = \sum_{j=0}^{\infty} (-1)^j \dim_k H_j(X;k)$ for any field k.

The Euler characteristic can be used to find the genus of a Seifert surface because a closed surface with boundary is determined by orientability, number of boundary components, and Euler characteristic. If M is an orientable closed surface of genus g with k boundary components then $\chi(M) = 2 - 2g - k$ and if N is a nonorientable closed surface of genus g with k boundary components then $\chi(N) = 2 - g - k$.

1.92. Moore spaces. Let G be an abelian group and n a natural number. A *Moore space* of type (G,n) is a space $M(G,n)$, simply connected if $n>1$, with reduced homology groups

$$\widetilde{H}_k(M(G,n)) = \begin{cases} G & k=n \\ 0 & k\neq n \end{cases}$$

We will show that Moore spaces exist. For instance, $M(\mathbf{Z},n)=S^n$ and $M(\mathbf{Z}/m)=S^n\cup_m D^{n+1}$ is the CW-complex with one $(n+1)$-cell attached to an n-sphere by a map of degree m (the mapping cylinder for a degree m self-map of the n-sphere). If $G=\mathbf{Z}\oplus\cdots\oplus\mathbf{Z}\oplus\mathbf{Z}/m_1\oplus\cdots\oplus\mathbf{Z}/m_t$ is a finitely generated abelian group, then

$$M(G,n)=S^n\vee\cdots\vee S^n\vee M(\mathbf{Z}/m_1,n)\vee\cdots\vee M(\mathbf{Z}/m_t,n)$$

can be constructed as a wedge of these special Moore spaces. For a general abelian group G, take a short exact sequence $0\to K\xrightarrow{d}F\to G\to 0$ where F and K are free abelian groups. Suppose that y_β is a basis of K, x_α a basis of F, and that $dy_\beta=\sum d_{\alpha\beta}x_\alpha$. Then $M(G,n)=\bigvee S^n_\alpha\cup\coprod D^{n+1}_\beta$ where the attaching map for the $(n+1)$-cell D^{n+1}_β is $\partial D^{n+1}_\beta\xrightarrow{\Delta}\bigvee\partial D^{n+1}_\beta\xrightarrow{\vee d_{\alpha\beta}}\bigvee S^n_\alpha$. According to the cellular boundary formula, the cellular chain complex for this CW-complex is $\cdots\to 0\to K\xrightarrow{d}F\to 0\to\cdots$ so that its only nonzero reduced homology group is G in degree n.

1.93. Corollary. For any given sequence H_i, $i>0$, of abelian groups, there exists a space X such that $H_i(X)=H_i$ for all $i>0$.

Proof. $X=\bigvee M(H_i,i)$. □

10. Homological algebra for beginners

Let R be a commutative ring with unit (such as $\mathbf{Z}$, $\mathbf{Q}$ or $\mathbf{F}_p$).

1.1. Morphisms on quotient modules. Let A be an R-module and $B\subseteq A$ a submodule. A homomorphism $B/A\to C$ on the quotient module is the same thing as homomorphism $A\to C$ that vanishes on B:

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[d, "f"'] & A/B \arrow[dl, dotted, "\bar f"] \\ C & \end{tikzcd}$$

There exists a morphism $\bar f$ making the diagram commute if and only if f vanishes on B.

1.2. Exactness.

1.94. Definition. A pair of R-module homomorphisms

$$A_0\xrightarrow{f_{\text{in}}}A_1\xrightarrow{f_{\text{out}}}A_2$$

is *exact* at A_1 if the image of f_{in} equals the kernel of f_{out}.

A *short exact sequence* is an exact diagram of R-modules of the form

$$0\longrightarrow A_0\longrightarrow A_1\longrightarrow A_2\longrightarrow 0$$

In a short exact sequence, $A_0\to A_1$ is injective, $A_1\to A_2$ is surjective, and $\operatorname{im}(A_0\to A_1)=\ker(A_1\to A_2)$ in A_1.

1.95. Proposition (Split short exact sequence). The following conditions are equivalent

(1) There exists a homomorphism $A_2\leftarrow A_1$ such that $A_2\to A_1\to A_2$ is the identity of A_2
(2) There exists a homomorphism $A_1\leftarrow A_0$ such that $A_0\to A_1\to A_0$ is the identity of A_0
(3) There exists an isomorphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & 0 \\ & & \| & & \cong\downarrow & & \| & & \\ 0 & \longrightarrow & A_0 & \longrightarrow & A_0 \oplus A_2 & \longrightarrow & A_2 & \longrightarrow & 0 \end{array}$$

where the morphisms in the lower short exact sequence are the obvious ones.

A *long exact sequence* is an exact diagram of R-modules of the form

$$\cdots \longrightarrow A_{i-1} \longrightarrow A_i \longrightarrow A_{i+1} \longrightarrow A_{i+2} \longrightarrow \cdots$$

In a long exact sequence we have

$$\begin{aligned} &A_i \to A_{i+1} \text{ is surjective } \iff A_{i+1} \to A_{i+2} \text{ is the 0-homomorphism} \\ &A_i \to A_{i+1} \text{ is injective } \iff A_{i-1} \to A_i \text{ is the 0-homomorphism} \\ &A_i \to A_{i+1} \text{ is an isomorphism } \iff A_{i-1} \to A_i \text{ and } A_{i+1} \to A_{i+2} \text{ are 0-homomorphisms} \end{aligned}$$

1.96. Lemma (The 5-lemma). If

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \cong\Big\downarrow\varphi_1 & & \cong\Big\downarrow\varphi_2 & & \Big\downarrow\varphi_3 & & \varphi_4\Big\downarrow\cong & & \varphi_5\Big\downarrow\cong \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

is a commutative diagram where the two rows are exact and the four outer vertical homomorphisms are isomorphisms, then also the middle vertical homomorphism φ_3 is an isomorphism.

1.3. The category of chain complexes. Let (A, ∂) and (B, ∂) be chain complexes. Suppose that $f_0, f_1 \colon A \to B$ are two chain maps.

1.97. Definition. A *chain homotopy* from f_0 to f_1 is a sequence of homomorphisms $T \colon A_n \to B_{n+1}$ so that $\partial T + T\partial = f_1 - f_0$.

We say that f_0 and f_1 are chain homotopic, and write $f_0 \simeq f_1$, if there exists a chain homotopy from f_0 to f_1.

1.98. Lemma. $f_0 \simeq f_1 \Longrightarrow H_*(f_0) = H_*(f_1) \colon H_*(A) \to H_*(B)$

Homology is a functor from the category of R-module chain complexes with chain homotopy classes of chain homomorphisms to the category of R-modules.

1.99. Lemma. If $f_0 \simeq f_1 \colon A \to B$ and $g_0 \simeq g_1 \colon B \to C$ then $g_0 f_0 \simeq g_1 f_1 \colon A \to C$.

Proof. Suppose that $\partial S + S\partial = f_1 - f_0$ and $\partial T + T\partial = g_1 - g_0$. Let $U = Tf_1 + g_0 S \colon A_* \to C_{*+1}$. Then

$$\begin{aligned} \partial U + U\partial &= \partial(Tf_1 + g_0S) + (Tf_1 + g_0S)\partial = (\partial T + T\partial)f_1 + g_0(\partial S + S\partial) \\ &= (g_1 - g_0)f_1 + g_0(f_1 - f_0) = g_1 f_1 - g_0 f_0 \end{aligned}$$

so that U is a chain homotopy from $g_0 f_0$ to $g_1 f_1$. □

1.100. Lemma (The fundamental lemma of homological algebra). Any short exact sequence

$$0 \to (A_*, \partial_A) \to (B_*, \partial_B) \to (C_*, \partial_C) \to 0$$

of chain complexes induces a long exact sequence

$$\cdots \to H_{n+1}(C) \xrightarrow{\partial} H_n(A) \to H_n(B) \to H_n(C) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

in homology.

Proof. Diagram chase. □

1.101. Corollary (The snake lemma). Any morphism

$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r] \arrow[d, "\varphi_1"'] & A_2 \arrow[r] \arrow[d, "\varphi_2"'] & A_3 \arrow[r] \arrow[d, "\varphi_3"'] & 0 \\ 0 \arrow[r] & B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & 0 \end{tikzcd}$$

between short exact sequences induces an exact sequence

$$0 \longrightarrow \ker\varphi_1 \longrightarrow \ker\varphi_2 \longrightarrow \ker\varphi_2 \longrightarrow \operatorname{coker}\varphi_1 \longrightarrow \operatorname{coker}\varphi_2 \longrightarrow \operatorname{coker}\varphi_3 \longrightarrow 0$$

of kernels and cokernels.

Proof. This is a special case of the Fundamental Lemma of Homological Algebra 1.100. □

5-lemma and the Snake Lemma.

1.4. Finitely generated abelian groups.

1.102. Definition. [**18**, 6.51] A *basis* for an abelian group F is a subset B of F such for any element x of F has a unique presentation as a linear combination $x = \sum_{b\in B} x_b b$ of the elements in B. An abelian group is *free* if it has a basis.

The bases for a free abelian group F all have the same cardinality, the *rank* of F.

1.103. Proposition. [**18**, 10.6] Any subgroup H of a free finite rank abelian group F is free finite rank and $\operatorname{rank}(H) \leq \operatorname{rank}(F)$.

1.104. Theorem (Fundamental theorem for finitely generated abelian groups). [**18**, 10.8, 10.9] *Any finitely generated abelian group A is isomorphic to an abelian group of the form*

Smith decomposition: $\mathbf{Z}/d_1 \oplus \cdots \oplus \mathbf{Z}/d_t$ *where* $d_1 \mid \cdots \mid d_t$
Primary decomposition: $\mathbf{Z} \oplus \cdots \oplus \mathbf{Z} \oplus \mathbf{Z}/q_1 \oplus \cdots \oplus \mathbf{Z}/q_t$ *where* $q_1, \ldots, q_t$ *are prime powers*

For instance, $\mathbf{Z}/2 \oplus \mathbf{Z}/76 \cong \mathbf{Z}/4 \oplus \mathbf{Z}/38 \cong \mathbf{Z}/2 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/19$.

The *torsion subgroup* $T(A)$ of A consists of all finite order elements in A. The *rank* of A is the rank of the free abelian group $A/T(A)$. The prime powers $q_1, \ldots, q_t$ are the primary abelian invariants of A. (There are also the Smith normal form abelian invariants of A.)

CHAPTER 2

Construction and deconstruction of spaces

Simplicial complexes are used in geometric and algebraic topology to construct and deconstruct spaces. There are several kinds of simplicial complexes. In order to underline the natural development of ideas we shall here follow the historical genesis from euclidian and abstract simplicial complexes, through ordered simplicial complexes to semi-simplicial sets, aka Δ-sets, and simplicial sets.

1. Abstract Simplicial Complexes

The abstract simplicial complexes are the simplicial complexes closest to geometry. See [**19**, Chp 3] for more information.

2.1. Definition (ASC). An *abstract simplicial complex* is a set of finite sets that is closed under passage to nonempty subsets.

Let K be an ASC. The elements of K are called *simplices*. The elements of a simplex $\sigma \in K$ are called its *vertices*. The subsets of a simplex $\sigma \in K$ are its *faces*. The dimension of a simplex is one less than its cardinality. The dimension of K is the maximal dimension of any simplex in K. The *vertex set* of K is the union $V(K) = \bigcup K$ of all the simplices.

A *simplicial map* $f\colon K \to L$ between two ASCs, K and L, is a map $f\colon V(K) \to V(L)$ between the vertex sets that takes simplices to simplices, ie $\forall \sigma \in K\colon f(\sigma) \in L$ or, simply, $f(K) \subset L$. Abstract simplicial complexes and simplicial maps determine a category. Two abstract simplicial complexes are *isomorphic* if there exist invertible simplicial maps between them.

An ASC K' is a *subcomplex* of K if $K' \subset K$ (meaning that every simplex of K' is a simplex of K). The inclusion is then a simplicial map $K' \to K$.

2.2. Example (The ASC generated by a finite set). For any finite nonempty set σ of cardinality $d+1$, for instance $\sigma = d_+ = \{0, 1, \dots, d\}$, let $D[\sigma]$ be the set of all nonempty subsets of σ. Then $D[\sigma]$ is a d-dimensional finite ASC. The subset $\partial D[\sigma]$ of all proper faces of σ is a $(d-1)$-dimensional subcomplex of $D[\sigma]$ (when $d > 0$). Any map $f\colon \sigma \to \tau$ between two finite nonempty sets induces a simplicial map $D[f]\colon D[\sigma] \to D[\tau]$ between the associated ASCs. Thus $D[-]$ is a functor from finite nonempty sets to abstract simplicial complexes. For any nonempty subset $\sigma' \subset \sigma$, there is a subcomplex $D[\sigma'] \subset D[\sigma]$, and $D[\sigma'] \cap D[\sigma''] = D[\sigma' \cap \sigma'']$ when $\emptyset \neq \sigma', \sigma'' \subset \sigma$.

Examples of subcomplexes of the ASC K are

- The k-skeleton of K is the subcomplex $K^{(k)} = \{\sigma \in K \mid \dim \sigma \leq k\}$ of all simplices of dimension $\leq k$. The skeleta form an ascending chain
$$\{\{v\} \mid v \in V(K)\} = K^{(0)} \subset K^{(1)} \subset \cdots \subset K^{(k)} \subset K^{(k+1)} \subset \cdots \subset K$$
of subcomplexes of K and $K = \bigcup K^{(k)}$ is the union of its skeleta.
- The *star* of a simplex $\sigma \in K$ is the subcomplex
$$\mathrm{star}(\sigma) = \{\tau \in K \mid \sigma \cup \tau \in K\}$$
- The *link* of a simplex $\sigma \in K$ is the subcomplex
$$\mathrm{link}(\sigma) = \{\tau \in K \mid \sigma \cup \tau \in K, \sigma \cap \tau = \emptyset\}$$
of the simplices that are disjoint from σ but together with σ span a simplex in K.
- The subcomplex generated by a subset L of K is the set
$$\bigcup_{\sigma \in L} \{\tau | \tau \subset \sigma\} = \bigcup_{\sigma \in L} D[\sigma]$$

of all faces of all simplices in L. The subcomplex generated by the simplex $\sigma \in K$, is the set $D[\sigma] \subset K$, of all faces of σ. The star of σ is the subcomplex generated by all supersimplices of σ. $K = \bigcup_{\sigma\in K} D[\sigma]$ is the union of the subcomplexes generated by its (maximal) simplices.
- The *deletion* of a a vertex $v \in V(K)$ is the subcomplex
$$K - v = \mathrm{dl}_K(v) = \{\sigma \in K \mid v \notin \sigma\}$$
of all simplices in K not having v as a vertex and the *link* of v is the subcomplex
$$\mathrm{lk}_K(v) = \{\sigma \in \mathrm{dl}_K(v) \mid \sigma \cup \{v\} \in K\}$$
of all simplices that are disjoint from v but together with v span a simplex of K
- The union or intersection of any set of subcomplexes is a subcomplex.

2.1. Realization. We first construct a functor, $\mathbf{R}[-]$, from the category of sets to the category of real vector spaces. For any set V let $\mathbf{R}[V]$ be the real vector space with basis V. Explicitly, we may let
$$\mathbf{R}[V] = \{t\colon V \to \mathbf{R} \mid \mathrm{supp}(t) \text{ is finite}\}$$
be the vector space of all coordinate functions on V. (The support of a function $t\colon V \to \mathbf{R}$ is the set $\mathrm{supp}(t) = \{v \in V \mid t(v) \neq 0\}$.) For each $v \in V$, we regard v also as the real function $v\colon V \to \mathbf{R}$ given by
$$v(v') = \begin{cases} 1 & v' = v \\ 0 & v' \neq v \end{cases}$$
so that V becomes an (unordered) basis for $\mathbf{R}[V]$. If V' is a subset of V, then $\mathbf{R}[V']$ is a subspace of $\mathbf{R}[V]$.

For any map $f\colon U \to V$ between two sets, U and V, let $\mathbf{R}[f]\colon \mathbf{R}[U] \to \mathbf{R}[V]$ be the linear map given by
$$\forall s \in \mathbf{R}[U]\, \forall v \in V\colon \mathbf{R}[f](s)(v) = \sum_{f(u)=v} s(u)$$
where the sum is finite since s has finite support. Alternatively, $\mathbf{R}[f]$ is the linear map given by $\mathbf{R}[f](u) = f(u)$ for any $u \in U$. From this description it is clear that $\mathbf{R}[g\circ f] = \mathbf{R}[g]\circ\mathbf{R}[f]$ for composable maps $U \xrightarrow{f} V \xrightarrow{g} W$. Thus $\mathbf{R}[-]$ is a functor which takes injective maps to injective linear maps and surjective maps to surjective linear maps.

Next, we construct the realization functor, $|-|$, from the category of abstract simplicial complexes to the category of topological spaces. To motiviate the definition it perhaps helps to consider the standard geometric simplex in Euclidean space. Let σ be a finite set of $d+1$ elements. Then $\mathbf{R}[\sigma] = \mathbf{R}^{d+1}$ and the standard geometric d-simplex is
$$\Delta^d = |D[\sigma]| = |\sigma| = \{\sigma \xrightarrow{t} \mathbf{R} \mid t(\sigma) \subset [0,1], \sum_{v\in\sigma} t(v) = 1\} \subset \mathbf{R}[\sigma] = \mathbf{R}^{d+1}$$
More generally, for any ASC K with vertex set $V = V(K)$, the *realization* of K, is the subset of $\mathbf{R}[V]$ given by
$$|K| = \{V \xrightarrow{t} \mathbf{R} \mid t(V) \subset [0,1], \mathrm{supp}(t) \in K, \sum_{v\in V} t(v) = 1\} \subset \mathbf{R}[V]$$
The number $t(v) \in [0,1]$ is called the vth barycentric coordinate of of the point t in $|K|$. If K' is a subcomplex of K, then $|K'|$ is a subset of $|K|$. For example, if v is a vertex of K then the realization of the subcomplex K_v is the set $|K_v| = \{t \in |K| \mid t(v) = 0\}$ of all points with vth barycentric coordinate equal to 0. In fact, the realization of K is the union
$$|K| = \bigcup_{\sigma\in K} |\sigma|$$
of the realizations
$$|\sigma| = \{t \in |K| \mid \mathrm{supp}(t) \subset \sigma\} = \{t \in |K| \mid \sum_{v\in\sigma} t(v) = 1\} = \{\sum_{v\in\sigma} t_v v \mid t_v \geq 0, \sum_{v\in\sigma} t_v = 1\},$$
of its subcomplexes $D[\sigma]$. We call $|\sigma| = |D[\sigma]|$ the *cell* of the simplex $\sigma \in K$. The cell is the set of all convex combinations of vertices of the simplex $\sigma \in K$.

For any simplicial map $f\colon K \to L$ between ASCs and simplices $\sigma \in K$, $\tau \in L$ where $f(\sigma) = \tau$, the linear map $\mathbf{R}[f]\colon \mathbf{R}[V(K)] \to \mathbf{R}[V(L)]$ takes the cell $|\sigma| \subset |K|$ onto the cell $|\tau| \subset |L|$,

$$|\sigma| \ni \sum_{u\in\sigma} t_u u \xrightarrow{\mathbf{R}[f]} \sum_{u\in\sigma} t_u f(u) \in |\tau|,$$

so that the linear map $\mathbf{R}[f]$ restricts to a map $|f|\colon |K| = \bigcup_{\sigma\in k} |\sigma| \to |L| = \bigcup_{\tau\in L} |\tau|$.

The next step is to equip $|K|$ with a metric topology. The vector space $\mathbf{R}[V]$ is a metric space with the usual metric

$$d(s,t) = \left(\sum_{v\in V} |s(v) - t(v)|^2\right)^{1/2}$$

and so also the subset $|K| \subset \mathbf{R}[\sigma]$ is a metric space. We let $|K|_d$ be the set $|K|$ with the metric topology.

2.3. Lemma. Let $\sigma = \{v_0, \ldots, v_k\} \subset V$ be a k-dimensional simplex of K. Then $|\sigma|$ is a compact subset of $\mathbf{R}[V]$ homeomorphic to the standard geometric k-simplex Δ^k.

Proof. $\mathbf{R}^{k+1} \supset \Delta^k \ni \sum t_i e_i \to \sum t_i v_i \in |\sigma| \subset \mathbf{R}[V]$ is a bijective and continuous map, even an isometry, and the domain is compact, the codomain Hausdorff, so it is a homeomorphism. □

However, there is another topology, better suited for our purposes, on the set $|K| = \bigcup_{\sigma\in K} |\sigma|$. For each simplex σ of K, the subset $|\sigma|$ is a compact and therefore closed subset of the Hausdorff space $|K|_d$. The topology coherent with the closed covering $\{|\sigma| \mid \sigma \in K\}$ of $|K|$ by its cells is the topology defined by

$$A \text{ is } \begin{Bmatrix}\text{open}\\ \text{closed}\end{Bmatrix} \text{ in } |K| \overset{\text{def}}{\iff} \forall \sigma \in K\colon A \cap |\sigma| \text{ is } \begin{Bmatrix}\text{open}\\ \text{closed}\end{Bmatrix} \text{ in } |\sigma|$$

for any subset $A \subset |K|$. It is immediate that this does indeed define a topology on $|K|$. In the following, we let $|K|$ stand for the set $|K|$ with the coherent topology. All sets that are open (or closed) in $|K|_d$ are also open (or closed) in $|K|$. In particular, $|K|$ is Hausdorff. ($|K|$ is in fact even normal.) Also, it is immediate from the definition that

$$|K| \to Y \text{ is continuous} \iff |\sigma| \subset |K| \to Y \text{ is continuous for all simplices } \sigma \in K$$

for any map $|K| \to Y$ out of $|K|$ into some topological space Y. In particular, for any simplicial map $f\colon K \to L$ the induced map $|f|\colon |K| \to |L|$ is continuous in the coherent topologies as it takes simplices linearly to simplices in that $|f|(\sum_{u\in\sigma} t_u u) = \sum t_u f(u) \in |\tau| \subset L$ where $f(\sigma) = \tau$.

Obviously,

$$|\sigma| \cap |\tau| = \begin{cases} |\sigma \cap \tau| & \sigma \cap \tau \neq \emptyset \\ \emptyset & \sigma \cap \tau = \emptyset \end{cases}$$

for any two simplices of K. If $L \subset K$ is a subcomplex we may consider $|L| = \bigcup_{\tau\in L} |\tau|$ as a subset of $|K| = \bigcup_{\sigma\in K} |\sigma|$. For any simplex $\sigma \in K$,

$$|L| \cap |\sigma| = \bigcup_{\tau\in L, \tau\subset\sigma} |\tau| \subset |\sigma|$$

is closed in $|\sigma|$ because it is the finite union (possibly empty) of the realizations of those faces of σ that are in L. This shows that the realization of a subcomplex is a closed subspace of the realization.

The *open star* of a a vertex v is the complement in $|K|$ to $|K_v|$:

$$\operatorname{st}(v) = |K| - |K_v| = |K| - \{t \in |K| \mid t(v) = 0\} = \{t \in |K| \mid t(v) > 0\}$$

of $|K|$. The *open cell* of the simplex σ is the subset

$$\langle\sigma\rangle = \{t \in |K| \mid \forall v \in V(K)\colon v \in \sigma \iff t(v) > 0\}$$

of the cell $|\sigma|$. The *barycenter* of the n-simplex σ is the point $\frac{1}{n+1}\sum_{v\in\sigma} v$ of the open cell $\langle\sigma\rangle$.

2.4. Lemma (Open stars and open simplices). Let K be an ASC.

(1) The open star $\operatorname{st}(v)$ is an open star-shaped neighborhood of the vertex $v \in |K|$.
(2) The open stars cover $|K|$.
(3) $\bigcap_{v\in\sigma} \operatorname{st}(v) \neq \emptyset \iff \sigma \in K$, for any finite nonempty set $\sigma \subset V(K)$ of vertices.
(4) $\langle\sigma\rangle \cap \operatorname{st}(v) \neq \emptyset \iff v \in \sigma$, for all simplices $\sigma \in K$.

(5) $|K| = \bigcup_{\sigma\in K}\langle\sigma\rangle$ (disjoint union) and $\mathrm{st}(v) = \bigcup_{\sigma\ni v}\langle\sigma\rangle$.

Proof. (1) Because evaluation $t \to t(v)$ is a continuous map $\mathbf{R}[V(K)] \to \mathbf{R}$, the open star of v is open in $|K|_d$ and thus also in $|K|$. For any $t \in \mathrm{st}(v)$, there is a simplex $\sigma \in K$ such that $|\sigma|$ contains t. Vertex v lies in simplex σ for otherwise $\sigma \in K_v$ and $t \in |\sigma| \subset |K_v|$. The continuous path $[0,1] \ni \lambda \to \lambda v + (1-\lambda)t \in |\sigma|$ connects v and t in $|\sigma|$ and in $\mathrm{st}(v)$ for $\lambda v + (1-\lambda)t(v) = \lambda + (1-\lambda)t(v) > 0$.
(2) It is clear from the definition of $\mathrm{st}(v)$ that $|K| = \bigcup_{v\in V(K)}\mathrm{st}(v)$.
(3) $\bigcap_{v\in\sigma}\mathrm{st}(v) \neq \emptyset \iff \exists t \in |K|\colon \sigma \subset \operatorname{supp} t \iff \sigma \in K$.
(4) Clear from the definition of $\mathrm{st}(v)$. □

An ASC is *locally finite* if every vertex belongs to only finitely many simplices.

2.5. Theorem. [19] *Fuglede Let K be an ASC and $|K|$ its realization.*

1. *$|K|$ is Hausdorff.*
2. *$|K|$ is locally path connected.*
3. *There is a bijection between the set of path components of $|K|$ and the set of equivalence classes $V(K)/\sim$ where $\sim$ is the equivalence relation on the vertex set generated by the 1-simplices. In particular, $|K|$ is path connected $\iff$ $|K^{(1)}|$ is path connected.*
4. *$|K|$ is compact $\iff$ K is finite*
5. *$|K|$ is locally compact $\iff$ K is locally finite $\iff$ $|K|$ is first countable $\iff$ $|K| = |K|_d$*

Proof. (1) $|K|$ is Hausdorff since it has more open sets than the metric space $|K|_d$ which is Hausdorff.
(2) See [19, Theorem 2, p 144], or the Solution to Problem 3 of Exam January 2007, or refer ahead to the fact $|K|$ is a CW-complex and that CW-complexes are locally contractible and locally path connected [10, Proposition A.4].
(3) See the Solution to Problem 2 of Exam April 2007. Since $\{\mathrm{st}(v) \mid v \in V(K)\}$ is an open covering of $|K|$ by connected open sets, two points, x and y of $|K|$, are in the same connected component if and only if there exist finitely many $v_0, v_1, \ldots, v_k \in V(K)$ such that $\mathrm{st}(v_{i-1}) \cap \mathrm{st}(v_i) \neq \emptyset$, or $\{v_{i-1}, v_i\} \in K$, for $1 \le i \le k$ and $x \in \mathrm{st}(v_0)$, $y \in \mathrm{st}(v_k)$. These observations imply that the path connected component of $\mathrm{st}(v_0)$ is $C(\mathrm{st}(v0)) = \bigcup_{v_0\sim v_1}\mathrm{st}(v_1)$ and that the map

$$V(K)/\sim \to \pi_0(|K|)\colon v \to C(\mathrm{st}(v)) = C(v)$$

is bijective.
(4) If K is finite, $|K|$ is the quotient space of a compact space and therefore itself compact. If K is infinite, let C be the set consisting of one point from each open cell $\langle\sigma\rangle$ for all $\sigma \in K$. Then C is closed and discrete because $|\sigma| \cap C'$ is finite for any $C' \subset C$ and for any $\sigma \in K$. Thus $|K|$ is not compact.
(5) If K is locally finite then $\{|\sigma| \mid \sigma \in K\}$ is a locally finite closed covering of $|K|_d$ as each open star intersects only finitely many closed simplices. Then $|K| = |K|_d$ by the Glueing Lemma (General Topology, 2.53) so $|K|$ is first countable as it is a metric space. $|K|$ is also locally compact for $\mathrm{st}(v)$ is an open set contained in the compact set which is the realization of the subcomplex generated by all simplices that contain v. If K is not locally finite, K contains a subcomplex isomorphic to $L = \{0, 1, 2, \ldots, \{0,1\}, \{0,2\}, \ldots\}$, and $|K|$ contains a closed subspace homeomorphic to the countable wedge $|L| = \bigvee \Delta^1$ which is not first countable or locally compact at the base point (General Topology, 2.97.(7), 2.171). Then $|K|$ is not first countable, for subspaces of first countable spaces are first countable, nor locally compact, for closed subspaces of locally compact spaces are locally compact. □

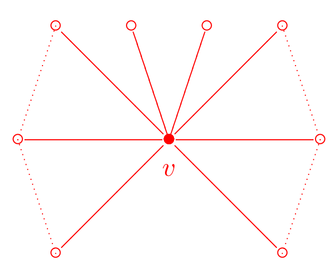


Figure 1. The open star of vertex v

2.6. Example (ESC). A *Euclidean simplicial complex* is a union C of a set $\mathcal{C}$ of geometric simplices in some Euclidean space such that

(1) All faces of all simplices in $\mathcal{C}$ are in $\mathcal{C}$.
(2) The intersection of any two simplices in $\mathcal{C}$ is either empty or a common face.
(3) $\mathcal{C}$ is a locally finite closed covering of C.

Let $K(C)$ be the ASC consisting of the set of vertices for the geometric simplices in C. There is an obvious bijective continuous map $|K(C)| \to C$, taking vertices to vertices and simplices to simplices, which is in fact a homeomorphism since the topologies on both complexes are coherent with their subspaces of simplices. Namely, observe that third condition above implies that the topology on C is coherent with the closed covering of C by its simplices in the sense that any subspace of C whose intersection with each simplex is open in that simplex is open in C. See the Solution to Problem 2 of the Exam January 2007.

2.7. Example. The real line $\mathbf{R}$ is an ESC because it is the locally finite union of the 1-simplices $[i, i+1]$ for $i \in \mathbf{Z}$. The associated ASC is is $K = \{\{i\} \mid i \in \mathbf{Z}\} \cup \{\{\{i, i+1\} \mid i \in \mathbf{Z}\}$ with realization $|K| = \mathbf{R}^1$. If $K = \{\{0, 1, \ldots, m\}, \{0, 1\}, \{1, 2\}, \ldots \{m-1, m\}, \{m, 0\}\}$ then $|K| = S^1$. If $\sigma = \{0, 1, \ldots, k\}$ then $|\Delta[\sigma]| = \Delta^k$ and $|\partial\Delta[k]| = \partial\Delta^k = S^{k-1}$. If K the 2-dimensional subcomplex of $D[\{1, 2, 3, 4, 5, 6\}]$ generated by

$$\{\{1, 4, 5\}, \{5, 2, 1\}, \{2, 5, 6\}, \{6, 3, 2\}, \{3, 6, 1\}, \{1, 4, 3\}\}$$

then $|K|$ embeds in $\mathbf{R}^3$ as a triangulated Möbius band.

The comb space $(\{0\} \cup \{1/n \mid n = 1, 2, \ldots\}) \times I$ is a subspace of $\mathbf{R}^2$ that is not an ESC because it is not locally path connected. Let $0 = 0 \times 0$ and $1_n = 1 \times (1 - \frac{1}{n})$ in the plane and let $K = \bigcup D[\{0, 1_n\}]$. Then $|K| = \bigvee [0, 1]$. The bijective continuous map $|K| \to \bigcup [0, 1_n] \subset \mathbf{R}^2$ is not a homeomorphism as $|K|$ is not first countable (General Topology, 2.97.(7)) and so does not embed into any Euclidean space.

Unfortunately, products do not commute with the realization functor. The product in the category **ASC** of abstract simplicial complexes of $K = (V_K, S_K)$ and $L = (V_L, S_L)$ is the simplicial complex $K \otimes L$ with vertex set $V_K \times V_L$ and with simplex set

$$S_{K \otimes L} = \{\sigma \subset V_K \times V_L \mid \mathrm{pr}_1(\sigma) \in S_K, \mathrm{pr}_1(\sigma) \in S_L\}$$

Note that $K \otimes L$ is indeed an abstract simplicial complex equipped with simplicial projections $\mathrm{pr}_1 \colon K \otimes L \to K$ and $\mathrm{pr}_2 \colon K \otimes L \to L$ inducing a bijection

$$\mathrm{Hom}_{\mathbf{ASC}}(M, K \otimes L) \to \mathrm{Hom}_{\mathbf{ASC}}(M, K) \times \mathrm{Hom}_{\mathbf{ASC}}(M, L)$$

For instance $D[m] \otimes D[n] = D[mn + m + n]$ so the induced map $|\mathrm{pr}_1| \times |\mathrm{pr}_2| \colon |K \otimes L| \to |K| \times |L|$ of realizations is usually not a homeomorphism.

2. Ordered Simplicial complexes

Very often, the vertex set of an ASC is an ordered set (Example 2.7). When we want to remember the ordering of the vertex set we do it formally by speaking about OSCs.

2.9. Definition (OSC). An ordered simplicial complex is an ASC with a partial ordering on its vertex set such that every simplex is totally ordered.

A *simplicial map* $f \colon K \to L$ between two OSCs is a poset map $f \colon V(K) \to V(L)$ between the vertex sets that takes simplices of K to simplices of L.

2.10. Example (Order complexes). To every poset P, we can associate an OSC, $\Delta(P)$, the *order complex* of P, consisting of all nonempty totally ordered finite subsets of P. $\Delta(-)$ is a functor from the category of posets to the category of OSCs. If σ is a finite totally ordered set containing $k+1$ elements, then the ASC $D[\sigma]$ has $(k+1)!$ automorphisms but the OSC $\Delta(\sigma)$ has just one automorphism.

2.11. Example (ASC $\xrightarrow{\mathrm{sd}}$ OSC $\to$ Δ-sets). To every ASC K we can associate a poset, $P(K)$, the face poset of K, which is K ordered by inclusion. $P(-)$ is a functor from the category of ASCs to the category of posets. The composite $\mathrm{sd} = \Delta \circ P$, called the *barycentric subdivision* functor (Figure 3), is a functor from ASCs to OSCs.

To every OSC K we can associate a Δ-set (even a simplicial set [**21**, 8.1.8]). Namely, let $K_n \subset K$ be the subset of simplices of dimension n and let $d_i \colon K_n \to K_{n-1}$, $0 \le i \le n$, be the map obtained by deleting vertex number i in each n-simplex $\sigma \in K_n$. This construction is a functor from the category of OSCs to the category of Δ-sets.

The vertex set of the OSC $\operatorname{sd} K$ is the simplex set of the ASC K and the k-simplices of $\operatorname{sd} K$ are length k chains of inclusions $s_0 \subset \cdots \subset s_k$ of simplices in K.

The realization of an OSC is the realization of the underlying ASC. What is the realization of the Δ-set associated to an OSC? What is the realization of the OSC associated to an ASC? The next lemma shows that any space that can be realized by an ASC can also be realized by an OSC.

2.12. Lemma. For any ASC K, barycenters give a homeomorphism $b\colon |\operatorname{sd} K| \to |K|$.

Proof. For each simplex s of K, let $b(s) \in \langle s \rangle \subset |K|$ be the barycenter. Then $s \to b(s)$ is a map from the 0-skeleton of $|\operatorname{sd} K|$ to $|K|$. Now extend this map linearly by

$$(s_0 \subset \cdots \subset s_k) \times \Delta^k \to s_k \times \Delta^k \to |K|\colon \Delta^k \ni \sum t_i e_i \to \sum t_i b(s_i) \in |K|$$

In other words, this is the map b given by $b|(s_0 \subset \cdots \subset s_k) = [b(s_0), \dots, b(s_k)]$. We exploit that it makes sense to take convex combinations of points of $|K|$ if they all belong to a simplex. It can be shown that $b\colon |\operatorname{sd} K| \to |L|$ is a homeomorphism [**19**, Thm 4 p 122]. $\square$

The inverse of the barycentric subdivision map $|K| \xrightarrow{b^{-1}} |\operatorname{sd} K|$ is a cellular map (1.73) (and *not* a simplicial map) as $|K|^n \subset |\operatorname{sd} K|^n$ for all n.

2.13. Theorem (Simplicial Approximation). *Let K and L be simplicial complexes where K is finite. For any map $f\colon |K| \to |L|$ there exist some $r \ge 0$ and a simplicial map $g\colon \operatorname{sd}^r K \to L$ such that*

1. *for every point $x \in |\operatorname{sd}^r K|$, there is a closed simplex of L such that $f(x)$ and $|g|(x)$ lie in that simplex*
2. *$|\operatorname{sd}^r K| \underset{\simeq}{\xrightarrow{b^r}} |K| \xrightarrow{f} |L|$ is homotopic to $|g|$.*

Proof. The realization $|K|$ is a compact metric space (2.5). One can show that the maximal diameter of any positive dimensional simplex of $|K|$ goes down under barycentric subdivision [**19**, Lemma 12 p 124].

Let ε be the Lebesgue number (General topology, 2.158) of the open covering of $|K|$ induced from the open covering of $|L|$ by open stars (2.4.2), $|K| = \bigcup_{u \in V(L)} f^{-1}\operatorname{st}(u)$. By replacing K by some iterated subdivision we may assume that all simplices in $|K|$ have diameter $< \varepsilon/2$. The triangle inequality implies that the open star $\operatorname{st}(v)$ of any vertex v in K has diameter $< \varepsilon$. Thus $\operatorname{st}(v) \subset f^{-1}\operatorname{st}(g(v))$ or $f(\operatorname{st}(v)) \subset \operatorname{st}(g(v))$ for some function $g\colon V(K) \to V(L)$ between the vertex sets.

We now want to prove that

1. g is a simplicial map
2. For any point in $x \in |K|$, $f(x)$ and $|g|(x)$ belong to the same simplex of L

For the first item, let s be a simplex in K. Since

$$\emptyset \neq f(\langle s \rangle) \subset f\big(\bigcap_{v \in s} \operatorname{st}(v)\big) \subset \bigcap_{v \in s} f(\operatorname{st}(v)) \subset \bigcap_{v \in s} \operatorname{st}(g(v)) = \bigcap_{u \in g(s)} \operatorname{st}(u)$$

the image $g(s) = \{g(v) \mid v \in s\}$ is a simplex of L (2.4.3). The second item will follow if we can show that

$$\forall x \in |K| \forall s_2 \in L\colon f(x) \in \langle s_2 \rangle \implies |g|(x) \in |s_2|$$

or, since $|K| = \bigcup_{s \in K} \langle s \rangle$, that

$$\forall s \in K \forall s_2 \in L\colon f\langle s_1 \rangle \cap \langle s_2 \rangle \neq \emptyset \implies g(s_1) \subset s_2$$

But this follows from

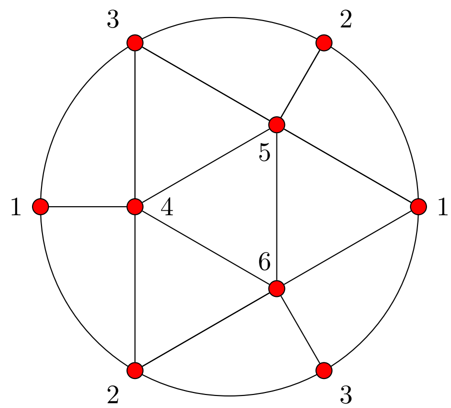
$$f\langle s_1 \rangle \cap \langle s_2 \rangle \subset f\big(\bigcap_{v \in s_1} \operatorname{st}(v)\big) \cap \langle s_2 \rangle \subset \bigcap_{v \in s_1} f(\operatorname{st}(v)) \cap \langle s_2 \rangle \subset \bigcap_{v \in s_1} \operatorname{st}(g(v)) \cap \langle s_2 \rangle = \bigcap_{u \in g(s_1)} \operatorname{st}(u) \cap \langle s_2 \rangle$$

for, as the latter set is nonempty, $g(s_1) \subset s_2$ (2.4.4).

We can now define a homotopy $F\colon I \times |K| \to |L|$ by $F(t,x) = tf(x) + (1-t)|g|(x)$. This is well-defined since if $f(x) \in \langle s_2 \rangle$, both $f(x)$ and $|g|(x)$ are in $|s_2|$ and so $F(t,x) \in |s_2|$ for all t. The continuity of F follows because the restriction of F to $I \times |s_1|$ is continuous for all simplices s_1 in K [**19**, 3.1.21]. $\square$

A *triangulation* of a topological space X consists of an OSC K and a homeomorphism $|K| \to X$. A *polyhedron* is a space that admits a triangulation. (A fundamental, but rarely proved, theorem says that any compact surface is a polyhedron [**7**].)

Here (and here) is a triangulation of $\mathbf{R}P^2$



The Hauptvermutung says that any two triangulations of a manifold have isomorphic subdivisions. This is true for surfaces and that is why the classification of surfaces is a combinatorial problem. However, the Hauptvermutung is not true in dimensions > 2, and therefore the classification of 3-manifolds can not be reduced to combinatorics. Here is a list of manifold triangulations. See [**15**, Chp 4] for triangulations of 3-manifolds.

ASCs can be investigated with the help of the program asc.prg

3. Partially ordered sets

We shall need some constructions with partially ordered sets.

2.14. Definition. A *partial order* (or partially ordered set or poset) is a set X with a binary relation $\leq$ that is

reflexive: $a \leq a$ for all $a \in X$
anti-symmetric: If $a \leq b$ and $b \leq a$ then $a = b$
transitive: If $a \leq b$ and $b \leq c$ then $a \leq c$

A *linear order* is a partial order where any two elements are comparable: For any $a, b \in X$, either $a \leq b$ or $b \leq a$.

A map $f \colon X \to Y$ between partially ordered sets is *order preserving* if $x_1 \leq x_2 \Longrightarrow f(x_1) \leq f(x_2)$. Let **POSET** denote the category of posets with order preserving maps, and **POSI** the category of posets with order preserving *injective* maps. If $f \colon X \to Y$ is an injecttive order preserving map, then f is strictly order preserving in the sense that $x_1 < x_2 \Longrightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in X$. (We write $x_1 < x_2$ if $x_1 \leq x_2$ and $x_1 \neq x_2$.)

2.15. Definition. The *standard n-simplex*, $n = 0, 1, 2, \ldots$, is the linear order

$$n_+ = \{0 < 1 < \cdots < n\}$$

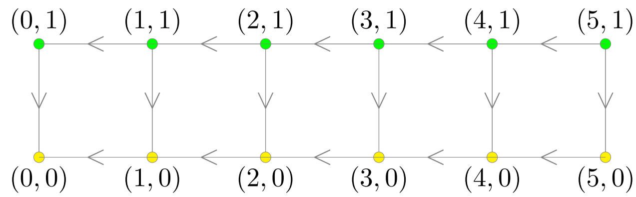
of $n+1$ points from the linear order $\mathbf{Z}$.

There are $\binom{n+1}{m+1} = \binom{n+1}{n-m}$ injective poset morphisms from m_+ to n_+.

The *cylinder* $X \times 1_+$ on the poset X is a poset with the *product order*

$$(x_1, t_1) \leq (x_2, t_2) \iff x_1 \leq x_2 \text{ and } t_1 \leq t_2$$

There are injective poset morphisms $i_0, i_1 \colon X \to X \times 1_+$ given by $i_0(x) = (x, 0)$ and $i_1(x) = (x, 1)$ embedding X as the bottom and top of the cylinder. As an example, here is the cylinder $5_+ \times 1_+$ on the 5-simplex



2.16. Definition. (1) The ith *coface*, $d^i_n \in \mathbf{POSI}((n-1)_+, n_+)$, $i \in n_+$, is the injective poset morphism

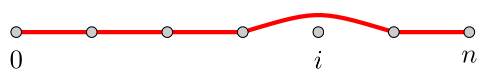
$$0 < 1 < \cdots < i-1 < i+1 < \cdots < n$$

that avoids i.

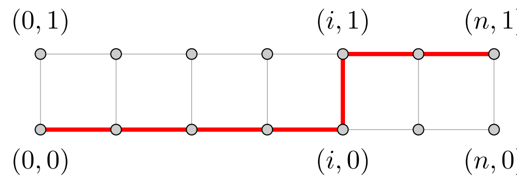
(2) The ith *prism*, $P^i_n \in \mathbf{POSI}((n+1)_+, n_+ \times 1_+)$, $i \in n_+$, is the maximal $(n+1)$-simplex in $n_+ \times 1_+$

$$P^i_n = (0,0) < \cdots < (0,i) < (1,i) < \cdots < (n,i)$$

with a jump from $(i,0)$ to $(i,1)$.



$d^i \colon (n-1)_+ \to n_+$



$P^i \colon (n+1)_+ \to n_+ \times 1_+$

We now investigate the composition map

$$\mathbf{POSI}((n-1)_+, n_+) \times \mathbf{POSI}(n_+, (n+1)_+) \xrightarrow[\text{composition}]{(d^i, d^j) \to d^j d^i} \mathbf{POSI}((n-1)_+, (n+1)_+)$$

in the category **POSI**. The domain of this composition map has double as many elements as the codomain: The domain has $(n+1)(n+2)$ elements and the codomain has $\binom{n+2}{2}$ elements. We divide the domain into two disjoint parts of equal size such that the composition map is a bijection on each part.

2.17. Lemma (Cosimplicial identities). The diagram

$$\begin{tikzcd}
\{(i,j) \mid n+1 \geq j > i \geq 0\} \arrow[rr, shift left, "R(i,j) = (j,i) + (-1,0)"] \arrow[dr, "(i,j) \to d^j d^i"'] & & \{(i,j) \mid n \geq i \geq j \geq 0\} \arrow[ll, shift left, "R^{-1}(i,j) = (j,i) + (0,1)"] \arrow[dl, "d^j d^i \leftarrow (i,j)"] \\
& \mathbf{POSI}((n-1)_+, (n+1)_+) &
\end{tikzcd}$$

is commutative. In other words,

$$d^j d^i = \begin{cases} d^i d^{j-1} & j > i \\ d^{i+1} d^j & j \leq i \end{cases} \tag{2.18}$$

when $i \in n_+$ and $j \in (n+1)_+$.

Proof. Let us first look at the green triangle, $n+1 \geq j > i \geq 0$, of Figure 2. We observe that $d^j d^i$ and $d^i d^{j-1}$ both equal the injective poset morphism $d^{\{i,j\}} \colon (n-1)_+ \to (n+1)_+$ that do not take the values i and j. This means that composition is invariant under the bijection R, $R(i,j) = (j-1,i)$, between the green and yellow triangle. In the yellow triangle, $0 \leq j \leq i \leq n$, $R^{-1}(i,j) = (j, i+1)$, and $d^j d^i = d^{i+1} d^j$. $\square$

Next, we investigate compositions of coface and prism maps

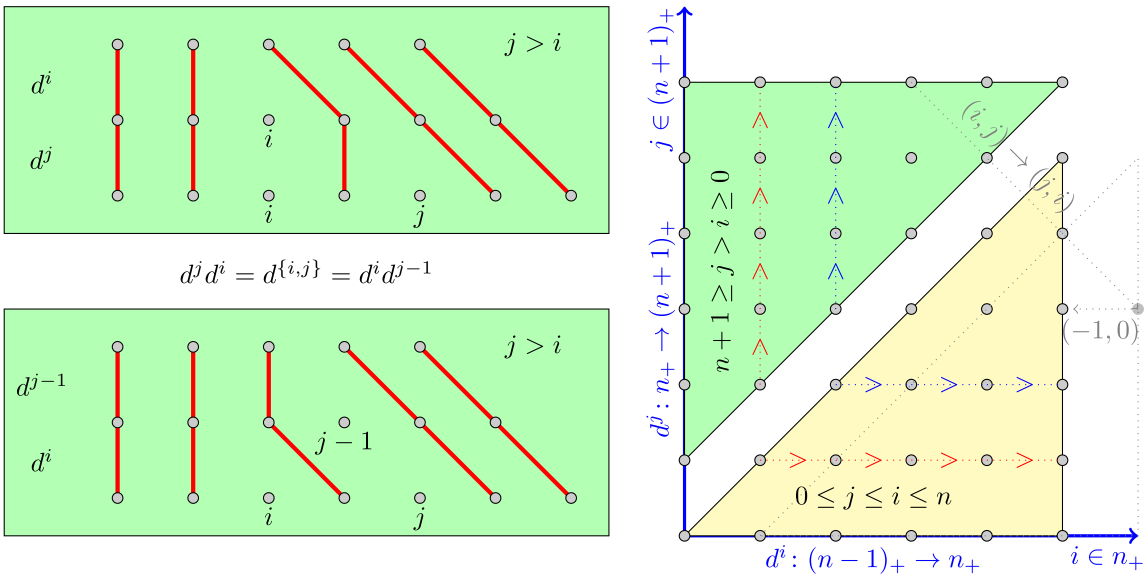


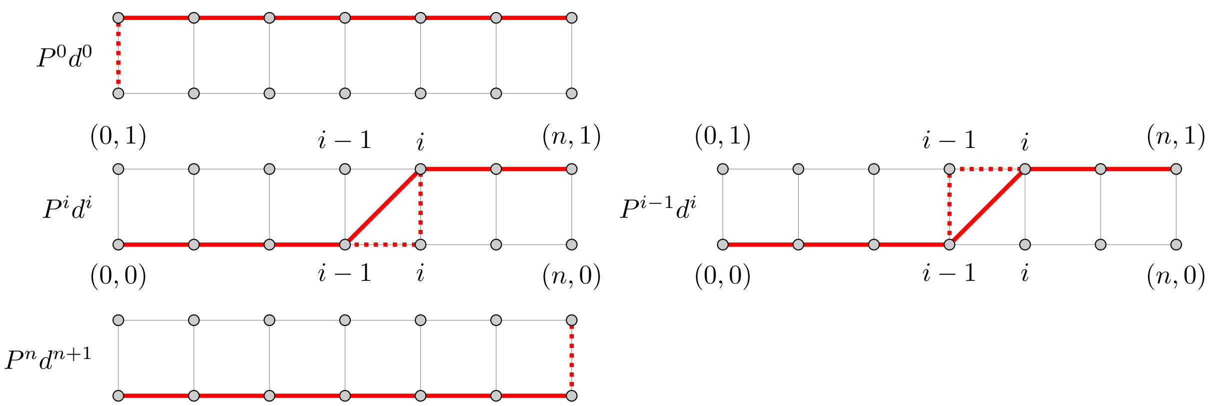
FIGURE 2. Cosimplicial identities

$$n_+ \xrightarrow{d^j} (n+1)_+ \xrightarrow{P^i_n} n_+ \times 1_+$$
$$n_+ \xrightarrow{P^i_{n-1}} (n-1)_+ \times 1_+ \xrightarrow{d^j \times 1} n_+ \times 1_+$$

2.19. LEMMA (Prism identities). $P^0_n d^0 = i_1$, $P^i_n d^i = P^{i-1}_n d^i$ for $1 \leq i \leq n$, and $P^n_n d^{n+1} = i_0$. For $j \notin \{i, i+1\}$

$$P^i_n d^j = \begin{cases} (d^{j-1} \times 1) P^i_{n-1} & n+1 \geq j > i+1 \geq 1 \\ (d^j \times 1) P^{i-1}_{n-1} & 0 \leq j < i \leq n \end{cases}$$

PROOF. We may illustrate the first identities like this



For instance, take $P^i \colon (n+1)_+ \to n_+ \times 1_+$ and precompose with $d^i \colon n_+ \to (n+1)_+$. The result, $P^i d^i$, is shown above. The remaining identities are indicated in the left part of Figure 3. $\square$

Let $BX_n = \mathbf{POSI}(n_+, X)$ be the set of all n-simplices in the poset X. The ith *face map* $d_i \colon BX_n \to BX_{n-1}$ is given by $d_i(n_+ \xrightarrow{\sigma} X) = (n-1)_+ \xrightarrow{d^i} n_+ \xrightarrow{\sigma} X = \sigma d^i$, $i \in n_+$. The nth *chain group* of X is the free abelian group $C_n(X) = \mathbf{Z}BX_n$ with basis BX_n. The *chain complex* of X is the chain complex $(C_*(X), \partial)$ with

$$C_n(X) = \mathbf{Z}BX_n, \qquad \partial = \sum_{i \in n_+} (-1)^i d_i \colon C_n(X) \to C_{n-1}(X), \qquad \partial\sigma = \sum_{i \in n_+} (-i)^i \sigma d^i, \quad \sigma \in BX_n$$

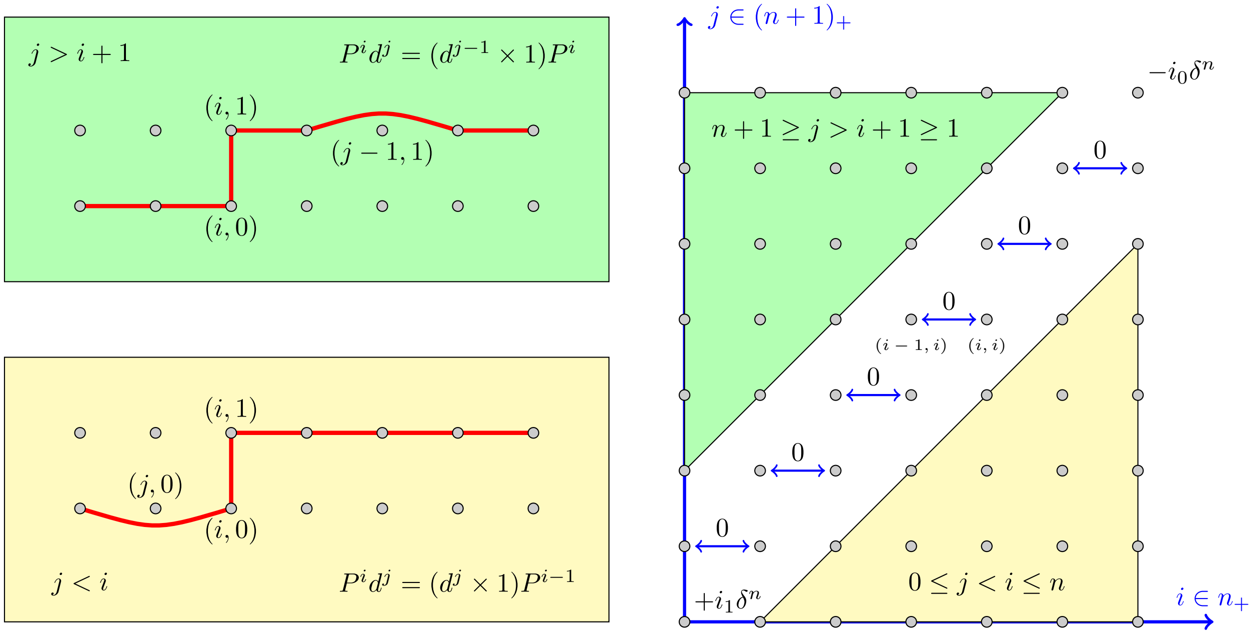


Figure 3. Prism identities

The next corollary shows that this is indeed a chain complex.

2.20. Corollary. The composition of two boundary maps

$$\begin{tikzcd} C_{n-1}(X) & C_n(X) \arrow[l, "\partial_{n-1}"'] & C_{n+1}(X) \arrow[l, "\partial_n"'] \arrow[ll, bend right, "\partial_{n-1}\partial_n = 0"'] \end{tikzcd}$$

is trivial so that $\mathrm{im}(\partial_n) \subseteq \ker(\partial_{n-1})$.

Proof. Since the boundary is obviously natural, $\partial\partial\sigma = \partial\partial\sigma\delta^{n+1} = \sigma\partial\partial\delta^{n+1}$, and it suffices to prove that $\partial\partial\delta^{n+1} = 0$. We find that

$$\partial\partial\delta^{n+1} = \partial \sum_{j \in (n+1)_+} (-1)^j d^j = \sum_{\substack{i \in n_+ \\ j \in (n+1)_+}} (-1)^{i+j} d^j d^i = 0$$

as the pairs from the green triangle cancel the pairs from the yellow triangle in Figure 2. $\square$

Define the *prism operator* $P_n \colon C_n(X) \to C_{n+1}(X \times 1_+)$ to be the $\mathbf{Z}$-linear homomorphism given by

$$P_n(n_+ \xrightarrow{\sigma} X) = \sum_{i \in n_+} (-1)^i \big((n+1)_+ \xrightarrow{P^i} n_+ \times 1_+ \xrightarrow{\sigma \times 1} X \times 1_+\big) = \sum_{i \in n_+} (-1)^i (\sigma \times 1) P^i_n \tag{2.21}$$

In particular, the prism on the identity n-simplex is

$$P_n\delta^n = \sum_{i \in n_+} (-1)^i P^i \in C_{n+1}(n_+ \times 1_+)$$

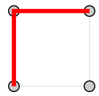
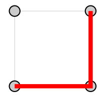
The prism operator is obviously natural in the sense that the diagram

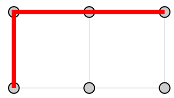
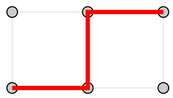
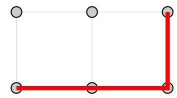
$$\begin{tikzcd} C_n(X) \arrow[r, "f_*"] \arrow[d, "P"'] & C_n(Y) \arrow[d, "P"] \\ C_n(X \times 1_+) \arrow[r, "(f \times 1)_*"'] & C_n(Y) \end{tikzcd}$$

commutes for any injective poset morphism $f \colon X \to Y$.

2.22. Example. Here are the values of $P_n\delta^n$ for $n = 0, 1, 2,$:

$P_0\delta^0 =$  $\in C_1(0_+ \times 1_+)$

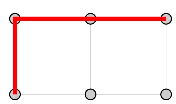
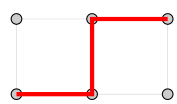
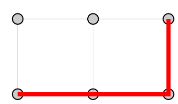
$P_1\delta^1 =$  -  $\in C_2(1_+ \times 1_+)$

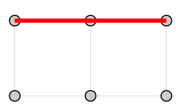
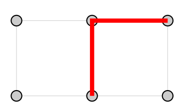
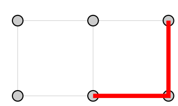
$P_2\delta^2 =$  -  +  $\in C_3(2_+ \times 1_+)$

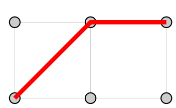
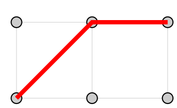
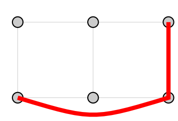
2.23. Example. We consider the special case $n = 2$ where we have the homomorphisms

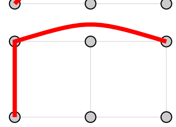
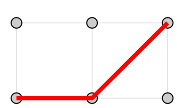
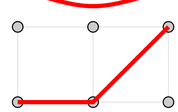
$$\begin{tikzcd} C_1(2_+) \arrow[dr, "P_1"'] & C_2(2_+) \arrow[l, "\partial"'] \arrow[d, shift right, "i_0"'] \arrow[d, shift left, "i_1"] \arrow[dr, "P_2"] & \\ & C_2(2_+ \times 1_+) & C_3(2_+ \times 1_+) \arrow[l, "\partial"] \end{tikzcd}$$

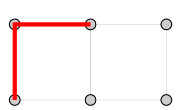
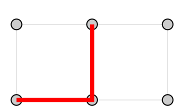
The formal proof follows the same the pattern as we see in this special case. Here is $\partial P_2\delta^2 = d_0P_2\delta^2 - d_1P_2\delta^2 + d_2P_2\delta^2 - d_3P_2\delta^2 \in C_2(2_+ \times 1_+)$ as computed from $P_2\delta^2 = P_2^0 - P_2^1 + P_2^2 \in C_3(2_+ \times 1_+)$,

$P_2\delta^2 = P_2^0 - P_2^1 + P_2^2$  -  + 

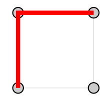
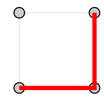
$d_0P_2^0 - d_0P_2^1 + d_0P_2^2$  -  +  -

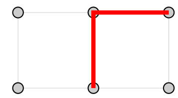
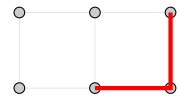
$-d_1P_2^0 + d_1P_2^1 - d_1P_2^2$  +  -  +

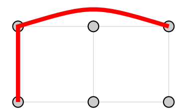
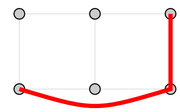
$+d_2P_2^0 - d_2P_2^1 + d_2P_2^2$  -  +  -

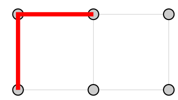
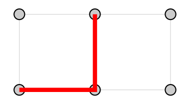
$-d_3P_2^0 + d_3P_2^1 - d_3P_2^2$  +  -

and here is $P_1\partial\delta^2 = P_1(d^0 - d^1 + d^2) = (d^0 \times 1)(P_1^0 - P_1^1) - (d^1 \times 1)(P_1^0 - P_1^1) + (d^2 \times 1)(P_1^0 - P_1^1) \in C_2(2_+ \times 1_+)$ as computed from $P_1\delta^1 = P_1^0 - P_1^1 \in C_2(1_+ \times 1_+)$,

$P_1\delta^1 = P_1^0 - P_1^1$  $-$ 

$(d^0 \times 1)P_1^0 - (d^0 \times 1)P_1^1$  $-$  $+$

$-(d^1 \times 1)P_1^0 + (d^1 \times 1)P_1^1$  $+$  $-$

$+(d^2 \times 1)P_1^0 - (d^2 \times 1)P_1^1$  $-$ 

We conclude that indeed $\partial P_2\delta^2 = i_1 - P_1\partial\delta^2 - i_0$. This means that the boundary of the prism on a simplex is the top minus the prism on the boundary minus the bottom.

2.24. Corollary (The boundary of a prism). In the diagram

$$\begin{tikzcd} C_{n-1}(X) \arrow[dr, "P"'] & C_n(X) \arrow[l, "\partial"'] \arrow[d, shift right, "i_0"'] \arrow[d, shift left, "i_1"] \arrow[dr, "P"] & \\ & C_n(X \times 1_+) & C_{n+1}(X \times 1_+) \arrow[l, "\partial"] \end{tikzcd}$$

$$\partial P = i_1 - P\partial - i_0 \colon C_n(X) \to C_n(X \times 1_+)$$

The boundary of a prism is the top, the prism on the boundary, and the bottom

Proof. As the homomorphisms of the diagram are natural and $\sigma = \sigma_*\delta^n$ for any n-simplex $\sigma \in C_n(X)$, it suffices to consider the case where $X = n_+$ is the standard n-simplex and $\sigma = \delta^n$ is the identity simplex. We need to show that $\partial P\delta^n = (i_1 - P_{n-1}\partial + i_0)\delta^n$. We observe that

$$\partial P\delta^n = \partial \sum_{i \in n_+} (-1)^i P^i = \sum_{\substack{i \in n_+ \\ j \in (n+1)_+}} (-1)^{i+j} P^i d^j, \quad P\partial\delta^n = P \sum_{j \in n_+} (-1)^j d^j \overset{(2.21)}{=} \sum_{\substack{i \in (n-1)_+ \\ j \in n_+}} (-1)^{i+j} (d^j \times 1) P^i$$

The sum that computes $\partial P\delta^n$ runs over all the pairs $(i,j) \in n_+ \times (n+1)_+$ shown in Figure 3. The contribution to this sum from the two lines $j = i$ and $j = i+1$, $i \in n_+$, is

$$\sum_{i \in n_+} (P^i d^i - P^i d^{i+1}) = i_1 - i_0$$

as all the horizontally connected pairs in Figure 3. The contribution to the sum from the green and the yellow triangle is

$$\begin{aligned} \partial P\delta^n - (i_1 - i_0) &= \sum_{n+1 \geq j > i+1 \geq 1} (-1)^{i+j} P^i d^j + \sum_{0 \leq j < i \leq n} (-1)^{i+j} P^i d^j \\ &= \sum_{n+1 \geq j > i+1 \geq 1} (-1)^{i+j} (d^{j-1} \times 1) P^i + \sum_{0 \leq j < i \leq n} (-1)^{i+j} (d^j \times 1) P^{i-1} && \text{(Lemma 2.19)} \\ &= -\sum_{n \geq j \geq i+1 \geq 1} (-1)^{i+j} (d^j \times 1) P^i - \sum_{0 \leq j \leq i \leq n-1} (-1)^{i+j} (d^j \times 1) P^i && \text{(re-indexing)} \\ &= -\sum_{n \geq j > i \geq 0} (-1)^{i+j} (d^j \times 1) P^i - \sum_{0 \leq j \leq i \leq n-1} (-1)^{i+j} (d^j \times 1) P^i && \text{(rewriting the first sum)} \\ &= -\sum_{\substack{i \in (n-1)_+ \\ j \in n_+}} (-1)^{i+j} (d^j \times 1) P^i && \text{(see formula above)} \\ &= -P\partial\delta^n \end{aligned}$$

We conclude that $\partial P\delta^n = i_1 - P\partial\delta^n - i_0$ and that's what we wanted to show. $\square$

The nth homology group of X,

$$H_n(X) = H_n(C_n(X), \partial)$$

is the nth homology group of the chain complex of X. Homology is a functor from the category **POSI** of posets with injective poset morphisms to the category of abelian groups.

2.25. Theorem (Homotopy invariance). $(i_0)_* = (i_1)_* \colon H_*(X) \to H_*(X \times 1_+)$.

Proof. Let $[z] \in H_n(X)$ be a homology class represented by an n-cycle $z \in \mathbf{Z}B_n(X)$. The homology classes $(i_0)_*[z] = [i_0 z]$ and $(i_0)_*[z] = [i_0 z]$ in $H_n(X \times 1_+)$ are identical because

$$[i_1 z] - [i_0 z] = [\partial P z + P\partial z] = [\partial P z] = 0$$

as $\partial z = 0$ and $\partial P z$ is a boundary. $\square$

4. Δ-sets

We now focus on a small full subcategory of the category **POSI** of posets with injective order preserving maps.

2.26. Definition. $\Delta_<$ is the full subcategory of **POSI** whose objects are the standard n-simplices n_+, $n \geq 0$.

2.27. Definition. A Δ-set[1] is a functor $S_\bullet \colon \Delta_<^{\mathrm{op}} \to \mathbf{SET}$, a contravariant functor from the category $\Delta_<$ of standard simplices to the category of sets. **ΔSET** is the category of Δ-sets. A co-Δ-set is a functor $S^\bullet \colon \Delta_< \to \mathbf{SET}$.

In other words, a Δ-set is a graded set $S_\bullet = \bigcup_{n=0}^{\infty} S_n$, where $S_n = S(n_+)$, with *face maps* $d_i = S(d^i) \colon S_n = S(n_+) \to S((n-1)_+) = S_{n-1}$, satisfying the *simplicial identities*

$$d_i d_j = \begin{cases} d_{j-1} d_i & j > i \\ d_{i+1} d_j & j \leq i \end{cases} \tag{2.28}$$

when $i \in n_+$ and $j \in (n+1)_+$. The simplicial identities, dual to the cosimplicial identities (2.18), can be visualized as a commutative diagram

$$\begin{tikzcd} \{(i,j) \mid n+1 \geq j > i \geq 0\} \arrow[rr, shift left, "{R(i,j) = (j,i) + (-1,0)}"] \arrow[dr, "{(i,j) \to d_i d_j}"'] & & \{(i,j) \mid n \geq i \geq j \geq 0\} \arrow[ll, shift left, "{R^{-1}(i,j) = (j,i) + (0,1)}"] \arrow[dl, "{d_i d_j \leftarrow (i,j)}"] \\ & \mathbf{SET}(S_{n+1}, S_{n-1}) & \end{tikzcd}$$

dual to the commutative diagram of Lemma 2.17.

The Δ-set S is N-dimensional if S_n is empty for $n > N$. If S and T are Δ-sets, a morphism $\varphi \colon S \to T$ is a sequence of maps $\varphi_n \colon S_n \to T_n$ commuting with the face maps. (Similarly, we may speak about Δ-spaces, Δ-G-spaces, Δ-groups, etc.)

2.1. The chain complex and simplicial homology groups of a Δ-set. The *chain complex* of the Δ-set S is the chain complex $(\mathbf{Z}[S], \partial)$

$$0 \longleftarrow \mathbf{Z}[S_0] \xleftarrow{\partial_0} \mathbf{Z}[S_1] \xleftarrow{\partial} \cdots \xleftarrow{\partial} \mathbf{Z}[S_{n-1}] \xleftarrow{\partial_{n-1}} \mathbf{Z}[S_n] \xleftarrow{\partial_n} \mathbf{Z}[S_{n+1}] \longleftarrow \cdots \tag{2.29}$$

with boundary operator

$$\mathbf{Z}[S_n] \leftarrow \mathbf{Z}[S_{n+1}] \colon \partial_n = \sum_{j \in (n+1)_+} (-1)^j d_j$$

Here, $\mathbf{Z}[S_n]$ is the free abelian group with basis S_n. ($\mathbf{Z}[S]$ is the Δ-abelian group which in degree n is the free abelian group $\mathbf{Z}[S_n]$.) That $\mathbf{Z}[S]$ is indeed a chain complex is a consequence of the simplicial identities (2.28):

[1] Some authors use the term "semi-simplicial set" instead since it is a "simplicial set" without degeneracies. See [21, Historical Remark 8.1.10]

2.30. Lemma. $\partial_{n-1}\partial_n = 0$

Proof. The composition of ∂_n followed by ∂_{n-1} is

$$\partial_{n-1}\partial_n = \sum_{\substack{j\in(n+1)_+ \\ i\in n_+}} (-1)^{i+j} d_i d_j$$

where the summation runs over the green and yellow triangles of Figure 2. The simplicial identities (2.28) show that the contributions from these two triangles will cancel as they occur with opposite signs in the above sum. □

2.31. Definition. The *simplicial homology groups* $H_n^\Delta(S)$ of the Δ-set S are the homology groups of the chain complex $(\mathbf{Z}[S], \partial)$ of S.

2.32. Example. Let S be the 1-dimensional Δ-set

$$\{v\} \underset{d_1}{\overset{d_0}{\leftleftarrows}} \{a\}$$

Then $|S| = S^1$ and the chain complex $(\mathbf{Z}S, \partial)$ is $0 \xleftarrow{0} \mathbf{Z} \xleftarrow{0} \mathbf{Z} \xleftarrow{0} 0$ with homology groups $H_0^\Delta(S) = \mathbf{Z}$ and $H_1^\Delta(S) = \mathbf{Z}$.

2.2. Sub-Δ-sets and quotient Δ-sets. Let S be a Δ-set. We discuss sub-Δ-sets and quotient Δ-sets of S.

2.33. Definition. A sub-Δ-set of S is a sequence of subsets A_n of S_n stable under the face maps.

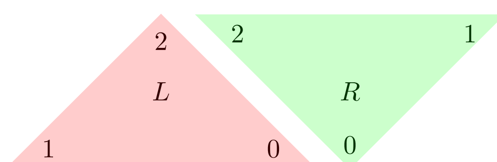
2.34. Definition. An equivalence relation $\sim$ on S consists of a sequence of equivalence relations on the sets S_n, $n \geq 0$, such that the face maps preserve the equivalence relation:

$$\forall a, b \in S_n \colon a \sim b \Longrightarrow d_i a \sim d_i b, \qquad 0 \leq i \leq n,$$

The quotient Δ-set is the Δ-set $S/\sim$ whose set of n-simplices is $\big(S/\sim\big)_n = S_n/\sim$ and whose face maps are induced by those of S.

- It is only possible to identify a simplex with another simplex of the same dimension; it is not possible to collapse an n-simplex to a point if $n > 0$.
- The Δ-set morphism $S \to S/\sim$ induces a quotient map $|S| \to |S/\sim|$ from the realization of S to the realization of its quotient (General topology, 2.79).
- If A is sub-Δ-set of S then there is a quotient Δ-set S/A with $(S/A)_n = S_n/A_n$ obtained by identifying all the n-simplices of A. We write $H_n(S, A)$ for the nth homology group of the quotient Δ-set S/A.

2.35. Example (A Δ-complex structure on the torus $M_1 = S^1 \times S^1$). Let S be the Δ-set $L \amalg R$ where $L = \Delta[2_+] = R$, realizing $\Delta^2 \amalg \Delta^2$. Let $\sim$ be the smallest equivalence relation on S that identifies $d_1 L = d_1 R$, $d_2 L = d_0 R$, and $d_0 L = d_2 R$.



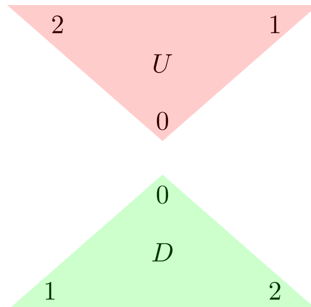
Let $T = S/\sim$ be the quotient Δ-set. The simplices of T are $T_2 = \{L, R\}$, $T_1 = \{d_0L, d_1L, d_2L\}$, and $T_0 = \{(0)\}$. The chain complex is

$$0 \leftarrow \mathbf{Z} \xleftarrow{\partial_1} \mathbf{Z}^3 \xleftarrow{\partial_2} \mathbf{Z}^2 \leftarrow 0, \qquad \partial_2 = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix}, \quad \partial_1 = 0$$

because $\partial_2 L = d_0 L - d_1 L + d_2 L$ and $\partial_2 R = d_2 R - d_1 R - d_2 R = d_0 L - d_1 L - d_2 L = \partial_2 L$. Thus the simplicial homology groups are $H_0(T) = \mathbf{Z}$, $H_1(T) = \mathbf{Z} \oplus \mathbf{Z}$, and $H_2(T) = \mathbf{Z}$. The 2nd homology group is generated by the 2-cycle $L - R$, and the 1st homology group by the 1-cycles $d_0 L$ and $d_2 L$. The 1-cycle $d_2 L$ is homologous to $d_0 L - d_1 L$.

Try to identify the 1-cycles $d_0 L$ and $d_1 L$ and the 2-cycle $L - R$ on the presentation of the torus from Chp 4 of Homotopy theory for beginners.

2.36. EXAMPLE (A Δ-complex structure on the crosscap $N_1 = \mathbf{R}P^2$). Let S be the Δ-set $U \amalg D$ where $U = \Delta[2_+] = D$, realizing $\Delta^2 \amalg \Delta^2$. Let $\sim$ be the smallest equivalence relation on S that identifies $d_2U = d_1D$, $d_1U = d_2D$, and $d_0U = d_0D$.



Let $P^2 = S/\sim$ be the quotient Δ-set. The simplices of P^2 are $P^2_2 = \{U, D\}$, $P^2_1 = \{d_0U, d_1U, d_2U\}$, and $P^2_0 = \{(0), (1)\}$. The chain complex is

$$0 \leftarrow \mathbf{Z}^2 \xleftarrow{\partial_1} \mathbf{Z}^3 \xleftarrow{\partial_2} \mathbf{Z}^2 \leftarrow 0, \qquad \partial_2 = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}, \quad \partial_1 = \begin{pmatrix} 0 & 0 \\ 1 & -1 \\ 1 & -1 \end{pmatrix}$$

because $\partial_2 U = d_0U - d_1U + d_2U = d_0U - (d_1U - d_2U)$, $\partial_1 D = d_0D - d_1D + d_2D = d_0U + (d_1U - d_2U)$ and $\partial_1 d_0U = \partial_1(12) = (2) - (1) = 0$, $\partial_1 d_1U = \partial_1(02) = (2) - (0)$, $\partial_1 d_2U = \partial_1(01) = (1) - (0)$. The 1-cycles are $Z_1(P^2) = \mathbf{Z}\{d_0U, d_1U - d_2U\} \subset \mathbf{Z}P^2_1$. The matrix for $\partial_2 \colon \mathbf{Z}P^2_2 \to Z_1(P^2)$ is

$$\partial_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

where $\simeq$ stands for row or column operation and thus $H^{\Delta}_0(S) = \mathbf{Z}$, $H^{\Delta}_1(S) = \mathbf{Z}/2\mathbf{Z}$, and $H^{\Delta}_2(S) = 0$. We have shown that

- d_0U is a 1-cycle, homologous to $d_1U - d_2U$ as $\partial U = d_0U - (d_1U - d_2U)$;
- d_0U is not a 1-boundary but $2d_0U = \partial(U + D)$ is;
- there are no 2-cycles in the Δ-set P^2.

Try to identify the 1-cycle d_0U on the presentations of $\mathbf{R}P^2$ from Chp 4 of Homotopy theory for beginners.

2.3. The topological realization of a Δ-set. The *topological realization* of the Δ-set S is a kind of tensor product of the Δ-set with the co-Δ-space $\Delta^{\bullet}$. More precisely, the realization of S is defined to be the quotient space of a collection of disjoint geometric simplices,

$$|S| = \coprod_{n \geq 0} S_n \times \Delta^n / \sim \tag{2.37}$$

where $\sim$ is the equivalence relation generated by $(\sigma, d^i y) \sim (d_i\sigma, y)$ for all $(\sigma, y) \in S_n \times \Delta^{n-1}$. This relation identifies the $(n-1)$-simplex $d_i\sigma \times \Delta^{n-1}$ with the ith face $\sigma \times d^i\Delta^{n-1}$ of the n-simplex $\sigma \times \Delta^n$ for all $\sigma \in S_n$. In this way a Δ-set is a recipe for building a space out of geometric simplices.

2.38. DEFINITION. A Δ-complex is the topological realization of a Δ-set. A Δ-complex structure on a topological space X consists of a Δ-set S and a homeomorphism between $|S|$ and X.

The data

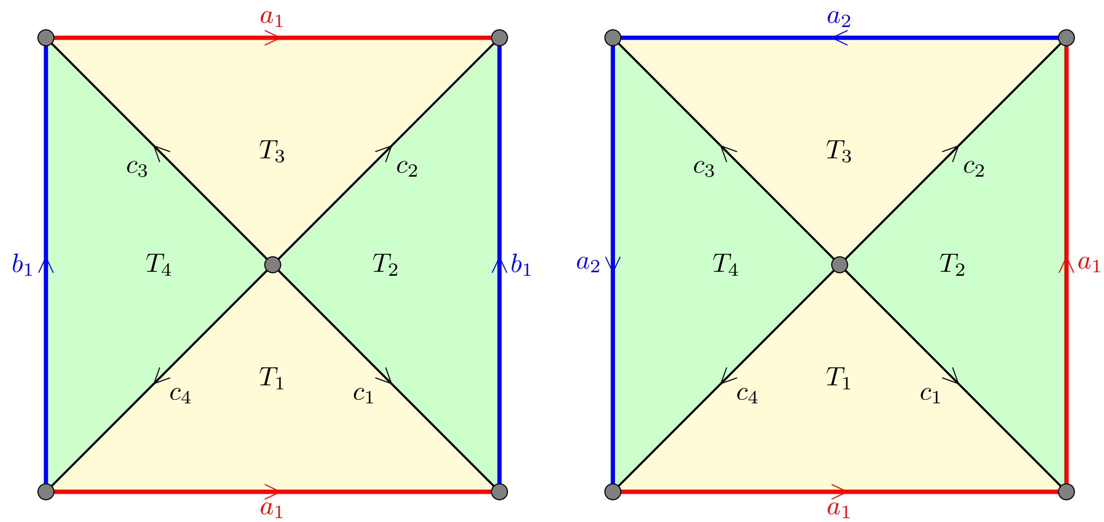


FIGURE 4. Δ-complex structures on the torus M_1 and the Klein bottle N_2

Realize the unit procede by counit   Sing the counit precede by unit

$$\begin{tikzcd} \mathbf{\Delta SET} \arrow[r, shift left, "|\cdot|"] & \mathbf{TOP} \arrow[l, shift left, "\mathrm{Sing}"] \end{tikzcd}$$

$$\begin{tikzcd} S \arrow[d, "\eta_S"'] \\ \mathrm{Sing}(|S|) \end{tikzcd} \qquad \begin{tikzcd} X \\ |\mathrm{Sing}(X)| \arrow[u, "\varepsilon_X"] \end{tikzcd} \qquad \begin{tikzcd} |S| \arrow[d, "|\eta_S|"] \arrow[dd, bend right, equal] \\ |\mathrm{Sing}(|S|)| \arrow[d, "\varepsilon_{|S|}"] \\ |S| \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Sing(X)} \\ \mathrm{Sing}(|\mathrm{Sing}(X)|) \arrow[u, "\mathrm{Sing}(\varepsilon_X)"] \\ \mathrm{Sing}(X) \arrow[u, "\eta_{\mathrm{Sing}(X)}"] \arrow[uu, bend right, equal] \end{tikzcd}$$

unit — η_S ε_X — counit

means that the realization functor and the singular functor are adjoint functors with $|\cdot|$ the left and Sing the right adjoint functor. The natural transformations, η and ε, the unit and counit of the adjunction, are defined by $\eta(\sigma)(x) = (\sigma, x)$ and $\varepsilon(\sigma, x) = \sigma(x)$ for all n-simplices $\sigma \colon \Delta^n \to X$ in X and all points $x \in \Delta^n$. Observe that

$$(d_i\eta_S(\sigma))(y) = (d_i\sigma, y) = (\sigma, d^iy) = \eta_S(\sigma)(d^iy), \qquad \sigma \in S_n,\ y \in \Delta^{n-1}$$

and that

$$\varepsilon_X(\sigma, d^iy) = \sigma(d^iy) = (d_i\sigma)(y) = \varepsilon_X(d_i\sigma, y), \qquad \sigma \in \mathrm{Sing}(X)_n,\ y \in \Delta^{n-1}$$

The first equation shows that the unit is a morphism of Δ-sets and the second one that the counit is a well-defined morphism of topological spaces. As always, an adjunction determines a bijection

$$\mathbf{\Delta SET}(S, \mathrm{Sing}(X)) = \mathbf{Top}(|S|, X)$$

where the continuous map $|S| \xrightarrow{f} X$ and the Δ-set morphism $S \xrightarrow{\varphi} \mathrm{Sing}(X)$ correspond to each other if $f(\sigma, x) = \varphi(\sigma)(x)$ for all $\sigma \in \mathrm{Sing}_n(X)$ and $x \in \Delta^n$.

2.39. EXAMPLE (A Δ-complex structure on D^n and S^{n-1}). The Δ-set $\Delta[n] = Bn_+$ of the poset n_+ consists of *all* nonempty subsets of n_+. Define $\partial\Delta[n]$ as the Δ-set of all *proper* subsets of n_+. The topological realizations of these two Δ-sets are $|\Delta[n]| = \Delta^n$, the n-simplex, and $|\partial\Delta[n]| = \partial\Delta^n$, the $(n-1)$-sphere.

2.40. EXAMPLE (A Δ-complex structure on M_1). Consider the 2-dimensional Δ-set $S = S_0 \leftleftarrows S_1 \Lleftarrow S_2$ with $S_0 = \{0, 1\}$, $S_1 = \{1, \ldots, 6\}$, $S_2 = \{1, \ldots, 4\}$ and face maps as listed. The realization of S is a torus as shown in Figure 4. The 0-simplex 0 is the middle vertex and the 0-simplex 1 is the vertex at the four corners of the square. The 1-simplices $1, \ldots, 4, 5, 6$ are $c_1, \ldots, c_4, a_1, b_1$. The homology of this Δ-set is $H_0^\Delta(S) = \mathbf{Z}$, $H_1^\Delta(S) = \mathbf{Z} \oplus \mathbf{Z}$, and $H_2^\Delta(S) = \mathbf{Z}$.

$S_1 \to S_0$	1	2	3	4	5	6
d_0	1	1	1	1	1	1
d_1	0	0	0	0	1	1

$S_2 \to S_1$	1	2	3	4
d_0	5	6	5	6
d_1	1	2	2	3
d_2	4	1	3	4

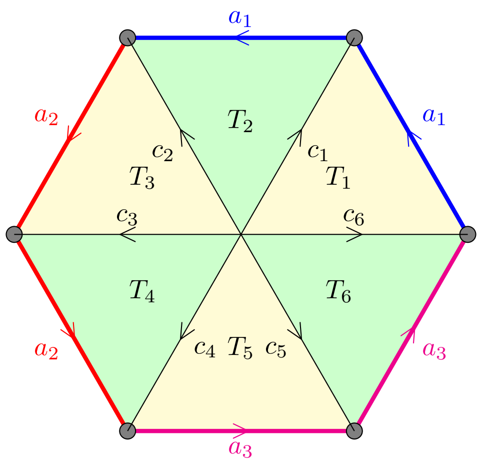


FIGURE 5. A Δ-complex structure on the nonorientable surface N_3

2.41. EXAMPLE (A Δ-complex structure on N_2). Consider the 2-dimensional Δ-set $S = (\, S_0 \leftleftarrows S_1 \Lleftarrow S_2 \,)$ with $S_0 = \{0, 1\}$, $S_1 = \{1, \dots, 6\}$, $S_2 = \{1, \dots, 4\}$ and face maps as listed. The realization of S is the nonori-

$S_1 \to S_0$	1	2	3	4	5	6
d_0	1	1	1	1	1	1
d_1	0	0	0	0	1	1

$S_2 \to S_1$	1	2	3	4
d_0	5	5	6	6
d_1	1	2	3	4
d_2	4	1	2	3

entable surface N_2 of genus 2 as indicated by Figure 4. The 0-simplex 0 is the middle vertex and the 0-simplex 1 is the vertex at the four corners of the square. The 1-simplices $1, \dots, 4, 5, 6$ are $c_1, \dots, c_4, a_1, a_2$. The homology of this Δ-set is $H_0^\Delta(S) = \mathbf{Z}$, $H_1^\Delta(S) = \mathbf{Z}/2 \oplus \mathbf{Z}$, and $H_2^\Delta(S) = 0$.

2.42. EXAMPLE (A Δ-complex structure on N_3). Consider the 2-dimensional Δ-set $S = (\, S_0 \leftleftarrows S_1 \Lleftarrow S_2 \,)$ with $S_0 = \{0, 1\}$, $S_1 = \{1, \dots, 9\}$, $S_2 = \{1, \dots, 6\}$ and face maps as listed. The realization of S is the nonori-

$S_1 \to S_0$	1	2	3	4	5	6	7	8	9
d_0	1	1	1	1	1	1	1	1	1
d_1	0	0	0	0	0	0	1	1	1

$S_2 \to S_1$	1	2	3	4	5	6
d_0	7	7	8	8	9	9
d_1	1	2	3	4	5	6
d_2	6	1	2	3	4	5

entable surface N_3 of genus 3. The 0-simplex 0 is the middle vertex and the 0-simplex 1 is the vertex at the four corners of the square. The 1-simplices $1, \dots, 4, 5, 9$ are $c_1, \dots, c_6, a_1, a_2, a_3$. The homology of this Δ-set is $H_0^\Delta(S) = \mathbf{Z}$, $H_1^\Delta(S) = \mathbf{Z}/2 \oplus \mathbf{Z} \oplus \mathbf{Z}$, and $H_2^\Delta(S) = 0$.

Try to find similar Δ-sets realizing M_2 and $N_1 = \mathbf{R}P^2$ (Example 4.41). The magma program deltaset.prg computes homology groups of Δ-sets. The Smith Normal Form of an integer matrix is often useful when computing homology.

2.4. Δ-sets are ubiquitous. We observe that sets, posets, categories, and topological spaces have associated Δ-sets.

2.43. EXAMPLE (The Δ-set of a set). The Δ-set BX of a set X is

$$B_n X = \mathbf{SET}(n_+, X) = X^{n+1}, \qquad d_i(n_+ \xrightarrow{\sigma} X) = (n-1)_+ \xrightarrow{d^i} n_+ \xrightarrow{\sigma} X$$

The set B_nX in degree n is the product X^{n+1} of $n+1$ copies of X and the ith face map

$$d_i(x_0,\dots,x_i,\dots,x_n) = (x_0,\dots,\widehat{x_i},\dots,x_n)$$

deletes the ith coordinate for $0 \leq i \leq n$

2.44. Example (The Δ-set of a poset). The Δ-set BX of a poset X is

$$BX_n = \mathbf{POSI}(n_+, X) = \{x_0 < \cdots < x_n | x_i \in X\}, \qquad d_i(n_+ \xrightarrow{\sigma} X) = (n-1)_+ \xrightarrow{d^i} n_+ \xrightarrow{\sigma} X$$

In other words, an n-simplex in X is a totally ordered subset

$$\sigma = \{x_0 < x_1 < \cdots < x_n\}$$

of $n+1$ points in X.

2.45. Example (The Δ-set of a small category). The Δ-set BX of the small category X has

$$BX_n = \mathbf{CAT}(n_+, X) = \{n_+ \xrightarrow{\sigma} X\}, \qquad d_i(n_+ \xrightarrow{\sigma} X) = (n-1)_+ \xrightarrow{d^i} n_+ \xrightarrow{\sigma} X$$

The set in degree n is

$$BX_n = \{c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_1} c_2 \to \cdots \to c_{n-1} \xrightarrow{f_{n-1}} c_n\}$$

and the face maps $d_0,\dots,d_n \colon BX_n \to BX_{n-1}$ are

$$d_i(c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_1} c_2 \to \cdots \to c_{n-1} \xrightarrow{f_{n-1}} c_n) = \begin{cases} c_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} c_n & i = 0 \\ c_1 \xrightarrow{f_1} \cdots \to c_{i-1} \xrightarrow{f_i f_{i-1}} c_{i+1} \cdots \xrightarrow{f_{n-1}} c_n & 0 < i < n \\ c_0 \xrightarrow{f_0} \cdots \xrightarrow{f_{n-2}} c_{n-1} & i = n \end{cases}$$

In the realization $|BX|$ of BX there is

- one vertex for each object c_0 of the category
- one 1-simplex connecting the vertices c_0 and c_1 for each morphism $c_0 \xrightarrow{f_1} c_1$ in the category
- one 2-simplex, glued onto the edges f_1, f_2, and f_1f_2

$$\begin{tikzcd} & c_1 \arrow[dr, "f_2"] & \\ c_0 \arrow[ur, "f_1"] \arrow[rr, "f_1f_2"'] & & c_2 \end{tikzcd}$$

for every pair of composable morphisms in the category X.

2.46. Example (The Δ-set of a topological space). The Δ-set of the topological space X is

$$\mathrm{Sing}(X)_n = \mathbf{TOP}(\Delta^n, X) = \{\Delta^n \xrightarrow{\sigma} X\}, \qquad d_i(\Delta^n \xrightarrow{\sigma} X) = \Delta^{n-1} \xrightarrow{d^i} \Delta^n \xrightarrow{\sigma} X$$

The set $\mathrm{Sing}(X)_n$ in degree n consists of all continuous maps $\Delta \xrightarrow{\sigma} X$ of the standard geometric n-simplex into X and the face maps $d_0,\dots,d_n \colon \mathrm{Sing}(X)_n \to \mathrm{Sing}(X)_{n-1}$ are induced by the coface maps $d^i \colon \Delta^{n-1} \to \Delta^n$. More explicitly,

$$(d_i\sigma)(t_0,\dots,t_{n-1}) = \begin{cases} \sigma(0,t_0,\dots,t_{n-1}) & i = 0 \\ \sigma(t_0,\dots,t_{i-1},0,t_i,\dots,t_{n-1}) & 0 < i < n \\ \sigma(t_0,\dots,t_{n-1},0) & i = n \end{cases}$$

for any n-simplex $\Delta^n \xrightarrow{\sigma} X$ in X. In short form, $d_i\sigma = \sigma d^i$ where $d^i \colon \Delta^{n-1} \to \Delta^n$ is the geometric coface map (1.1).

2.47. Example (The Δ-set of a discrete group). [**3**, §I.5, Exercise 3 p 19] [**10**, Example 1B.7] [**21**, Example 8.1.7]. We are going to define a functor

$$B \colon \mathbf{GRP} \to \Delta\mathbf{SET}$$

from the category $\mathbf{GRP}$ of groups to the category of Δ-sets.

Let G be a group. Define EG to be the Δ-set BG of G viewed as a set (2.43): $EG_n = G^{n+1}$ and with face maps $d_i[g_0, \dots, g_n] = [g_0, \dots, \widehat{g}_i, \dots, g_n]$ that simply forgets one of the coordinates. The realization (2.37) $|EG|$ is contractible by the homotopy $h_t \colon |EG| \to |EG|$ which on $G^{n+1} \times \Delta^n$ is given by

$$([g_0, \dots, g_n], x) \to ([e, g_0, \dots, g_n], (1-t)d^0x + te_0)$$

This homotopy is well-defined because

$$\begin{aligned} h_t(d_j[g_0, \dots, g_n], x) &= ([e, g_0, \dots, \widehat{g}_j, \dots, g_n], (1-t)d^0x + te_0) = (d_{j+1}[e, g_0, \dots, g_n], (1-t)d^0x + te_0) \\ &\sim ([e, g_0, \dots, g_n], (1-t)d^{j+1}d^0x + td^{j+1}e_0) = ([e, g_0, \dots, g_n], (1-t)d^0d^jx + te_0) = h_t([g_0, \dots, g_n], d^jx) \end{aligned}$$

for any $x \in \Delta^{n-1}$ and all $j \geq 0$. The start value of the homotopy is the identity map and the end value is

$$([e, g_0, \dots, g_n], e_0) = ([e, g_0, \dots, g_n], d^ne_0) \simeq ([e, g_0, \dots, g_{n-1}], e_0) \simeq \cdots \simeq ([e], e_0)$$

which is a single point. But EG is not just a Δ-set; it is a Δ-G-set. This means that the sets EG_n are (left) G-sets with G-action defined coordinate-wise and the face maps are G-maps. Therefore the associated Δ-abelian group is in fact a Δ-$\mathbf{Z}G$-module and the simplicial chain complex $\mathbf{Z}EG_*$

$$0 \longleftarrow \mathbf{Z}EG_0 \xleftarrow{\partial} \mathbf{Z}EG_1 \xleftarrow{\partial} \cdots \xleftarrow{\partial} \mathbf{Z}EG_{n-1} \xleftarrow{\partial} \mathbf{Z}EG_n \xleftarrow{\partial} \cdots$$

is a chain complex of $\mathbf{Z}G$-modules. Since it computes the homology of the contractible space EG it has the homology of a point and so it is a free resolution over $\mathbf{Z}G$ of the trivial $\mathbf{Z}G$-module $\mathbf{Z}$, called the *standard resolution*.

Define $BG = G\backslash EG$ to be the quotient Δ-set (§2.2), the projective version of EG. The n-simplices of BG are $BG_n = G\backslash EG_n = G\backslash G^{n+1}$. The simplicial n-chains of BG are

$$\mathbf{Z}BG_n = \mathbf{Z}[G\backslash G^{n+1}] = \mathbf{Z} \otimes_{\mathbf{Z}G} \mathbf{Z}G^{n+1} = \mathbf{Z} \otimes_{\mathbf{Z}G} \mathbf{Z}EG_n$$

Here we use the general formula $\mathbf{Z} \otimes_{\mathbf{Z}G} \mathbf{Z}[S] = \mathbf{Z}[G\backslash S]$ where $\mathbf{Z}[S]$ stands for the free $\mathbf{Z}$-module on the set G-set S. (Use the universal property of the tensor product to prove this identity.) Thus the simplicial chain complex $\mathbf{Z}BG_* = \mathbf{Z} \otimes_{\mathbf{Z}G} EG_*$ has homology $H_*(BG) = \mathrm{Tor}_*^{\mathbf{Z}G}(\mathbf{Z}, \mathbf{Z})$. This *group homology* is a new invariant of the group G.

We can enumerate the n-simplices of BG by G^n using the bijection

$$G^n \to G\backslash G^{n+1} \colon [g_1|\cdots|g_n] \to G(e, g_1, g_1g_2, \dots, g_1\cdots g_n)$$

In this context, the elements of G^n are traditionally written in bar notation $(g_1, \dots, g_n) = [g_1|\cdots|g_n]$. Using that BG is the quotient Δ-set of EG, we see that the face maps $d_i \colon BG_n \to BG_{n-1}$, $0 \leq i \leq n$, are

$$d_i[g_1|\cdots|g_n] = \begin{cases} [g_2|\cdots|g_n] & i = 0 \\ [g_1|\cdots|g_ig_{i+1}|\cdots|g_n] & 0 < i < n \\ [g_1|\cdots|g_{n-1}] & i = n \end{cases}$$

using bar notation. For instance,

$$\begin{aligned} d_0[g_1|\cdots|g_n] = Gd_0(e, g_1, g_1g_2, \dots, g_1\cdots g_n) &= G(g_1, g_1g_2, \dots, g_1\cdots g_n) \\ &= G(e, g_2, \dots, g_2\cdots g_n) = [g_2|\cdots|g_n] \end{aligned}$$

and the other cases are proved similarly. This means that BG is the Δ-set of G viewed as a one-object category (2.45).

The Kan–Thurston theorem [**12**] says that any connected space has the homology of BG for some group G.

2.48. Example (The Δ-space of a functor). Slightly more generally, we now define the classifying space BF of a functor $F \colon \mathcal{C} \to \mathbf{Sets}$ or $F \colon \mathcal{C} \to \mathbf{Top}$ with values in the category of sets or even in the category of topological spaces (in such a way that the classifying space of the category will be the classifying space of the constant functor with value a point). Let BF be the Δ-set or Δ-space with 0-simplices $BF_0 = \coprod_{c \in \mathrm{Ob}(C)} F(c)$ equal to the set of $[c, x]$ where c is an object of $\mathcal{C}$ and x is an element of $F(c)$. For $n > 0$, let

$$BF_n = \coprod_{c_0 \to c_1 \to \cdots \to c_n} F(c_0)$$

be the set or space of all strings $[x, c_0 \xrightarrow{g_1} c_1 \to \cdots \xrightarrow{g_n} c_n]$ where $x \in F(c_0)$ and the morphisms are composable in $\mathcal{C}$. The face maps $d_i\colon BF_n \to BF_{n-1}$ are

$$d_i[x, c_0 \xrightarrow{g_1} c_1 \to \cdots \xrightarrow{g_n} c_n] = \begin{cases} [F(g_1)x, c_1 \xrightarrow{g_2} c_2 \to \cdots \xrightarrow{g_n} c_n] & i = 0 \\ [x, c_0 \to \cdots c_{i-1} \xrightarrow{g_{i+1}g_i} c_{i+1} \to \cdots \to c_n] & 0 < i < n \\ [x, c_0 \xrightarrow{g_1} c_1 \to \cdots \xrightarrow{g_{n-1}} c_{n-1}] & i = n \end{cases}$$

It is clear that formula (2.37) for the realization of a Δ-set works equally well for a Δ-space. In this particular case the realization of the Δ-space BF is the space

$$|BF| = \coprod_{n\geq 0} BF_n \times \Delta^n / \sim$$

where $BF_{n-1} \times \Delta^{n-1} \ni (d_i[x, c_0 \xrightarrow{g_1} c_1 \to \cdots \xrightarrow{g_n} c_n], y) \sim ([x, c_0 \xrightarrow{g_1} c_1 \to \cdots \xrightarrow{g_n} c_n], d^i y) \in BF_n \times \Delta_n$. BF is more commonly known as the homotopy colimit, $\operatorname{hocolim} F$, of the functor F [**9**, 5.12]. What is the classifying space BS of a Δ-set $S\colon \Delta_< \to \mathbf{Sets}$?

Note that there is a map $BF \to B\mathcal{C}$ of Δ-spaces, taking the subspace $[F(c_0), c_0 \to c_1 \to \cdots \to c_n]$ to $[c_0 \to c_1 \to \cdots \to c_n]$, where the fibre over each simplex is of the form $F(c_0)$ for some object c_0 of $\mathcal{C}$. This map can be used to express the homology of BF in terms of the homology of $B\mathcal{C}$ and the homology functor $H_jF\colon \mathcal{C} \to \mathbf{Ab}$ given by $H_jF(c) = H_j(F(c))$ for any object c of $\mathcal{C}$. In the special case where the index category is a group G, the functor is a G-space X and we get a map $BX \to BG$ where the fibre over any point is X. The space BX, usually denoted X_{hG}, is the homotopy orbit space of the G-space.

2.49. Example (Homology of a category with coefficients in a functor). Let $\mathcal{C}$ be a small category and $A\colon \mathcal{C} \to \mathbf{Ab}$ a functor from $\mathcal{C}$ to abelian groups (a $\mathcal{C}$-module). The associated classifying Δ-abelian group BA has $BA_0 = \bigoplus_{c\in \mathrm{Ob}(\mathcal{C})} A(c)$ and

$$BA_n = \bigoplus_{c_0 \to \cdots \to c_n} A(c_0), \qquad n > 0,$$

and with face maps $d_i\colon BA_n \to BA_{n-1}$ given by

$$d_0[a, c_0 \xrightarrow{g_1} c_1 \to \cdots \to c_n] = \begin{cases} [g_1 a, c_1 \to \cdots \to c_n] & i = 0 \\ [a, c_0 \to \cdots \to c_{i-1} \to c_{i+1} \to \cdots \to c_n] & 0 < i < n \\ [a, c_0 \to \cdots \to c_{n-1}] & i = n \end{cases}$$

We define $H_*(\mathcal{C}; A)$, the homology of the category $\mathcal{C}$ with coefficients in the functor A, to be the homology of the Δ-abelian group BA with differential $\partial = \sum (-1)^i d_i$. Note that $H_0(\mathcal{C}; A)$, the cokernel of the homomorphism

$$BA_0 = \bigoplus_{c\in \mathrm{Ob}(\mathcal{C})} A(c) \xleftarrow{\partial} \bigoplus_{c_0 \xrightarrow{g} c_1} A(c_0) = BA_1, \qquad \partial[a, g] = [ga, c_1] - [a, c_0]$$

is the colimit $\operatorname{colim} A$ [**21**, p 54] of the functor A. Because of this, the homology group $H_i(\mathcal{C}; A)$ is often denoted $\operatorname{colim}_i A$.

The homology groups $H_i(\mathcal{C}; H_jF)$, $i + j = n$, of the $\mathcal{C}$-modules H_jF from Example 2.48 are a first approximation to the nth homology group of the classifying space BF of the space valued functor $F\colon \mathcal{C} \to \mathbf{Top}$.

5. Simplicial sets

Let Δ_* be the category whose objects are the finite ordered sets n_+, $n \geq 0$ (Definition 2.15), and whose morphisms are *all* order preserving maps. (A map φ is order preserving if $i \leq j \implies \varphi(i) \leq \varphi(j)$.) In particular, Δ_* contains the order preserving maps

$$(n-1)_+ \xrightarrow{d^0, \ldots, d^n} n_+ \xleftarrow{s^0, \ldots, s^n} (n+1)_+$$

where d^i is the order preserving map that does not hit i (as before) and $s^j(0 < 1 < \cdots < n+1) = (0 < 1 < \cdots j \leq j < \cdots < n)$ is the nondecreasing map that hits j twice. Any morphism in Δ_* is a composition of these maps [**14**, VIII.5.1].

These order preserving maps correspond to unique linear maps

$$\Delta^{n-1} \xrightarrow{d^0,\dots,d^n} \Delta^n \xleftarrow{s^0,\dots,s^n} \Delta^{n+1}$$

of the standard simplices (1.1) where for instance s^j is the linear map that sends vertex $e_k \in \Delta^{n+1}$ to vertex $e_{s^j k} \in \Delta^n$.

2.50. DEFINITION. A *simplicial* set is a functor $S\colon \Delta_*^{\mathrm{op}} \to \mathbf{SET}$, a contravariant functor from the category Δ_* to the category **SET** of sets. A *simplicial morphism* between two simplicial sets is a natural transformation of functors.

In other words, a simplicial set is a graded set $S = \bigcup_{n\geq 0} S_n$ where the set S_n in level n is equipped with $n+1$ face and $n+1$ degeneracy maps

$$S_{n-1} \xleftarrow{d_0,\dots,d_n} S_n \xrightarrow{s_0,\dots,s_n} S_{n+1}$$

satisfying the simplicial identities

$$\begin{cases} d_i d_j = d_{j-1} d_i & i < j \\ d_i s_j = s_{j-1} d_i & i < j \\ d_j s_j = 1 = d_{j+1} s_j & \\ d_i s_j = s_j d_{i-1} & i > j+1 \\ s_i s_j = s_{j+1} s_i & i \leq j \end{cases}$$

We can present the simplicial set S as

$$S_0 \rightleftarrows S_1 \rightleftarrows S_2 \qquad \cdots$$

The topological realization of a simplicial set (or space) S is a quotient space of a set of disjoint simplices,

$$|S| = \coprod_{n\geq 0} S_n \times \Delta^n / \sim \tag{2.51}$$

where $\sim$ is the equivalence relation generated by the relation $(d_i x, y) \sim (x, d^i y)$ for $x \in S_n$, $y \in \Delta^{n-1}$ and $(x, s^j y) \sim (s_j x, y)$ for each $x \in S_n$, $y \in \Delta^{n+1}$. This relation identifies the $(n-1)$-simplex $d_i\sigma \times \Delta^{n-1} \subset S_{n-1} \times \Delta^{n-1}$ and the ith face of the n-simplex $\sigma \times \Delta^n \subset S_n \times \Delta^n$ for all points $\sigma \in S_n$ and it collapses the $(n+1)$-simplex $s_j\sigma \times \Delta^{n+1}$ onto the n-simplex $\sigma \times \Delta_n$.

A simplicial morphism is a map $S \to T$ of graded sets commuting with the face and degeneracy maps. Simplicial sets with simplicial morphisms form the category **sSET** of simplicial sets. A simplicial map $S \to T$ induces a (continuous) map $|S| \to |T|$ between the topological realizations.

2.52. EXAMPLE. The Δ-sets Δ_n, $\mathrm{Sing}(X)$, EG, BG, and $B\mathcal{C}$ (2.46, 2.47, 2.45) are in fact simplicial *sets* and the Δ-space BF (2.48) is a simplicial *space*. (Supply the definition of a simplicial space!) The simplicial set of a category $\mathcal{C}$ is the simplicial set $B\mathcal{C}$ and the classifying space of a category is the realization $|B\mathcal{C}|$ of this simplicial set. Thus there are functors

$$\mathbf{GRP} \subset \mathbf{CAT} \xrightarrow{B} \mathbf{sSET} \xrightarrow{|\cdot|} \mathbf{TOP}$$

Degeneracy maps for EG_n are defined by repeating one of the entries,

$$s^i(g_0,\dots,g_n) = (g_0,\dots,g_i,g_i,g_{i+1},\dots,g_n)$$

and the degeneracy maps for BG_n are defined by inserting the neutral element e at the different places in the bar notation, $s^i[g_1|\cdots|g_n] = [g_1|\cdots|g_i|e|g_{i+1}|\cdots|g_n]$. Degeneracy maps for BF are defined by inserting identity morphisms.

2.53. EXERCISE. Let C_2 be the group of order two. The simplicial set EC_2 has two nondegenerate simplices in each dimension and BC_2 has one nondegenerate simplex in each dimension. Can you identify these spaces and the map $|EC_2| \to |BC_2|$?

Suppose that $\mathcal{C} = 0 \to 1$ is the category with two objects and one nonidentity morphism. What is BF for a functor $F\colon \mathcal{C} \to \mathbf{Top}$? Such a functor is a map $A \xrightarrow{f} B$ between topological spaces. Remember that BF is a quotient of $(A \coprod B) \times \Delta^0 \coprod A \times \Delta^1$ and that there are identifications $[a, 0 \to 1, d^0 y] \sim [f(a), y]$ and $[a, 0 \to 1, d^1 y] \sim [a, y]$ for all $y \in \Delta^0$. Consider also space valued functors with index categories $\mathcal{C} = \bullet \rightrightarrows \bullet$, $\mathcal{C} = \bullet \longleftarrow \bullet \longrightarrow \bullet$, and $\mathcal{C} = \bullet \longrightarrow \bullet \longrightarrow \cdots$.

2.54. Exercise. Can you make the Δ-sets from [**10**, p 102] into simplicial sets (without changing their realizations)?

An elementary illustrated inroduction to simplicial sets

CHAPTER 3

Applications of singular homology

We consider a few classical application of singular homology.

1. Lefschetz fixed point theorem

Let X be a finite CW-complex and $f\colon X \to X$ a self-map of X. The *Lefschetz number of f* with coefficients in the field F is the alternating (finite) sum

$$\Lambda(f;F) = \sum_k (-1)^k \operatorname{tr}\left(H_k(X;F) \xrightarrow{f_*} H_k(X;F)\right)$$

of the traces of the induced map in each degree. We showed in 1.56 that if a self map of a sphere has no fixed points then its Lefschetz number is 0. This is in fact true in much greater generality.

3.2. Theorem (Lefschetz fixed point theorem). *Let $f\colon X \to X$ be a self-map of a (retract of a) finite polyhedron. Then*

$$f \text{ has no fixed points} \implies \Lambda(f;F) = 0$$

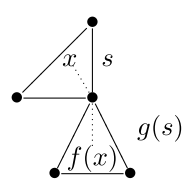
for any field F.

Proof. Assume that X is the realization of some finite simplicial complex L. Equip $|L|$ with a metric like in the proof of Theorem 2.13. Since $|L|$ is compact and f has no fixed points there is some $a > 0$ such that $d(x, f(x)) \geq a$ for all $x \in |L|$. By subdividing, if necessary, we may assume that $\operatorname{diam}|s| \leq a/3$ for all simplices $s \in S_L$ [**19**, Lemma 12 p 124]. By the Simplicial Approximation Theorem 2.13 there exists a subdivision K of L and a simplicial map $g\colon K \to L$ such that $f(x)$ and $|g|(x)$ are in the same simplex of L for any point $x \in |K|$. Then $d(f(x), |g|(x)) \leq a/3$, and $f \simeq |g|$. (To help the intuition, consider for instance a simplicial self-map $\operatorname{sd}\Delta^2 \to \Delta^2$.)

The main idea of the proof is that the simplices of K are so small that g moves each simplex $|s|$ of $|K|$ completely off itself, ie that

$$\forall s \in S_K\colon\ \underbrace{|s|}_{K} \cap \underbrace{|gs|}_{L} = \emptyset \tag{3.3}$$

Indeed, if the simplex $|s|$ (of $|K|$) has some overlap with its image simplex $|gs|$ (of $|L|$) and x is any point in $|s|$, then $d(x, f(x)) \leq \operatorname{diam}|s| + \operatorname{diam}|gs| \leq a/3 + a/3 < a$, since $f(x)$ lies in the simplex $g(s)$, contradicting the choice of a.



Since $|g|$ is the realization of a simplicial map $K \to L$, it takes $|K|^n$ to $|L|^n$. And since K is a subdivision of L, the identity map, or rather the inverse of the iterated barycentric map (2.12), $|L| \to |K|$ is cellular as $|L|^n \subset |K|^n$. (Look at a drawing of the barycentric subdivision of Δ^2.) Thus $|K| \xrightarrow{|g|} |L| \to |K|$ is a cellular self-map of the CW-complex $|K|$ so it induces a self-map

$$H_n(|K|^n, |K|^{n-1}) \xrightarrow{|g|_*} H_n(|L|^n, |L|^{n-1}) \xrightarrow{\mathrm{id}_*} H_n(|K|^n, |K|^{n-1})$$

of the cellular chain complex for $|K|$ (with coefficients in the field F). The first map, $|g|_*$, is the map $g\colon S_K \to S_L$ (restricted to n-simplices), and the second map, induced by the identity map, takes a simplex of L to the sum of the simplices in its iterated subdivision. But still, since $|s|$ (in $|K|$) and $|g(s)|$ (in $|L|$) are disjoint (3.3), the simplex $s \in H_n(|K|^n, |K|^{n-1})$ is not in the sum $\mathrm{id}_* g(s) \in H_n(|K|^n, |K|^{n-1})$ for the simplex $g(s) \in H_n(|L|^n, |L|^{n-1})$ and therefore the above map has zero trace as all diagonal entries in its matrix are zero.

By the Hopf trace formula (3.4), Lefschetz numbers are invariant under homology , so that the induced map $|g|_* \colon H_*(|K|) \to H_*(|K|)$ on singular homology also has $\Lambda(|g|_*) = 0$. But $f_* = |g|_*$ as $f \simeq |g|$ (1.13) so $\Lambda(f_*) = 0$.

Suppose now that X is a retract of a finite polyhedron $\overline{X}$. Let $i \colon X \to \overline{X}$ be the inclusion and $r \colon \overline{X} \to X$ the retraction. We can extend the self-map f of X to a self-map $\overline{f}$ of $\overline{X}$ if we map $\overline{X}$ into X and use f there, $\overline{f} = ifr$. Now observe that

- f and $\overline{f}$ have the same fixed points
- f_* and $\overline{f}_*$ have the same trace and the same Lefschetz number

Firstly, since $\overline{f}$ takes $\overline{X}$ into X and agrees with f on X, it has the same fixed points as f. Secondly, since $\overline{f} \colon (\overline{X}, X) \to (\overline{X}, X)$ factors through (X, X), the induced map $\overline{f}_* \colon H_k(\overline{X}, X) \to H_k(\overline{X}, X)$ is trivial. The commutative square

$$\begin{tikzcd}
0 \arrow[r] & H_k(X) \arrow[r] \arrow[d, "f_*"'] & H_k(\overline{X}) \arrow[r] \arrow[d, "\overline{f}_*"] & H_k(\overline{X}, X) \arrow[r] \arrow[d, "\overline{f}_*=0"] & 0 \\
0 \arrow[r] & H_k(X) \arrow[r] & H_k(\overline{X}) \arrow[r] & H_k(\overline{X}, X) \arrow[r] & 0
\end{tikzcd}$$

shows that $\mathrm{tr}(f_*) = \mathrm{tr}(\overline{f}_*)$ as traces are additive. Thus

$$f \text{ has no fixed points } \iff \overline{f} \text{ has no fixed points} \implies \Lambda(\overline{f}) = 0 \iff \Lambda(f) = 0$$

so the theorem also holds for self-maps of the retract X of the finite simplicial complex $\overline{X}$. □

The converse of the Lefschetz fixed point theorem [4] says that any self-map $f \colon X \to X$ of a finite polyhedron (satisfying some extra conditions) with Lefschetz number $\Lambda(f) = 0$ is homotopic to a map with no fixed points.

Consider a self-map ϕ of a finitely generated chain complex C over some field F,

$$\begin{tikzcd}
0 \arrow[r] & C_n \arrow[r] \arrow[d, "\phi_n"'] & \cdots \arrow[r] & C_k \arrow[r] \arrow[d, "\phi_k"'] & \cdots \arrow[r] & C_0 \arrow[r] \arrow[d, "\phi_0"'] & 0 \\
0 \arrow[r] & C_n \arrow[r] & \cdots \arrow[r] & C_k \arrow[r] & \cdots \arrow[r] & C_0 \arrow[r] & 0
\end{tikzcd}$$

This means that each C_k is a finite dimensional vector space over F and only finitely many are nonzero. Define the Lefschetz number of ϕ to be the alternating sum

$$\Lambda(\phi) = \sum_{k=0}^{n} (-1)^k \,\mathrm{tr}(\phi_k)$$

of the traces.

3.4. Theorem (Hopf trace formula). [19, 4.7.6] $\Lambda(\phi) = \Lambda(H_*(\phi))$

The Lefschetz number of the identity map is the Euler characteristic so we get that $\chi(C) = \chi(H(C))$ as a special case. A special case of that is the dimension formula for a linear map between two vector spaces.

2. Jordan–Brouwer separation theorem and the Alexander horned sphere

Recall the Mayer-Vietoris exact sequence in reduced homology (1.39)

$$\cdots \to \widetilde{H}_j(A \cap B) \to \widetilde{H}_j(A) \oplus \widetilde{H}_j(B) \to \widetilde{H}_j(A \cup B) \to \widetilde{H}_{j-1}(A \cap B) \to \cdots$$

for two open subspaces, A and B, of a topological space.

3.5. Lemma. Let M^n be an n-manifold and m a point of M. If $n \geq 2$ then there is a bijection between the path-components of $M - m$ and the path-components of M.

Proof. Let $A = M - m$ and let $B = \operatorname{int} D^n$ be an open embedded n-disc containing m. Then $B \simeq \mathbf{R}^n$ is contractible, $A \cup B = M$ and $A \cap B = \mathbf{R}^n - 0 \simeq S^{n-1}$. The long exact Mayer–Vietoris sequence ends with

$$\widetilde{H}_0(S^{n-1}) \to \widetilde{H}_0(M-m) \to \widetilde{H}_0(M) \to \widetilde{H}_{-1}(S^{n-1})$$

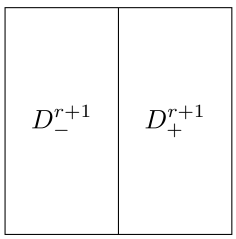
where $\widetilde{H}_0(S^{n-1}) = 0$ and $\widetilde{H}_{-1}(S^{n-1}) = 0$ since $n-1 \geq 1$. Thus $\widetilde{H}_0(M-m) \cong \widetilde{H}_0(M)$. Then also $H_0(M-m) \cong H_0(M)$ by the natural short exact sequence relating reduced and unreduced homology (§ 1.7). Because H_0 detects path-components (Proposition 1.4) this means that the path-components of $M-m$ and M are in bijection. $\square$

Let S^n be the n-sphere and $D^r \subset S^n$ a subspace homeomorphic to the r-disc. Then $S^n - D^r \neq \emptyset$.

3.6. Lemma. Any r-disc D^r in S^n, $n \geq 0$, has acyclic complement: $\widetilde{H}_*(S^n - D^r) = 0$.

Proof. The theorem is proved by induction over r. It is obviously true for $r = 0$.

Let now $r \geq 0$. Assume that there is a reduced k-cycle, $k \geq 0$, z in $S^n - D^{r+1}$ that is not a boundary, $0 \neq [z] \in \widetilde{H}_k(S^n - D^{r+1})$. Cut the $(r+1)$-disc into two 'halves' and write $D^{r+1} = D_-^{r+1} \cup D_+^{r+1}$ as the union of two 'smaller' $(r+1)$-discs with intersection $D_-^{r+1} \cap D_+^{r+1} = D^r$. (Think of D^{r+1} as I^{r+1}.) Using the Mayer–Vietoris sequence in reduced homology and the induction hypothesis:



$$D^{r+1} = D_-^{r+1} \cup D_+^{r+1}, \quad D^r = D_-^{r+1} \cap D_+^{r+1}$$
$$A = S^n - D_-^{r+1}, \quad B = S^n - D_+^{r+1}$$
$$A \cap B = S^n - D^{r+1}, \quad A \cup B = S^n - D^r$$
$$\widetilde{H}_k(S^n - D^{r+1}) \cong \widetilde{H}_k(S^n - D_-^r) \oplus \widetilde{H}_k(S^n - D_+^r)$$

Thus the homology class $[z]$ must also be nontrivial in the homology of at least one of the bigger spaces $S^n - D_\pm^{r+1}$. Continuing this way we obtain an descending chain of $(r+1)$-discs

$$D^{r+1} = D_0^{r+1} \supset D_1^{r+1} \supset \cdots \supset D_t^{r+1} \supset \cdots$$

where $\bigcap_{t=0}^\infty D_t^{r+1} = D^r$ is an r-disc and such that the reduced cycle $z \in C_k(S^n - D^{r+1})$ is not a boundary in any of the bigger spaces $S^n - D_t^{r+1}$, $t \geq 0$. However, $z = \partial w$ is the boundary of some $(k+1)$-chain w in $\bigcup(S^n - D_t^{r+1}) = S^n - \bigcap D_t^{r+1} = S^n - D^r$ by induction hypothesis. Now, the support $|w|$ of the is a compact space so that $|w| \subset S^n - D_T^{r+1}$ for some $T \gg 0$. This is a contradiction. $\square$

We now consider spheres embedded in spheres. We show that the homology of the complement of any r-sphere in an n-sphere does not depend of the embedding. For the standard embedding $S^r \subset S^n$, of the r-sphere into the n-sphere, $r \leq n$, the complement is

$$\begin{aligned} S^n - S^r = (\mathbf{R}^n \cup \{\infty\}) - (\mathbf{R}^r \cup \{\infty\}) = \mathbf{R}^n - \mathbf{R}^r &= (\mathbf{R}^{n-r} \times \mathbf{R}^r) - (\{0\} \times \mathbf{R}^r) \\ &= (\mathbf{R}^{n-r} - \{0\}) \times \mathbf{R}^r \simeq S^{n-1-r} \times \mathbf{R}^r \simeq S^{n-r-1} \end{aligned}$$

so that the reduced homology of the complement is

$$\widetilde{H}_j(S^n - S^r) = \widetilde{H}_j(S^{n-r-1}) = \begin{cases} \mathbf{Z} & j = n-1-r \\ 0 & \text{otherwise} \end{cases}$$

3.7. Corollary. Let now S^r be *any* subspace of S^n homeomorphic to the r-sphere. Then $n \geq r$ and the complement $S^n - S^r$ is a homology $(n-r-1)$-sphere.

Proof. Write the r-sphere $S^r = D_-^r \cup D_+^r$ as the union of two r-discs with intersection $D_-^r \cap D_-^r = S^{r-1}$. The complements of these discs are acyclic by Lemma 3.6 and as

$$(S^n - D_-^r) \cup (S^n - D_+^r) = S^n - S^{r-1}, \qquad (S^n - D_-^r) \cap (S^n - D_+^r) = S^n - S^r$$

the Mayer–Vietoris sequence in reduced homology shows that $\widetilde{H}_{j+1}(S^n - S^{r-1}) \cong \widetilde{H}_j(S^n - S^r)$. Apply this result r times and conclude that $\widetilde{H}_j(S^n - S^r) \cong \widetilde{H}_{j+r}(S^n - S^0) \cong \widetilde{H}_{j+r}(S^{n-1})$. Since $S^n - S^r$ has nonzero reduced homology in degree $j = n-r-1$, this number is ≥ -1 so that $n \geq r$. $\square$

For instance, all knot complements $S^3 - S^1$ have the homology of S^1 (but not the same fundamental group).

3.8. Corollary. Let $f\colon S^r \to S^n$ be an injective continuous map. Then $n \geq r$ and

- if $r = n$, then f is a homeomorphism,
- if $r = n-1$, then the complement $S^n - f(S^{n-1})$ has two acyclic open path-components,
- if $r < n-1$, then the complement $S^n - f(S^r)$ is path-connected.

Proof. We already know that $n \geq r$. And

- if $r = n$, the complement $S^n - S^r$ is empty because it has nonzero reduced homology in degree -1,
- if $r = n-1$, the complement $S^n - S^r$ has the homology of S^0,
- if $r < n-1$, the complement $S^n - S^r$ has the homology of a sphere of positive dimension

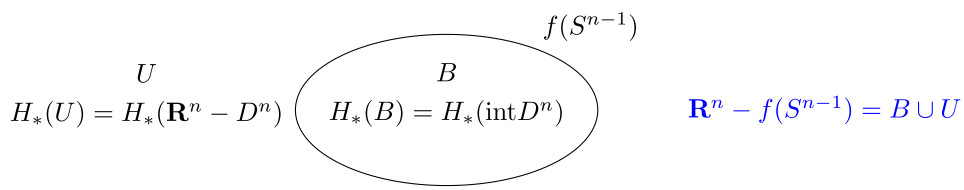
When $r = n-1$, $\widetilde{H}_0(S^n - S^{n-1}) = \widetilde{H}_0(S^0) = \mathbf{Z}$ and the complement has two path-connected components (Proposition 1.4), U_0 and U_1. The path-connected components are open in S^n because $S^n - S^{n-1}$ is locally path-connected, even locally euclidean. For $j > 0$

$$0 = H_j(S^n - S^{n-1}) = H_j(U_0) \oplus H_j(U_1)$$

by Proposition 1.3. Thus U_0 and U_1 are acyclic. □

The complement $S^n - S^{n-1}$ of any $(n-1)$-sphere in an n-sphere has two acyclic path components. One might guess that these components are open discs as they are in case of the standard embedding of S^{n-1} into S^n. This is indeed true when $n = 2$ (Schönflies Theorem) but not when $n = 3$. The Alexander horn sphere (Example 3.10) is an embedding of S^2 into $\mathbf{R}^3$ such that the unbounded component of the complement has infinite fundamental group.

We shall now discuss the complement of an $(n-1)$-sphere in $\mathbf{R}^n$. This is not so difficult because the complement $\mathbf{R}^n - S^{n-1}$ is just $S^n - S^{n-1}$ with one point removed. We show that all embeddings of S^{n-1} into $\mathbf{R}^n$ look like the standard embedding through the eyes of singular homology.



3.9. Corollary (Jordan– Brouwer Separation Theorem). (Cf 4.94) Let $f\colon S^{n-1} \to \mathbf{R}^n$ be an injective continuous map where $n \geq 1$. The complement $\mathbf{R}^n - S^{n-1}$ has two path components, B and U, where

(1) B is bounded and acyclic
(2) U is an unbounded homology $(n-1)$-sphere
(3) B and U are open in $\mathbf{R}^n$ and $\partial B = S^{n-1} = \partial U$

Proof. We shall assume that $n \geq 2$ as the case $n = 1$ is easy. We know from Corollary 3.8 that $(\mathbf{R}^n \cup \{\infty\}) - S^{n-1} = S^n - S^{n-1}$ has two acyclic path components, U_0 and U_1. Then also $\mathbf{R}^n - S^{n-1} = (S^n - S^{n-1}) - \{\infty\}$ has two path-components by Lemma 3.5. We may assume that the point ∞ belongs to U_1. Then

$$\mathbf{R}^n - S^{n-1} = (U_0 \cup U_1) - \{\infty\} = B \cup U, \quad \text{where} \quad B = U_0, \quad U = U_1 - \{\infty\},$$

are the two path components of $\mathbf{R}^n - S^{n-1}$. The bounded component B, homeomorphic to U_0, is acyclic, and the unbounded component, $U = U_1 - \{\infty\}$, has homology

$$\widetilde{H}_j(U) = \widetilde{H}_j(U_1 - \{\infty\}) \cong \widetilde{H}_{j+1}(U_1, U_1 - \{\infty\}) \cong \widetilde{H}_{j+1}(S^n, S^n - \{\infty\}) \cong \widetilde{H}_{j+1}(S^n) \cong \widetilde{H}_j(S^{n-1})$$

Here we use that U_1 is acyclic, we excise the closed set $U_0 \cup S^{n-1} = S^n - U_1$ from the pair $(S^n, S^n - \{\infty\})$, and we use that $S^n - \{\infty\} = \mathbf{R}^n$ is acyclic.

Both path components, B and U, are open in $\mathbf{R}^n$ as $\mathbf{R}^n - S^{n-1}$ is locally path connected (it is a manifold). $\mathbf{R}^n$ is the disjoint union of S^{n-1}, B, and U. The union $U \cup S^{n-1} = \mathbf{R}^n - B$ is a closed set containing U and hence containing the closure of U. Thus $\partial U = \mathrm{cl}\, U - U \subseteq (U \cup S^{n-1}) - U \subseteq S^{n-1}$. Similarly, $\partial B \subseteq S^{n-1}$. By methods from general topology, one may show that every point of S^{n-1} is a boundary point for U and for B. □

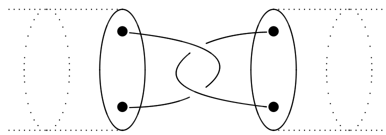
It is a delicate problem, known as the The Schönflies problem, to decide if B is homemorphic to an n-disc.

3.10. EXAMPLE (The Alexander horned sphere. Article from 1924. Illustration. Illustration). Let G be the nonabelian group that is the union

$$F_1 \hookrightarrow F_2 \hookrightarrow \cdots \hookrightarrow F_{2^n} \hookrightarrow F_{2^{n+1}} \hookrightarrow \cdots$$

of a sequence of free groups F_{2^n} on 2^n generators where the inclusion of F_{2^n} into $F_{2^{n+1}}$ takes the 2^n generators α_i of F_{2^n} to the commutators $[\beta_i, \gamma_i]$ of the $2 \cdot 2^n$ generators of $F_{2^{n+1}}$. G is an infinite group with trivial abelianization (a perfect group).

The Alexander horned disc is an embedding $D^3 \hookrightarrow \mathbf{R}^3$ such that $\pi_1(\mathbf{R}^3 - D^3) = G$ constructed in this way: Let X_0 be a solid torus in $\mathbf{R}^3$. Cut out an open segment, $(0,1) \times D^2$, of the torus, what remains is $B_0 = D^3$, and insert instead L



where the arcs are supposed to be solid tubes. Call the result X_1. Cut out an open segment of each of the two newly inserted tubes, what remains is $B_1 = D^3$, and insert instead copies of L. Call the result X_2. Continue this way to get sequences of compact spaces

$$X_0 \supset X_1 \supset \cdots \supset X_n \supset X_{n+1} \supset \cdots, \qquad B_0 \subset B_1 \subset \cdots \subset B_n \subset B_{n+1} \subset \cdots$$

where X_n is obtained from B_{n-1} by attaching 2^n-handles. Observe that $X = \bigcup B_n$ is homeomorphic to D^3 and that $\bigcup B_n = \bigcap X_n$. The fundamental group of the complement $\mathbf{R}^3 - X = \bigcup(\mathbf{R}^3 - X_n)$ is, by compactness, the union of the groups $\pi_1(\mathbf{R}^3 - X_n) = F_{2^n}$. The van Kampen theorem can be used to show that the inclusion $\mathbf{R}^3 - X_n \subset \mathbf{R}^3 - X_{n+1}$ induces the inclusion $F_{2^n} \hookrightarrow F_{2^{n+1}}$ used above. (Consider the first stage $\mathbf{R}^3 - X_0 \subset \mathbf{R}^3 - X_1$. Put a loop α around X_0 as indicated by the dotted circle to the left (or right) and put loops, β and γ, around the two handles added to B_0 to form X_1. Then $\pi_1(\mathbf{R}^3 - X_0) = \langle \alpha \rangle = F_1$, $\pi_1(\mathbf{R}^3 - X_1) = \langle \beta, \gamma \rangle = F_2$, and the induced homomorphism $\pi_1(\mathbf{R}^3 - X_0) \to \pi_1(\mathbf{R}^3 - X_1)$ takes α to the commutator $[\beta, \gamma]$ because α is the boundary of a disc that has been removed from a torus; it is homotopic to the commutator of a meridinal and a longitudinal circle on the torus.)

3.11. COROLLARY. Let $f\colon D^n \to \mathbf{R}^n$ be an injective continuous map where $n \geq 2$. Then $f(\operatorname{int} D^n)$ is the bounded component of $\mathbf{R}^n - f(S^{n-1})$. In particular, $f(\operatorname{int} D^n)$ is open in $\mathbf{R}^n$.

PROOF. Let B and U be the path components of $\mathbf{R}^n - S^{n-1}$. We have

$$B \cup U = \mathbf{R}^n - f(S^{n-1}) = \mathbf{R}^n - f(D^n - \operatorname{int} D^n) = \mathbf{R}^n - (f(D^n) - f(\operatorname{int} D^n)) = (\mathbf{R}^n - f(D^n)) \cup f(\operatorname{int} D^n)$$

The space $\mathbf{R}^n - f(D^n)$ is open, path connected, and unbounded. That $\mathbf{R}^n - f(D^n)$ is path connected follows from Lemma 3.5 as $\mathbf{R}^n - f(D^n) = S^n - f(D^n) - \{\infty\}$ and $S^n - f(D^n)$ is path connected, even acyclic by Lemma 3.6.

Thus $\mathbf{R}^n - f(D^n)$ is contained in the unbounded path component, U, of $\mathbf{R}^n - f(S^{n-1})$. The space $f(\operatorname{int} D^n)$ is a path connected subspace of $\mathbf{R}^n - f(S^{n-1})$, so it is contained in either B of U. But since $(\mathbf{R}^n - f(D^n)) \cup f(\operatorname{int} D^n) = B \cup U$ and both U and B are nonempty, we must in fact have that $\mathbf{R}^n - f(D^n)$ equals U and $f(\operatorname{int} D^n)$ equals B. □

3.12. COROLLARY. Let U be an open subspace of $\mathbf{R}^n$, where $n \geq 2$. Any injective continuous map $f\colon U \to \mathbf{R}^n$ is open.

PROOF. Let V be any open subset of U. There are (scaled) closed discs D^n so that $V = \bigcup D^n = \bigcup \operatorname{int} D^n$. Then $f(V) = \bigcup f(\operatorname{int} D^n)$ is open as a union of open sets (3.11). □

In particular, $f(U)$ is open and f is an embedding, a homeomorphism $f\colon U \to f(U)$.

3.13. COROLLARY. Let U and V be homeomorphic subspaces of $\mathbf{R}^n$, $n \geq 2$. Then

$$U \text{ is open in } \mathbf{R}^n \iff V \text{ is open in } \mathbf{R}^n$$

PROOF. Let $f\colon U \to V$ be a homeomorphism. The composition $U \xrightarrow{f} V \subset \mathbf{R}^n$, of f followed by the inclusion of V into $\mathbf{R}^n$, is an injective continuous map, so it is open. In particular is $f(U) = V$ open. □

In Corollary 3.12 we may replace U and $\mathbf{R}^n$ by arbitrary manifolds.

3.14. COROLLARY (Invariance of domain). Let M and N be topological n-manifolds, $n \geq 2$.

(1) Any injective continuous map $f\colon M \to N$ is open.
(2) Any bijective continuous map $f\colon M \to N$ is a homeomorphism.

PROOF. (1) Suppose first that N is $\mathbf{R}^n$. The manifold M is a union $M = \bigcup U_i$ of open subspaces U_i homeomorphic to $\mathbf{R}^n$. From Corollary 3.12 we know that $f(U_i)$ is open in $\mathbf{R}^n$. Thus $f(M) = \bigcup f(U_i)$ is also open in $\mathbf{R}^n$.

Now to the general case. Write $N = \bigcup V_j$ as as a union of open subspaces V_j homeomorphic to $\mathbf{R}^n$. Then $M = \bigcup f^{-1}(V_j)$ and, as $ff^{-1}(V_j)$ is open in $V_j = \mathbf{R}^n$ and in N, $f(M) = \bigcup ff^{-1}(V_j)$ is open. Of course, we may replace M by any open subset of M. Thus f is an open map.
(2) Since any bijective continuous map $M \to N$ is open by (1), it is a homeomorphism. □

3.15. COROLLARY. Let $f\colon M \to N$ be an injective continuous map between n-manifolds. If M is compact and N is connected, then f is a homeomorphism.

PROOF. Since M is compact and N is Hausdorff, the image $f(M)$ is closed. By Corollary 3.14, $f(M)$ is open. Since N is connected, $f(M) = N$. Thus f is a bijection and hence a homeomorphism by 3.14.(2). □

3.16. COROLLARY. A compact n-manifold cannot embed in $\mathbf{R}^n$.

PROOF. If the compact n-manifold M embeds in the connected manifold $\mathbf{R}^n$ then $M = \mathbf{R}^n$ by Corollary 3.15. But this is absurd since M is compact and $\mathbf{R}^n$ noncompact. □

For example, S^n does not embed in $\mathbf{R}^n$. It follows that $\mathbf{R}^n$ cannot contain a subspace homeomorphic to $\mathbf{R}^m$ for $m > n$ for then it would also contain a copy of $S^n \subset \mathbf{R}^{n+1} \subset \mathbf{R}^m$.

3. Group homology and Eilenberg –MacLane Complexes $K(G,1)$

Let G be a group.

3.17. DEFINITION. A $K(G,1)$ is a connected CW-complex with fundamental group isomorphic to G whose universal covering space is contractible.

The circle S^1 is a $K(\mathbf{Z},1)$ because the universal covering space $\mathbf{R}$ is contractible.

The infinite projective space $\mathbf{R}P^\infty$ is a $K(C_2,1)$ and the infinite lense space $L^\infty(m)$ is a $K(C_m,1)$ because the infinite sphere S^∞ is contractible [**10**, Example 1B.3].

Knot complements are $K(G,1)$s [**10**, Example 1B6].

The orientable surfaces M_g of genus $g \geq 1$ and the nonorientable surfaces N_g of genus $g \geq 2$ are $K(G,1)$s where G is the fundamental group [**3**, §II.4] [**10**, Example 1B2]. It is clear that the torus $S^1 \times S^1$ is a $K(\mathbf{Z}\times\mathbf{Z},1)$. One now proceeds by induction. There is a theorem that says that the push-out of a diagram $K(G_1,1) \leftarrow K(H,1) \to K(G_2,1)$ of $K(G,1)$s and maps that are injective on π_1 is again a $K(G,1)$ [**3**, Thm II.7.3] [**10**, Thm 1B.11]. This theorem can be used here [**10**, Example 1B.14]. Alternatively, see [**10**, Exercise 4.2.16]. (The orientable surface S^2 of genus 0 and the nonorientable surface $\mathbf{R}P^2$ of genus 1 are not $K(G,1)$s for the universal covering space S^2 is not contractible as $H_2(S^2)$ is nontrivial.) Also noncompact surfaces and surfaces with boundary are $K(G,1)$ [**3**, §II.4, Examples].

$K(G,1)\times K(H,1) = K(G\times H,1)$ and $K(G,1)\vee K(H,1) = K(G*H,1)$ by the theorem [**3**, Thm II.7.3] [**10**, Thm 1B.11] mentioned above.

The double mapping cylinder X_{mn} of the degree m and the degree n self-map of the circle is a $K(G,1)$ [**10**, 1B.12].

For any group G there is a $K(G,1)$, namely the simplicial complex BG (2.47), and all $K(G,1)$s are homotopy equivalent [**10**, Thm 1B.8]; they represent the same homotopy type. We define the kth group homology of G to be $H_k(K(G,1))$, often denoted simply $H_k(G)$. Thus we have computed the group homology of all cyclic groups. In the language of homological algebra,

$$H_k(G) = \mathrm{Tor}_k^{\mathbf{Z}G}(\mathbf{Z},\mathbf{Z})$$

since the simplicial chain complex for BG is $\Delta_*(BG) = EG_* \otimes_{\mathbf{Z}G} \mathbf{Z}$ where EG_* is a free resolution of $\mathbf{Z}$ over $\mathbf{Z}G$.

See [**3**, Chp II] or [**10**, Chp 1.B] for more on group homology. For cohomology of finitely generated abelian groups see Cohomology of finitely generated abelian groups.

CHAPTER 4

Singular cohomology

1. Cohomology

The *singular cochain complex* of the space X with coefficients in the abelian group G is the dual

$$0 \longrightarrow C^0(X;G) \xrightarrow{\delta} C^1(X;G) \xrightarrow{\delta} \cdots \xrightarrow{\delta} C^k(X;G) \xrightarrow{\delta} C^{k+1}(X;G) \xrightarrow{\delta} \cdots$$

of the singular chain complex of Chapter 1. Here, $C^k(X;G) = \mathrm{Hom}_{\mathbf{Z}}(C_k(X),G) = G\langle S_k(X)\rangle$ is the abelian group of all functions from the set $S_k(X)$ of singular k-simplices in X to G. The *coboundary* map, which is is the dual of the boundary map, takes the k-cochain $\varphi\colon C_k(X) \to G$ to the $(k+1)$-cochain $\delta\phi = \phi\varphi$ as in the diagram

$$\begin{tikzcd} C_{k+1}(X) \arrow[r, "\partial"] \arrow[dr, "\delta\varphi"'] & C_k(X) \arrow[r, "\partial"] \arrow[d, "\varphi"] & C_{k-1}(X) \\ & G & \end{tikzcd}$$

of group homomorphisms.

4.1. Definition. The kth singular cohomology group of X is the quotient group

$$H^k(X;G) = Z^k(X;G)/B^k(X;G)$$

of the k-cocyles $Z^k(X;G) = \ker\delta = \{\varphi\colon C_k(X) \to G \mid \varphi(B_k(X)) = 0\}$ by the k-coboundaries $B^k(X;G) = \mathrm{im}\,\delta = \{\psi\,\partial \mid \psi\colon C_{k-1}(X) \to G\}$.

4.2. Evaluation. Cochains act (most naturally from the right) on chains by the *evaluation map*

$$C_k(X;G) \times C^k(X;G) \xrightarrow{\langle\ ,\ \rangle} G, \quad \langle c,\phi\rangle = \phi(c)$$

Since $\langle\partial c,\phi\rangle = \langle c,\delta\phi\rangle$ (for $c \in C_{k+1}(X;G)$ and $\phi \in C^k(X;G)$) we have that $\langle B_k, Z^k\rangle = 0 = \langle Z_k, B^k\rangle$ so there is an induced bilinear evaluation map

$$H_k(X;G) \times H^k(X;G) \xrightarrow{\langle\ ,\ \rangle} G, \quad \langle [z],[\phi]\rangle = \phi(z), \qquad z \in Z_k(X), \phi \in Z^k(X;G),$$

on homology. We may view this bilinear map as a linear map

$$H^k(X;G) \xrightarrow{h} \mathrm{Hom}_{\mathbf{Z}}(H_k(X;G),G), \quad h([\phi])([z]) = \langle [z],[\phi]\rangle = \phi(z)$$

from cohomology to the dual of homology. Is h an isomorphism?

4.3. Ext **and the UCT for cohomology.** We investigate the relation between dualizing and taking homology. we begin by making two observations. First we dualize short exact sequences and realize that the dual sequence may not be exact any more. The second observation is an example.

4.4. Lemma. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian groups and let G be an abelian group. Then

$$0 \to \mathrm{Hom}(C,G) \to \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G)$$

is exact. If the short exact sequence is split exact (eg if C is free) then

$$0 \to \mathrm{Hom}(C,G) \to \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G) \to 0$$

is also split exact.

4.5. Example. We look at the chain complex C like this

$$0 \longleftarrow \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z} \xleftarrow{(x,2y) \leftarrow (x,y)} \mathbf{Z} \oplus \mathbf{Z} \longleftarrow 0$$

where the nonzero groups are in degree 1 and 2. We compute its homology and cohomology

	$i=1$	$i=2$
$H_i(C;\mathbf{Z})$	$\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$	0
$H^i(C;\mathbf{Z})$	$\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$

On the basis of this one example, formulate a conjecture about the relation between the homology and the cohomology of a chain complex (of free abelian groups)! The conjecture is formalized in the Universal Coefficient Theorem (or UCT for short) which uses the functor Ext that we now define.

Let G and H be two abelian groups. Choose a short exact sequence

$$0 \to F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} H \to 0 \tag{4.6}$$

where F_0 and F_1 are free abelian groups. If we apply the functor $\mathrm{Hom}_{\mathbf{Z}}(-,G)$ to this short exact sequence we get an exact sequence (4.4)

$$0 \to \mathrm{Hom}_{\mathbf{Z}}(H,G) \xrightarrow{\partial_0^*} \mathrm{Hom}_{\mathbf{Z}}(F_0,G) \xrightarrow{\partial_1^*} \mathrm{Hom}_{\mathbf{Z}}(F_1,G) \to \mathrm{Ext}_Z(H,G) \to 0 \tag{4.7}$$

where we write $\mathrm{Ext}_Z(H,G)$ for the abelian group

$$\mathrm{Ext}_{\mathbf{Z}}(H,G) = \operatorname{coker} \partial_1^* = \mathrm{Hom}_Z(F_1,G)/\operatorname{im} \partial_1^* \tag{4.8}$$

to the right. This notation is justified since the isomorphism type of this group does not depend on the choice of (4.6).

4.9. Lemma. (Lifting lemma) Let $0 \to F_1' \xrightarrow{\partial_1'} F_0' \xrightarrow{\partial_0'} G \to 0$ be another short exact sequence where F_0' and F_1' are abelian groups (not necessarily free). Any group homomorphism $\alpha_{-1}\colon H \to H$ lifts to a morphism

$$\begin{tikzcd}
0 \arrow[r] & F_1 \arrow[r, "\partial_1"] \arrow[d, "\alpha_1"] & F_0 \arrow[r, "\partial_0"] \arrow[d, "\alpha_0"] & H \arrow[r] \arrow[d, "\alpha_{-1}"] & 0 \\
0 \arrow[r] & F_1' \arrow[r, "\partial_1"] & F_0' \arrow[r, "\partial_0"] & H \arrow[r] & 0
\end{tikzcd}$$

of short exact sequences and the lift is unique up to chain homotopy.

If we apply the lemma to the identity map of H we see that any two short exact sequences as in (4.6) are chain homotopy equivalent. The dual chain complexes (4.7) are then also chain homotopy equivalent and therefore the isomorphism class of the group (4.8) does not depend on the choice of (4.6).

4.10. Example. ($\mathrm{Ext}_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Z}/n\mathbf{Z})$) Suppose that $H = \mathbf{Z}/m\mathbf{Z}$ is cyclic of order m and $G = \mathbf{Z}/n\mathbf{Z}$ is cyclic of order n. We may take (4.6) to be $0 \to \mathbf{Z} \xrightarrow{\cdot m} \mathbf{Z} \to \mathbf{Z}/m\mathbf{Z} \to 0$ and then (4.7) becomes

$$0 \longrightarrow \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Z}/n\mathbf{Z}) \longrightarrow \mathbf{Z}/n\mathbf{Z} \xrightarrow{\cdot m} \mathbf{Z}/n\mathbf{Z} \longrightarrow \mathrm{Ext}_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Z}/n\mathbf{Z}) \longrightarrow 0$$

The two groups, $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Z}/n\mathbf{Z})$ and $\mathrm{Ext}_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Z}/n\mathbf{Z})$, at the ends of this exact sequence are cyclic groups of the same order since for a homomorphism between finite abelian groups the kernel and the cokernel always have equal orders. We see that

$$\mathrm{Ext}_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Z}/n\mathbf{Z}) = \frac{\mathbf{Z}/n}{m \cdot \mathbf{Z}/n} = \frac{\mathbf{Z}}{m\mathbf{Z} + n\mathbf{Z}} = \mathbf{Z}/(m,n)\mathbf{Z}$$

where (m,n) denotes the greatest common divisor of m and n.

4.11. Example. ($\mathrm{Ext}_{\mathbf{Z}}(H,G)$ for any finitely generated abelian group H) When $H = \mathbf{Z}$ is infinite cyclic or $H = \mathbf{Z}/m\mathbf{Z}$ is cyclic of finite order m, it is immediate from the definition that $\mathrm{Ext}_{\mathbf{Z}}(\mathbf{Z},G) = 0$ and that $\mathrm{Ext}_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z},G) = G/mG$ for any abelian group G. Since also $\mathrm{Ext}_{\mathbf{Z}}(-,G)$ commutes with finite direct sums (why?), we have computed $\mathrm{Ext}_{\mathbf{Z}}(H,G)$ for any finitely generated abelian group H and any abelian group G.

In particular, when $G = \mathbf{Z}$, $\mathrm{Ext}(H,\mathbf{Z})$ is isomorphic to the torsion subgroup of H and $\mathrm{Hom}(H,\mathbf{Z})$ to the free component of H.

Here is an indication of the connection between Ext and extensions. Let $0 \to G \to A \to H \to 0$ be an extension of H by G. The Lifting Lemma gives a map

$$\begin{tikzcd} 0 \arrow[r] & F_1 \arrow[r, "\partial_1"] \arrow[d, "\alpha_1"] & F_0 \arrow[r, "\partial_0"] \arrow[d, "\alpha_0"] & H \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & G \arrow[r] & A \arrow[r] & H \arrow[r] & 0 \end{tikzcd}$$

and the map $\alpha_1 \in \mathrm{Hom}(F_1, G)$ represents an element of $\mathrm{Ext}(H, G)$. In this way, any group extension represents an element of the Ext-group.

The Universal Coefficient Theorem (UCT) expresses to what extent the two functors homology and dualizing commute.

4.12. Theorem (UCT). *Let (C_*, ∂) be a chain complex of free abelian groups. Form the dual cochain complex $(\mathrm{Hom}(C_*; G), \delta)$ of homomorphisms of the chain complex into some abelian group G. Then there is a natural short exact sequence*

$$0 \longrightarrow \mathrm{Ext}_{\mathbf{Z}}(H_{k-1}(C_*), G) \longrightarrow H^k(\mathrm{Hom}(C_*, G)) \xrightarrow{h} \mathrm{Hom}_{\mathbf{Z}}(H_k(C_*), G) \longrightarrow 0$$

The sequence always splits but not naturally.

Proof. First note that the short exact sequence $0 \to Z_k \xrightarrow{i_k} B_k \to H_k \to 0$ dualizes (4.7) to the exact sequence

$$0 \to \mathrm{Hom}(H_k, G) \to B_k^* \xrightarrow{i_k^*} Z_k^* \to \mathrm{Ext}(H_k, G) \to 0$$

Next, observe that we have a short exact sequence of chain complexes

$$\begin{tikzcd} 0 \arrow[r] & Z_{k+1} \arrow[r] \arrow[d, "0"] & C_{k+1} \arrow[r, "\partial"] \arrow[d, "\partial"] & B_k \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & Z_k \arrow[r] & C_k \arrow[r, "\partial"] & B_{k-1} \arrow[r] & 0 \end{tikzcd}$$

and because the groups are free abelian the dual diagram

$$\begin{tikzcd} 0 \arrow[r] & B_k^* \arrow[r] & C_{k+1}^* \arrow[r] & Z_{k+1}^* \arrow[r] & 0 \\ 0 \arrow[r] & B_{k-1}^* \arrow[r] \arrow[u, "0"] & C_k^* \arrow[r] \arrow[u, "\delta"] & Z_k^* \arrow[r] \arrow[u, "0"] & 0 \end{tikzcd}$$

is again a short exact sequence of chain complexes. The Fundamental Theorem of Homological Algebra produces a long exact sequence

$$\cdots \to B_{k-1}^* \xrightarrow{i_{k-1}^*} Z_{k-1}^* \to H^k(C_*) \to Z_k^* \xrightarrow{i_k^*} B_k^* \to \cdots$$

where the connecting homomorphism turns out to be the restriction map from B_k^* to Z_k^*. This long exact sequence determines short exact sequences

$$0 \to \mathrm{Ext}(H_{k-1}, G) = \operatorname{coker} i_{k-1}^* \to H^k(C_*) \to \ker i_k^* = \mathrm{Hom}(H_k, G) \to 0$$

of the UCT.

<u>The UCT splits</u>: Use the maps

$$\begin{tikzcd} & B_k \arrow[d] & C_{k-1} \arrow[l, "\partial"'] \arrow[d, "\partial"] & & \\ 0 \arrow[r] & Z_k \arrow[r] \arrow[d, "\pi"'] & C_k \arrow[l, bend right, dotted, "\sigma"'] \arrow[r] & B_{k-1} \arrow[r] & 0 \\ G & H_k(C) \arrow[l, "\varphi"'] & & & \end{tikzcd}$$

where, in particular, σ is a splitting so that σ is the identity on the subgroup Z_k of C_k. Construct the splitting as in

$$\begin{tikzcd}
\mathrm{Hom}(H_k(C), G) \arrow[r, "\varphi \to \varphi\pi\sigma"] \arrow[dr, dotted] & \mathrm{Hom}(C_k, G) & \\
& Z^k(C; G) \arrow[u, hook] \arrow[r, two heads] & H^k(C; G)
\end{tikzcd}$$

Note that $\delta(\varphi\pi\sigma) = \varphi\pi\sigma\partial = 0$ so that the cochain $\varphi\pi\sigma$ is actually a cocycle. □

Note that the UCT applies to the singular chain complex of a space or a pair of spaces, the cellular chain complex of a CW-complex (1.68), and the simplicial chain complex of a Δ-complex (2.29). For a space with finitely generated integral homology groups, the UCT says that the integral cohomology group in degree k is isomorphic to the direct sum of the free component of the homology group in degree k and the torsion of the homomology group in degree $k-1$.

4.13. Corollary. If a chain map between chain complexes of free abelian groups induces an isomorphism on homology then it induces an isomorphism on cohomology with coefficients in any abelian group.

Proof. Use the UCT and the 5-lemma. □

What makes the proof of the UCT work is the fact (which we have used several times) that any subgroup of a free abelian group is free.

4.14. Proposition. Any subgroup of a free abelian groups is free.

Proof. Let us first consider *finitely generated* free abelian groups. We use induction over the cardinality of a basis. To start the induction, let $\mathbf{Z}$ be free of rank 1. Any subgroup of the abelian group $\mathbf{Z}$ is free for it is of the form $m\mathbf{Z}$ for some $m \in \mathbf{Z}$ and $m\mathbf{Z}$ is isomorphic to $\mathbf{Z}$ or to 0. Let now $\mathbf{Z}^{n+1} = \mathbf{Z}^n \oplus \mathbf{Z}$ be free of rank $n+1$. Let G be a subgroup. There is a similar splitting $G = G_n \oplus G_1$ of G. We know that G_1 and G_n are free by induction. Thus G is free. Use transfinite induction for the general case. □

Now this proof works equally well for any PID R since the submodules of the module R, the ideals in the ring R, are of the form mR for some $m \in R$. So we have the more general

4.15. Proposition. Any submodule of a free module over a PID is free.

Thus there is a more general form of the UCT for chain complexes of free modules over a PID R. Just replace $\mathbf{Z}$ by R in 4.12. In particular, we could let $R = k$ be a field. Over a field there is not even an Ext-term since all modules, vector spaces, are free themselves.

4.16. Corollary. Let k be a field and let (C_*, ∂) be a chain complex of vector spaces over k. Then there is an isomorphism

$$H^i(C_*; V) = \mathrm{Hom}_k(H_i(C_*; k), V)$$

for any k-vector space V.

In particular, $H^i(X; k) \cong \mathrm{Hom}_k(H_i(X; k), k)$; over a field, cohomology is the dual of homology.

4.17. Exercise. Prove the Lifting Lemma 4.9.

4.18. Exercise. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian groups and let G be an abelian group. Show that there is a 6-term exact

$$0 \to \mathrm{Hom}(C, G) \to \mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G) \to \mathrm{Ext}(C, G) \to \mathrm{Ext}(B, G) \to \mathrm{Ext}(A, G) \to 0$$

of abelian groups. What is $\mathrm{Ext}(\mathbf{Q}, \mathbf{Z})$?

4.19. Tor and the UCT for homology. We briefly discuss the universal coefficient theorem for homology. Let H and G be two abelian groups. Apply the functor $-\otimes_{\mathbf{Z}} G$ to the short exact sequence (4.6) and get the exact sequence

$$0 \to \operatorname{Tor}_{\mathbf{Z}}(H,G) \to F_0 \otimes G \to F_1 \otimes G \to H \otimes G \to 0 \tag{4.20}$$

where we define $\operatorname{Tor}_{\mathbf{Z}}(H,G)$ as the kernel.

4.21. Exercise. Show that the cyclic groups $\mathbf{Z}/m\mathbf{Z} \otimes \mathbf{Z}/n\mathbf{Z}$ and $\operatorname{Tor}_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Z}/n\mathbf{Z})$ have the same order, $\mathrm{GCD}(m,n)$.

Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian groups and let G be an abelian group. Show that there is a 6-term exact

$$0 \to \operatorname{Tor}(A,G) \to \operatorname{Tor}(B,G) \to \operatorname{Tor}(C,G) \to A \otimes G \to B \otimes G \to C \otimes G \to 0$$

of abelian groups.

4.22. Theorem (UCT for homology). *Let (C_*, ∂) be a chain complex of free abelian groups and G and abelian group. Then there is a natural short exact sequence*

$$0 \longrightarrow H_k(C_*) \otimes G \longrightarrow H_k(C_* \otimes G) \longrightarrow \operatorname{Tor}(H_{k-1}(C_*), G) \longrightarrow 0$$

The sequence splits but not naturally.

Proof. Dual to the proof for the UCT in cohomology. $\square$

4.23. Reduced cohomology. By definition, the reduced cohomology groups of $X \neq \emptyset$ are the homology groups $\widetilde{H}^n(X;G)$ of the dual of the augmented chain complex (1.7). There is no difference between reduced and unreduced cohomology in positive degrees. In degree 0 there is a split exact sequence

$$0 \to G \xrightarrow{\varepsilon^*} H^0(X;G) \to \widetilde{H}^0(X;G) \to 0$$

which is $0 \to \operatorname{Hom}(\mathbf{Z}, G) \xrightarrow{\varepsilon^*} \ker \partial_1^* \to \ker \partial_1^* / \operatorname{im} \varepsilon^* \to 0$. $H^0(X;G) = Z^0(X;G) = \ker d_1^*$ is abelian group $\mathrm{map}(\pi_0(X), G)$ of maps of the set of path-components of X to G. The map $H^0(X;G) \ni \varphi \to \varphi(x_0)$ where x_0 is some point of X is a left inverse of ε^*. Note that $\widetilde{H}^*(\{x_0\};G) = 0$ for the space consisting of a single point. The long exact sequence in reduced cohomology (4.23) for the pair (X, x_0) gives that $\widetilde{H}^n(X;G) \cong H^n(X, x_0; G)$. The long exact sequence in (unreduced) cohomology (1.10) breaks into short split exact sequences because the point is a retract of the space and it begins with

$$0 \longrightarrow H^0(X, x_0; G) \longrightarrow H^0(X;G) \longrightarrow H^0(x_0;G) \longrightarrow 0$$

so that $\widetilde{H}^0(X;G) \cong H^0(X, x_0;G) \cong \ker\big(H^0(X;G) \to H^0(x_0;G)\big)$.

4.23. The long exact cohomology sequence for a pair. Suppose that (X, A) is a pair of spaces. The cohomology long exact sequence

$$\cdots \longrightarrow H^{k-1}(A;G) \xrightarrow{\delta} H^k(X,A;G) \xrightarrow{j^*} H^k(X;G) \xrightarrow{i^*} H^k(A;G) \longrightarrow H^k(X,A;G) \longrightarrow \cdots$$

is the long exact sequence of the short exact sequence

$$0 \to C^*(X,A;G) \xrightarrow{j^*} C^*(X;G) \xrightarrow{i^*} C^*(A;G) \to 0$$

of chain complexes obtained by dualizing the the short exact sequence $0 \to C_*(A) \xrightarrow{i} C_*(X) \xrightarrow{j} C_*(X,A) \to 0$ of *free* abelian groups (4.4). The *connecting homomorphism* δ takes the cohomology class $[\varphi] \in H^{k-1}(A;G)$ of the cocycle $\varphi\colon C_{k-1}(A) \to G$ to the cohomology class of the cocycle $\delta\varphi$ as in the diagram

$$\begin{tikzcd}
C_k(A) \arrow[r, "i"] \arrow[d, "\partial"] & C_k(X) \arrow[r, "j"] \arrow[d, "\partial"] & C_k(X,A) \arrow[ddl, dotted, "\delta\varphi"] \\
C_{k-1}(A) \arrow[r, "i"] \arrow[dr, "\varphi"'] & C_{k-1}(X) \arrow[d, "\overline{\varphi}"] & \\
& G &
\end{tikzcd}$$

of abelian groups.

The UCT applies also to relative cohomology. The homomorphisms of the cohomology long exact sequence correspond under the homomorphism h to the homomorphisms of the homology long exact sequence (1.10). This is clear for the restriction maps i^* and j^* by naturality and for δ we explicitly verify that this is so.

4.24. Lemma. The diagram

$$\begin{tikzcd} H^{k-1}(A;G) \arrow[r, "\delta"] \arrow[d, "h"] & H^k(X,A;G) \arrow[d, "h"] \\ \mathrm{Hom}(H_{k-1}(A),G) \arrow[r, "\partial^*"] & \mathrm{Hom}(H_k(X,A),G) \end{tikzcd}$$

is commutative: $(\delta[\varphi])[z] = [\varphi](\partial[z])$ or $\langle \partial[z9, [\varphi]\rangle = \langle [z], \delta[\varphi]\rangle$ for $[\varphi] \in H^{k-1}(A;G)$, $[z] \in H_k(X,A)$.

The reduced cohomology long exact sequence

$$\cdots \longrightarrow \widetilde{H}^{k-1}(A;G) \xrightarrow{\delta} H^k(X,A;G) \xrightarrow{j^*} \widetilde{H}^k(X;G) \xrightarrow{i^*} \widetilde{H}^k(A;G) \longrightarrow H^k(X,A;G) \longrightarrow \cdots$$

is obtained by dualizing the augmented chain complexes.

Similarly, the cohomology long exact sequence of a triple (X,A,B)

$$\cdots \to H^{k-1}(A,B;G) \xrightarrow{\delta} H^k(X,A;G) \xrightarrow{j^*} H^k(X,B;G) \xrightarrow{i^*} H^k(A,B;G) \to H^k(A,B;G) \to \cdots$$

is obtained by dualizing the short exact sequence $0 \to C_*(A,B) \to C_*(X,B) \to C_*(X,A) \to 0$ of relative chain complexes.

4.25. Homotopy invariance and excision. If $f_0 \simeq f_1 \colon X \to Y$, then the induced chain maps $C_*(f_0) \simeq C_*(f_1) \colon C_*X -$
The dual chain maps are then also chain homotopic and thus $H^*(f_0) = H^*(f_1) \colon H^*(Y;G) \to H^*(X;G)$.

When $X \supset A \supset \mathrm{int}(A) \supset \mathrm{cl}(U) \supset U$ then $H^k(X,A;G) \cong H^k(X-U, A-U;G)$ because the inclusion induces an isomorphism on homology and also on cohomology by the UCT (4.13).

4.26. Example. Using that the suspension SX of X is the union of two contractible cones we get that $\widetilde{H}^{k+1}(SX;G) = \widetilde{H}^{k+1}(C_+X \cup C_-X, C_-X;G) \cong \widetilde{H}^{k+1}(C_+,X;G) \cong \widetilde{H}^k(X;G)$.

4.27. Cup and cap products. In the following we use coefficients in some commutative ring R, typically $R = \mathbf{Z}, \mathbf{Z}/n\mathbf{Z}, \mathbf{Q}$. The product in R allows us to define products, the *cup* and the *cap* product, which are the bilinear maps

$$C^k(X;R) \times C^\ell(X;R) \xrightarrow{\cup} C^{k+\ell}(X;R), \quad C_{k+\ell}(X;R) \times C^k(X;R) \xrightarrow{\cap} C_\ell(X;R)$$

given by the formulas

$$\langle \sigma, \phi \cup \psi\rangle = \langle \sigma|[e_0 \cdots e_k]), \phi\rangle \cdot \langle \sigma|[e_k \cdots e_{k+\ell}], \psi\rangle, \quad \sigma \cap \phi = \langle \sigma|[e_0 \cdots e_k], \phi\rangle \sigma|[e_k \cdots e_{k+\ell}]$$

where $\sigma \in C_{k+\ell}(X;R)$ is a singular $(k+\ell)$ simplex in X. These two pairings are related since

$$\langle c, \phi \cup \psi\rangle = \langle c \cap \phi, \psi\rangle, \qquad c \in C_{k+\ell}(X;R), \phi \in C^k(X;R), \psi \in C^\ell(X;R) \tag{4.28}$$

$$c \cap (\phi \cup \psi) = (c \cap \phi) \cap \psi, \qquad c \in C_{m+k+\ell}(X;R), \phi \in C^k(X;R), \psi \in C^\ell(X;R) \tag{4.29}$$

These two pairings are natural in the sense that if $f \colon X \to Y$ is a map then

$$f^*(\phi \cup \psi) = f^*\phi \cup f^*\psi, \qquad c \in C_{k+\ell}(X;R), \phi \in C^k(Y;R), \psi \in C^\ell(Y;R) \tag{4.30}$$

$$f_*(c \cap f^*\phi) = f_*c \cap \phi, \qquad c \in C_{k+\ell}(X;R), \phi \in C^k(Y;R) \tag{4.31}$$

Cup and cap products behave nicely under the (co)boundary operator.

4.32. Lemma. $\delta(\phi\psi) = \delta\phi \cup \psi + (-1)^k \phi \cup \delta\psi$ and $(-1)^k \partial(c \cap \phi) = \partial c \cap \phi - c \cap \delta\phi$.

Proof. Verify that the first formula is true when $k = 1$, $\ell = 2$. Then figure out the general argument. The second formula follows from the first: $\langle (-1)^k \partial(c \cap \varphi), \psi\rangle = \langle c, \varphi \cup (-1)^k \delta\psi\rangle = \langle c, \delta(\varphi \cup \psi) - \delta\varphi \cup \psi\rangle = \langle \partial c \cap \varphi - c \cap \delta\varphi, \psi\rangle$. $\square$

These boundary formulas imply that there are induced pairings on homology

$$H^k(X;R) \times H^\ell(X;R) \xrightarrow{\cup} H^{k+\ell}(X;R), \qquad H_{k+\ell}(X;R) \times H^k(X;R) \xrightarrow{\cap} H_\ell(X;R)$$
$$[\phi] \cup [\psi] = [\phi \cup \psi] \qquad [z] \cap [\phi] = [z \cap \phi]$$

such that $\langle [z], [\phi] \cup [\psi] \rangle = \langle [z] \cap [\phi], [\psi] \rangle$.

If we let $C^*(X;R) = \bigoplus C^k(X;R)$ and $H^*(X;R) = \bigoplus H^k(X;R)$, then $(C^*(X;R), \delta, \cup)$ is a differential graded R-algebra and $(H^*(X;R), \cup)$ a graded R-algebra. The graded homology group $H_*(X;R) = \bigoplus H_k(X;R)$ is a graded module over the graded R-algebra $H^*(X;R)$ (4.29). If $f\colon X \to Y$ is any map then (4.31) $f^*\colon H^*(Y;R) \to H^*(X;R)$ is a ring homomorphism and $f_*\colon H_*(X;R) \to H_*(Y;R)$ is an $H^*(Y;R)$-module homomorphism; this means that the diagram

$$\begin{tikzcd}
H_{k+\ell}(X;R) \times H^k(X;R) \arrow[r, "\cap"] & H_\ell(X;R) \arrow[dd, "f_*"] \\
H_{k+\ell}(X;R) \times H^k(Y;R) \arrow[u, "\mathrm{id}\times f^*"'] \arrow[d, "f_*\times \mathrm{id}"] & \\
H_{k+\ell}(Y;R) \times H^k(Y;R) \arrow[r, "\cap"] & H_\ell(Y;R)
\end{tikzcd}$$

commutes or that $f_*(\zeta \cap f^*c) = f_*\zeta \cap c$ for all $\zeta \in H_{k+\ell}(X;R)$ and all $c \in H^k(Y;R)$ as in (4.31).

If X is path-connected, we have identifications $H_0(X;R) \underset{\cong}{\xrightarrow{\varepsilon}} R$ and $R \underset{\cong}{\xrightarrow{\varepsilon^*}} H^0(X;R)$ (1.7, 4.23) under which the cup product $R \times H^\ell(X;R) \xrightarrow{\cup} H^\ell(X;R)$ and the cap product $H_\ell(X;R) \times R \xrightarrow{\cap} H_\ell(X;R)$ are simply scalar multiplication and the cap product $H_k(X;R) \times H^k(X;R) \to H_0(X;R) = R$ is evaluation $\langle\ ,\ \rangle$ (4.2).

4.33. Relative cup and cap products. Suppose that $A \subset X$. The same formula that we used above for the cup product also defines bilinear maps

$$C^k(X;R) \times C^\ell(X,A;R) \xrightarrow{\cup} C^{k+\ell}(X,A;R), \qquad C^k(X,A;R) \times C^\ell(X,A;R) \xrightarrow{\cup} C^{k+\ell}(X,A;R)$$

For if $\phi \in C^k(X;R)$ is any cochain, and $\psi \in C^\ell(X;R) = \mathrm{Hom}_R(C_\ell(X;R), R)$ is a cochain that vanishes on $C_\ell(A,R)$, then the cochain $\phi \cup \psi \in \mathrm{Hom}_R(C_{k+\ell}(X;R), R)$ vanishes on $C_{k+\ell}(A,R)$. Similarly, the same formula that we used above for the cap product defines bilinear maps

$$C_{k+\ell}(X,A;R) \times C^k(X;R) \xrightarrow{\cap} C_\ell(X,A;R), \qquad C_{k+\ell}(X,A;R) \times C^k(X,A;R) \xrightarrow{\cap} C_\ell(X;R)$$

This is because, first, $C_{k+\ell}(A;R) \cap \phi \subset C_\ell(A;R)$ for all $\phi \in C^k(X;R)$, and, second, $C_{k+\ell}(A;R) \cap \phi = 0$ if ϕ vanishes on $C_k(A;R)$.

Since the boundary formulas (4.32) still hold there are induced bilinear maps

$$H^k(X;R) \times H^\ell(X,A;R) \xrightarrow{\cup} H^{k+\ell}(X,A;R), \qquad H^k(X,A;R) \times H^\ell(X,A;R) \xrightarrow{\cup} H^{k+\ell}(X,A;R)$$
$$H_{k+\ell}(X,A;R) \times H^k(X;R) \xrightarrow{\cap} H_\ell(X,A;R) \qquad H_{k+\ell}(X,A;R) \times H^k(X,A;R) \xrightarrow{\cap} H_\ell(X;R)$$

There are also cap products

$$H_{k+\ell}(X;R) \times H^k(X;R) \xrightarrow{\cap} H_\ell(X,A;R), \qquad H_{k+\ell}(X;R) \times H^k(X,A;R) \xrightarrow{\cap} H_\ell(X;R)$$

obtained by composing the above cap products with a map induced from an inclusion. Thus $H^*(X;R)$ and $H^*(X,A;R)$ are graded commutative R-algebras with $H_*(X;R)$ and $H_*(X,A;R)$ as graded modules. Let $f\colon (X,A) \to (Y,B)$ be any map. The induced maps $f^*\colon H^*(Y;R) \to H^*(X;R)$ and $f^*\colon H^*(Y,B;R) \to H^*(X,A;R)$ on cohomology are algebra maps and the induced maps $f_*\colon H_*(X;R) \to H_*(Y;R)$ and $f_*\colon H_*(X,A;R) \to H_*(Y,B;R)$ on homology are homomorphisms of modules (4.31).

If X is path-connected, we have identifications $H_0(X;R) \xrightarrow{\simeq} R$ and $R \xrightarrow{\simeq} H^0(X;R)$ under which the cup product $R \times H^\ell(X,A;R) \xrightarrow{\cup} H^\ell(X,A;R)$ and the cap product $H_\ell(X,A;R) \times R \xrightarrow{\cap} H_\ell(X;R)$ are simply scalar multiplication and the cap product $H_k(X,A;R) \times H^k(X,A;R) \to R$ is evaluation $\langle\ ,\ \rangle$ for relative (co)homology groups.

4.34. The cellular cochain complex of a CW-complex. Let X be a CW-complex with skeletal filtration $\emptyset = X^{-1} \subset X^0 \subset \cdots \subset X^n \subset X^{n+1} \subset \cdots \subset X$ and let G be an abelian group. The *cellular cochain complex* of X with coefficients in G is the dual complex

$$\cdots \longrightarrow H^{n-1}(X^{n-1}, X^{n-2}; G) \xrightarrow{d^n} H^n(X^n, X^{n-1}; G) \xrightarrow{d^{n+1}} H^{n+1}(X^{n+1}, X^n; G) \longrightarrow \cdots$$

of the cellular chain complex (1.68) and the *cellular cohomology* $H^n_{CW}(X;G)$ of X is its cohomology. Lemma 4.24 and the definition of the cellular boundary map shows that cellular coboundary map d^{n+1} is the map

$$H^n(X^n, X^{n-1}; G) \to H^n(X^n; G) \xrightarrow{\delta} H^{n+1}(X^{n+1}, X^n; G)$$

Just as in 1.65 we have

4.35. Theorem (Cf 1.69). *There is an isomorphism $H^n_{CW}(X;G) \cong H^n(X;G)$ which is natural wrt cellular maps.*

which follows from

4.36. Lemma (Cf 1.66). Let X be a CW-complex and $X^n = X^{n-1} \cup_\phi \coprod D^n_\alpha$ the n-skeleton. Then

(1) $H^k(X^n, X^{n-1}) = \mathrm{Hom}(H_k(X^n, X^{n-1}), G) = \begin{cases} \prod H_n(D^n_\alpha, S^{n-1}_\alpha) & k = n \\ 0 & k \neq n \end{cases}$
(2) $H^k(X^n) = 0$ for $k > n \geq 0$
(3) $H^k(X^n) \cong H^k(X)$ for $0 \leq k < n$.

4.37. Exercise. Compute the cohomology groups of the compact surfaces (1.75, 1.76) and projective spaces (1.77, 1.79).

It is, however, not so easy to compute cup and cap products in this way.

4.38. The cochain complex of a Δ-set. Let $S = \bigcup S_n$ be a Δ-complex (2.27) and G an abelian group. Write $G\langle S_n\rangle = \{S_n \xrightarrow{\phi} G\} = \mathrm{Hom}(\mathbf{Z}S_n, G)$ for the abelian group consisting of all functions of the set S_n of n-simplices into the abelian group G. The *simplicial cochain of S with coefficients in G*, $(G\langle S\rangle, \delta)$ or $(\Delta^*(|S|, \delta)$, is the dual

$$0 \xrightarrow{\delta} G\langle S_0\rangle \xrightarrow{\delta} G\langle S_1\rangle \xrightarrow{\delta} \cdots \xrightarrow{\delta} G\langle S_{n-1}\rangle \xrightarrow{\delta} G\langle S_n\rangle \xrightarrow{\delta} \cdots$$

of the simplicial chain complex (2.29). Thus $\delta(\phi)(\sigma) = \phi(\partial\sigma) = \sum(-1)^i\phi(d_i\sigma)$ for all $\phi\colon S_{n-1} \to G$ and all $\sigma \in S_n$. The simplicial cochain complex is a quotient of the singular cochain complex by the projection map

$$(G\langle S_*\rangle, \delta) \leftarrow (C^*(|S|), \delta) \tag{4.39}$$

dual to the unit transformation of the simplicial chain complex to the singular chain complex of Section 2.3). The *simplicial cohomology group* $H^n_\Delta(S)$ is the nth cohomology group of the simplicial cochain complex.

If the coefficient group $G = R$ is a ring then there are simplicial cup and cap products

$$R\langle S_k\rangle \times R\langle S_\ell\rangle \xrightarrow{\cup} R\langle S_{k+\ell}\rangle, \quad R[S_{k+\ell}] \times R\langle S_k\rangle \xrightarrow{\cap} R[S_\ell]$$

given by the same formulas as before (in disguise)

$$\langle \sigma, \phi \cup \psi\rangle = \langle \underbrace{d_{k+1}\cdots d_{k+1}}_{\ell}\sigma, \phi\rangle \cdot \langle \underbrace{d_0\cdots d_0}_{k}\sigma, \psi\rangle, \quad \sigma \cap \phi = \langle \underbrace{d_{k+1}\cdots d_{k+1}}_{\ell}\sigma, \phi\rangle\, \underbrace{d_0\cdots d_0}_{k}\sigma$$

where $\sigma \in S_{n+\ell}$ is a simplex in S and $\phi \in R\langle S_k\rangle, \psi \in R\langle S_\ell\rangle$. (Since $d_0\sigma$ means that we delete vertex 0 from σ, $d_0\cdots d_0\sigma$ is the back of σ.)

4.40. Theorem (Δ-sets have ∪ and ∩ products). *$(R[S], \partial)$ is a differential graded module over the differential graded ring $(R\langle S\rangle, \delta, \cup)$. The quotient morphism (4.39) of differential graded rings induces an isomorphism (natural wrt simplicial maps)*

$$H^*_\Delta(S;R) \xleftarrow{\cong} H^*(|S|, R)$$

of graded R-algebras. The map (4.39) of differential graded modules induces an isomorphism

$$H_*^\Delta(S;R) \xrightarrow{\cong} H_*(|S|;R)$$

of graded R-modules over the graded R-algebra $H^*(|S|;R)$.

Proof. The first two statements are immediate since the simplicial cup product and the singular cup product are in fact defined by the same formula. Since we already know that we have an isomorphism on homology (1.84) the (corollary to the) UCT (4.13) implies that we also have an isomorphism on cohomology. □

This theorem shows that there is a computer program that computes the cohomology ring of any finite Δ-complex.

4.41. Example. (The cohomology ring $H^*(\mathbf{R}P^2;\mathbf{F}_2)$) $\mathbf{R}P^2 = |S|$ is the realization of the Δ-set $S = (\ \{x_0,x_1\} \leftleftarrows \{a,b_1,b_2\} \Lleftarrow \{c_1,c_2\}\ )$ where $d_0(a,b_1,b_2) = (x_1,x_1,x_1)$, $d_1(a,b_1,b_2) = (x_1,x_0,x_0)$, $d_0(c_1,c_2) = (a,a)$, $d_1(c_1,c_2) = (b_1,b_2)$, $d_2(c_1,c_2) = (b_2,b_1)$ as shown in Figure 1. The simplicial chain and cochain com-

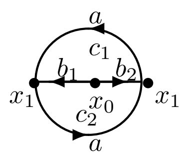


Figure 1. $\mathbf{R}P^2$ as a Δ-complex

plexes, $(\mathbf{F}_2[S],\partial)$ and $(\mathbf{F}_2\langle S\rangle,\delta)$, are

$$0 \longleftarrow \mathbf{F}_2\{x_0,x_1\} \xleftarrow{\partial_1} \mathbf{F}_2\{a,b_1,b_2\} \xleftarrow{\partial_2} \mathbf{F}_2\{c_1,c_2\} \longleftarrow 0$$

$$0 \longrightarrow \mathbf{F}_2\{x_0^t,x_1^t\} \xrightarrow{\partial_1^t} \mathbf{F}_2\{a^t,b_1^t,b_2^t\} \xrightarrow{\partial_2^t} \mathbf{F}_2\{c_1^t,c_2^t\} \longrightarrow 0$$

where y^t is the homomorphism that is dual to y and

$$\partial_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \partial_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and ∂_1^t and ∂_2^t are the transposed matrices. We read off the homology groups

$$H_1^\Delta(\mathbf{R}P^2;\mathbf{F}_2) = Z_1/B_1 = \mathbf{F}_2\{a, b_1+b_2\}/\mathbf{F}_2\{a+b_1+b_2\} \cong \mathbf{F}_2\{a\}$$
$$H_2^\Delta(\mathbf{R}P^2;\mathbf{F}_2) = Z_2 = \mathbf{F}_2\{c_1+c_2\}$$

The nonzero homology class $[\mathbf{R}P^2] \in H_2^\Delta(\mathbf{R}P^2;\mathbf{F}_2)$ represented by 2-cycle $\mathbf{R}P^2 = c_2 + c_2$ is called the orientation class. We also read off the cohomology groups

$$H^1_\Delta(\mathbf{R}P^2;\mathbf{F}_2) = Z^1/B^1 = \mathbf{F}_2\{a^t+b_1^t, a^t+b_2^t\}/\mathbf{F}_2\{b_1^t+b_2^t\} \cong \mathbf{F}_2\{[a^t+b_1^t]\}$$
$$H^2_\Delta(\mathbf{R}P^2;\mathbf{F}_2) = Z^2/B^2 = \mathbf{F}_2\{c_1^t,c_2^t\}/\mathbf{F}_2\{c_1^t+c_2^t\} \cong \mathbf{F}_2\{[c_2^t]\}$$

There is just one interesting cup product namely the square of the cohomology class represented by the 1-cocycle $\alpha = a^t + b_1^t$. The cup product $\alpha \cup \alpha$ is the 2-cocycle with values

$$\langle c_1, \alpha\cup\alpha\rangle = \langle d_2c_1,\alpha\rangle\langle d_0c_1,\alpha\rangle = \langle b_2,\alpha\rangle\langle a,\alpha\rangle = 0\cdot 1 = 0$$
$$\langle c_2, \alpha\cup\alpha\rangle = \langle d_2c_2,\alpha\rangle\langle d_0c_2,\alpha\rangle = \langle b_1,\alpha\rangle\langle a,\alpha\rangle = 1\cdot 1 = 1$$

on the basis $\{c_1,c_2\}$ for the 2-chains $\mathbf{F}_2[S_2]$. This means that that $\alpha\cup\alpha = c_2^t$ in the differential graded algebra $\mathbf{F}_2\langle S\rangle$ and that $[\alpha]\cup[\alpha] = [c_2^t]$ in the graded $\mathbf{F}_2$-algebra $H^*_\Delta(\mathbf{R}P^2;\mathbf{F}_2)$. Note that the cohomology class $[\alpha]$ and the homology class $[a]$ are the dual to each other under the UCT isomorphism $H^1(\mathbf{R}P^2;\mathbf{F}_2) \cong \mathrm{Hom}_{\mathbf{F}_2}(H_1(\mathbf{R}P^2;\mathbf{F}_2),\mathbf{F}_2)$ (4.16). We conclude that $H^*_\Delta(\mathbf{R}P^2;\mathbf{F}_2) \cong \mathbf{F}_2[\alpha]/\alpha^3$ is a truncated polynomial algebra on the nonzero class $\alpha = [a]^t$ in degree 1. We also note that cap product with the orientation class

$$[\mathbf{R}P^2]\cap -\colon H^k(\mathbf{R}P^2;\mathbf{F}_2) \to H_{2-k}(\mathbf{R}P^2;\mathbf{F}_2), \quad 0 \le k \le 2,$$

is an isomorphism in that $[\mathbf{R}P^2]\cap[\alpha]=[a]$ because $\langle[\mathbf{R}P^2]\cap[\alpha],[\alpha]\rangle=\langle[\mathbf{R}P^2],[\alpha]\cup[\alpha]\rangle=\langle c_1+c_2,c_2^t\rangle=1$. Or, alternatively, because

$$\begin{aligned} c_1\cap a^t&=\langle d_2c_1,a^t\rangle d_0c_1=0, & c_1\cap b_1^t&=\langle d_2c_1,b_1^t\rangle d_0c_1=0, & c_1\cap b_2^t&=\langle d_2c_1,b_2^t\rangle d_0c_1=0,\\ c_2\cap a^t&=\langle d_2c_2,a^t\rangle d_0c_2=0, & c_2\cap b_1^t&=\langle d_2c_2,b_1^t\rangle d_0c_2=a, & c_2\cap b_2^t&=\langle d_2c_2,b_2^t\rangle d_0c_2=a \end{aligned}$$

so that $\mathbf{R}P^2\cap\alpha=(c_1+c_2)\cap(a^t+b_1^t)=a$ according to the formulas from 4.38.

4.42. Example. (The cohomology algebra $H^*(N_2;\mathbf{F}_2)$) Using the Δ-complex structure on $N_2=\mathbf{R}P^2\#\mathbf{R}P^2$ (1.76) indicated by Figure 2 (or or see Example 2.41) we get a simplicial chain complex $\mathbf{F}_2[S]$ and a simplicial

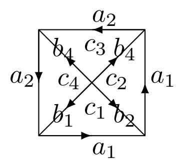


Figure 2. N_2 as a Δ-complex

cochain complex $\mathbf{F}_2\langle S\rangle$ of the form

$$0\longleftarrow \mathbf{F}_2\{x_0,x_1\}\xleftarrow{\partial_1}\mathbf{F}_2\{a_1,a_2,b_1,b_2,b_3,b_4\}\xleftarrow{\partial_2}\mathbf{F}_2\{c_1,c_2,c_3,c_4\}\longleftarrow 0$$

$$0\longrightarrow \mathbf{F}_2\{x_0^t,x_1^t\}\xrightarrow{\partial_1^t}\mathbf{F}_2\{a_1^t,a_2^t,b_1^t,b_2^t,b_3^t,b_4^t\}\xrightarrow{\partial_2^t}\mathbf{F}_2\{c_1^t,c_2^t,c_3^t,c_4^t\}\longrightarrow 0$$

where

$$\partial_1=\begin{pmatrix}0&0&1&1&1&1\\0&0&1&1&1&1\end{pmatrix}\qquad\text{and}\qquad \partial_2=\begin{pmatrix}1&1&0&0\\0&0&1&1\\1&0&0&1\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{pmatrix}$$

and ∂_1^t and ∂_2^t are the transposed matrices of ∂_1 and ∂_2. We read off the homology groups

$$\begin{aligned} H_1^{\Delta}(N_2;\mathbf{F}_2)&=\frac{\mathbf{F}_2\{a_1,a_2,b_1+b_2,b_2+b_3,b_3+b_4\}}{\mathbf{F}_2\{a_1+b_1+b_2,a_1+b_2+b_3,a_2+b_3+b_4\}}\cong\mathbf{F}_2\{[a_1],[a_2]\}\\ H_2^{\Delta}(N_2;\mathbf{F}_2)&=Z_2=\mathbf{F}_2\{[c_1+c_2+c_3+c_4]\}\end{aligned}$$

The homology class represented by the 2-cycle $N_2=c_1+c_2+c_3+c_4$ is the orientation class. We also read off the cohomology groups

$$\begin{aligned} H^1_{\Delta}(N_2;\mathbf{F}_2)&=\frac{\mathbf{F}_2\{a_1^t+b_2^t,a_2^t+b_4^t,b_1^t+b_2^t+b_3^t+b_4^t\}}{\mathbf{F}_2\{b_1^t+b_2^t+b_3^t+b_4^t\}}\cong\mathbf{F}_2\{[a_1^t+b_2^t],[a_2^t+b_4^t]\}\\ H^2_{\Delta}(N_2;\mathbf{F}_2)&=Z^2/B^2=\mathbf{F}_2\{c_1^t,c_2^t,c_3^t,c_4^t\}/\mathbf{F}_2\{c_1^t+c_2^t,c_3^t+c_4^t,c_1^t+c_4^t\}\cong\mathbf{F}_2\{[c_1^t]\}\end{aligned}$$

The 1-cocycles $\alpha_1=a_1^t+b_2^t$ and $\alpha_2=a_2^t+b_4^t$ are dual to the 1-cycles a_1 and b_1 in the sense that $a_1\cap\alpha_1=x_1$ and $b_1\cap\beta_1=x_1$. The 2-cocycle c_1^t is dual to the orientation class as $N_2\cap c_1^t=x_1$. The only interesting cup products are the products of the cohomology classes in degree 1. Using the table

σ	$d_2\sigma$	$d_0\sigma$
c_1	b_1	a_1
c_2	b_2	a_1
c_3	b_3	a_2
c_4	b_4	a_2

listing the front and back faces of the four 2-simplices of T, we find that

$$\langle c_i,\alpha_1\cup\alpha_1\rangle=\begin{cases}1 & i=2\\ 0 & i\neq 2\end{cases}\qquad\text{and}\qquad \langle c_i,\alpha_2\cup\alpha_2\rangle=\begin{cases}1 & i=4\\ 0 & i\neq 4\end{cases}$$

and therefore $\alpha_1 \cup \alpha_1 = c_2^t \sim c_1^t$ and $\alpha_2 \cup \alpha_2 = c_4^t \sim c_1^t$ in the $\mathbf{F}_2$-DGA $\mathbf{F}_2\langle S\rangle$. Similarly, $\alpha_1 \cup \alpha_2 = 0 = \alpha_2 \cup \alpha_1$. We conclude that the cup product $H^1_\Delta(N_2;\mathbf{F}_2) \times H^1_\Delta(N_2;\mathbf{F}_2) \xrightarrow{\cup} H^2_\Delta(N_2;\mathbf{F}_2)$ is a nondegenerate bilinear form with matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with respect to the basis $\{\alpha_1, \alpha_2\}$ for the 2-dimensional $\mathbf{F}_2$-vector space $H^1_\Delta(N_2;\mathbf{F}_2)$. Alternatively, we see that cap product with the orientation class

$$[N_2] \cap -\colon H^k(N_2;\mathbf{F}_2) \to H_{2-k}(N_2;\mathbf{F}_2), \quad 0 \le k \le 2,$$

is an isomorphism in that $N_2 \cap \alpha_1 = a_1$ and $N_1 \cap \alpha_2 = a_2$ because

$$\begin{aligned} \langle N_2 \cap \alpha_1, \alpha_1\rangle &= \langle N_2, \alpha_1 \cup \alpha_1\rangle = 1 & \qquad \langle N_2 \cap \alpha_2, \alpha_1\rangle &= \langle N_2, \alpha_2 \cup \alpha_1\rangle = 0 \\ \langle N_2 \cap \alpha_1, \alpha_2\rangle &= \langle N_2, \alpha_1 \cup \alpha_2\rangle = 0 & \qquad \langle N_2 \cap \alpha_2, \alpha_2\rangle &= \langle N_2, \alpha_2 \cup \alpha_2\rangle = 1 \end{aligned}$$

We have now computed the $\mathbf{F}_2$-cohomology rings for the nonorientable surfaces N_g of genus $g = 1, 2$. Can you guess what is the cohomology ring for N_g in general?

4.43. Example. (The integer cohomology ring $H^*(T;\mathbf{Z})$ of the torus)
Using the Δ-complex structure on The torus $T = M_1 = |S|$ is the realization of the Δ-set S indicated in Figure 3 (or in Example 2.40). The maps in S are $d_0(D_1, D_2, D_3, D_4) = (a_1, b_1, a_1, b_1)$, $d_1(D_1, D_2, D_3, D_4) =$

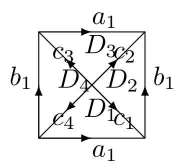


Figure 3. M_1 as a Δ-complex

(c_1, c_3, c_3, c_1), and $d_2(D_1, D_2, D_3, D_4) = (c_4, c_1, c_2, c_4)$ etc. The chain complex $\mathbf{Z}[S]$ and the cochain complex $\mathbf{Z}\langle S\rangle$ are

$$0 \longleftarrow \mathbf{Z}\{x_0, x_1\} \xleftarrow{\partial_1} \mathbf{Z}\{a_1, b_1, c_1, c_2, c_3, c_4\} \xleftarrow{\partial_2} \mathbf{Z}\{D_1, D_2, D_3, D_4\} \longleftarrow 0$$

$$0 \longrightarrow \mathbf{Z}\{x_0^t, x_1^t\} \xrightarrow{\partial_1^t} \mathbf{Z}\{a_1^t, b_1^t, c_1^t, c_2^t, c_3^t, c_4^t\} \xrightarrow{\partial_2^t} \mathbf{Z}\{D_1^t, D_2^t, D_3^t, D_4^t\} \longrightarrow 0$$

where

$$\partial_1 = \begin{pmatrix} 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 & -1 \end{pmatrix} \quad \text{and} \quad \partial_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

and ∂_1^t and ∂_2^t are the transposed matrices. We read off the homology groups

$$\begin{aligned} H_1^\Delta(T;\mathbf{Z}) &= \frac{\mathbf{Z}\{a_1, b_1, c_1 - c_4, c_2 - c_4, c_3 - c_4\}}{\mathbf{Z}\{a_1 - c_2 + c_3, b_1 - c_3 + c_4, c_1 - c_2 + c_3 - c_4\}} \cong \mathbf{Z}\{[a_1], [b_1]\} \\ H_2^\Delta(T;\mathbf{Z}) &= Z^2 = \mathbf{Z}\{[D_1 + D_2 - D_3 - D_4]\} \end{aligned}$$

The homology class represented by the 2-cycle $T = D_1 + D_2 - D_3 - D_4$ is called the *orientation class* of the manifold T. We also read off the cohomology groups

$$\begin{aligned} H^1_\Delta(T;\mathbf{Z}) &= \frac{\mathbf{Z}\{a_1^t - c_3^t - c_4^t, b_1^t + c_2^t + c_3^t, c_1^t + c_2^t + c_3^t + c_4^t\}}{\mathbf{Z}\{c_1^t + c_2^t + c_3^t + c_4^t\}} \cong \mathbf{Z}\{[a_1^t - c_3^t - c_4^t], [b_1^t + c_2^t + c_3^t]\} \\ H^2_\Delta(T;\mathbf{Z}) &= \frac{\mathbf{Z}\{D_1^t, D_2^t, D_3^t, D_4^t\}}{\mathbf{Z}\{D_1^t + D_4^t, D_2^t + D_4^t, D_3^t - D_4^t\}} \cong \mathbf{Z}\{[D_1^t]\} \end{aligned}$$

Since all homology groups are free abelian groups, evaluation $H_k(T;\mathbf{Z}) \times H_k(T;\mathbf{Z}) \to \mathbf{Z}$ is a nondegenerate pairing in this case. The 1-cocycles $\alpha_1 = a_1^t - c_3^t - c_4^t$ and $\beta_1 = b_1^t + c_2^t + c_3^t$ represent cohomology classes dual to the homology classes $[a_1], [b_1] \in H_1(T;\mathbf{Z})$ and the 2-cocycle D_1^t represents a cohomology class dual to the orientation class $[T]$.

The cap product $\mathbf{Z}[S_2] \times \mathbf{Z}\langle S_1\rangle \xrightarrow{\cap} \mathbf{Z}[S_1]$ satisfies

$$D_1 \cap c_4^t = a_1, \quad D_2 \cap c_1^t = b_1, \quad D_3 \cap c_2^t = a_1, \quad D_4 \cap c_4^t = b_1$$

while the products between all other combinations of generators are 0. For instance, $D_1 \cap c_4^t = \langle d_2 D_1, c_4^t\rangle d_0 D_1 = \langle c_4, c_4^t\rangle a_1 = a_1$. Hence

$$\begin{aligned} T \cap \alpha_1 &= (D_1 + D_2 - D_3 - D_4) \cap (a_1^t - c_3^t - c_4^t) = b_1, \\ T \cap \beta_1 &= (D_1 + D_2 - D_3 - D_4) \cap (b_1^t + c_2^t + c_3^t) = -a_1, \end{aligned}$$

and we see that $[T] \cap -\colon H^1_\Delta(T;\mathbf{Z}) \to H_1^\Delta(T;\mathbf{Z})$ is an isomorphism. The cup products are

$$[\alpha_1] \cup [\alpha_1] = 0 = [\beta_1] \cup [\beta_1], \quad [\alpha_1] \cup [\beta_1] = [D_1^t] = -[\beta_1] \cup [\alpha_1]$$

because

$$\begin{aligned} &\langle T \cap \alpha_1, \alpha_1\rangle = \langle T, \alpha_1 \cup \alpha_1\rangle = 0 && \langle T \cap \beta_1, \alpha_1\rangle = \langle T, \beta_1 \cup \alpha_1\rangle = -1 \\ &\langle T \cap \alpha_1, \beta_1\rangle = \langle T, \alpha_1 \cup \beta_1\rangle = 1 && \langle T \cap \beta_1, \beta_1\rangle = \langle T, \beta_1 \cup \beta_1\rangle = 0 \end{aligned}$$

We conclude that the cup product $H^1_\Delta(T;\mathbf{Z}) \times H^1_\Delta(T;\mathbf{Z}) \xrightarrow{\cup} H^2_\Delta(T;\mathbf{Z})$ is a nondegenerate bilinear form with matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

with respect to the basis $\{\alpha_1, \beta_1\}$ for the rank 2 free abelian group $H^1_\Delta(T;\mathbf{Z})$.

4.44. Example. The torus T and the wedge sum $X = S^1 \vee S^1 \vee S^2$ have isomorphic homology and cohomology groups (for any choice of coefficients) but they do not have the same cup product structure (nor the same fundamental group).

4.45. Example. The Moore space $M(\mathbf{Z}/m, 1) = S^1 \cup_m D^2$ has an obvious Δ-complex structure [**10**, Example 3.9]. The simplicial chain and cochain complexes with coefficients in $\mathbf{Z}/m$ are

$$0 \longleftarrow \mathbf{Z}/m\{v_0, v_1\} \xleftarrow{\partial_1} \mathbf{Z}/m\{e, e_0, \dots, e_{m-1}\} \xleftarrow{\partial_2} \mathbf{Z}/m\{T_0, \dots, T_{m-1}\} \longleftarrow 0$$

$$0 \longrightarrow \mathbf{Z}/m\{v_0^t, v_1^t\} \xrightarrow{\partial_1^t} \mathbf{Z}/m\{e^t, e_0^t, \dots, e_{m-1}^t\} \xrightarrow{\partial_2^t} \mathbf{Z}/m\{T_0^t, \dots, T_{m-1}^t\} \longrightarrow 0$$

where (for $m = 4$)

$$\partial_1 = \begin{pmatrix} 0 & -1 & -1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} \qquad \partial_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

We read off the homology

$$\begin{aligned} H_1^\Delta(M;\mathbf{Z}/m) &= \frac{\ker \partial_1}{\operatorname{im} \partial_2} \cong \mathbf{Z}/m\{[e]\} \\ H_2^\Delta(M;\mathbf{Z}/m) &= \ker \partial_2 = \mathbf{Z}/m\{[T_0 + T_1 + T_2 + T_3]\} \end{aligned}$$

and the cohomology groups

$$\begin{aligned} H^1_\Delta(M;\mathbf{Z}/m) &= \frac{\ker \partial_2^t}{\operatorname{im} \partial_1^t} = \mathbf{Z}/m\{[e^t + e_1^t + 2e_2^t + 3e_3^t]\} \\ H^2_\Delta(M;\mathbf{Z}/m) &= \frac{\mathbf{Z}/m\{T_0^t, T_1^t, T_2^t, T_3^t\}}{\mathbf{Z}/m\{T_0^t - T_3^t, T_1^t - T_3^t, T_2^t - T_3^t\}} \cong \mathbf{Z}/m\{[T_0^t]\} \end{aligned}$$

Let $\alpha = e^t + e_1^t + 2e_2^t + 3e_3^t$ be the generating cocycle in degree 1 and $\beta = T_0^t$ the generating cocycle in degree 2. From the table

σ	$\sigma\|[v_0,v_1]$	$\sigma\|[v_1,v_2]$	$\alpha\cup\alpha(\sigma)$
T_0	e_0	e	0
T_1	e_1	e	1
T_2	e_2	e	2
T_3	e_3	e	3

we conclude that $\alpha\cup\alpha = T_1^t + 2T_2^t + 3T_3^t \sim (1+2+3)T_0^t = 2\beta$ when $m=4$. In general we get that the cup product is given by

$$[\alpha]\cup[\alpha] = (1+2+\cdots+(m-1)) = \begin{cases} 0 & m \text{ is odd} \\ \frac{m}{2}[\beta] & m \text{ is even} \end{cases}$$

because the terms $k+(m-k)=m=0$ cancel.

These examples indicate that the cohomology algebra is graded commutative and that there is a duality between homology and cohomology in complementary degrees for compact manifolds.

4.46. Theorem (The cohomology ring is graded commutative). *If R is a commutative ring then $H^*(X;R)$ is graded commutative in the sense that*

$$\alpha\cup\beta = (-1)^{|\alpha||\beta|}\beta\cup\alpha$$

for homogeneous elements α and β.

Proof. The proof is surprisingly complicated. Let ϵ_n denote the sign $(-1)^{\frac{1}{2}n(n+1)}$ (the determinant of the linear map $\mathbf{R}^{n+1}\to\mathbf{R}^{n+1}\colon (x_0,\dots,x_n)\to(x_n,\dots,x_0)$). Then $\epsilon_{k+\ell}=(-1)^{k\ell}\epsilon_k\epsilon_\ell$.

Define a linear map

$$C_n(X)\xrightarrow{\rho} C_n(X),\qquad \rho(\sigma)=\epsilon_n\overline{\sigma}$$

where $\overline{\sigma}(v_i)=v_{n-i}$ is the simplex σ with vertices in the reverse order.

It turns out that ρ is a chain map chain homotopic to the identity. One considers the prism $\Delta^n\times I$, puts σ at the bottom Δ^n and $\rho(\sigma)$ at the top Δ^n and writes down a chain homotopy using the Δ-complex structure on $\Delta^n\times I$.

The dual cochain map $\rho^*\colon C^n(X)\to C^n(X)$ is also chain homotopic to the identity so that it induces the identity map on cohomology. Direct computation shows that

$$\epsilon_k\epsilon_\ell(\rho^*\phi\cup\rho^*\psi)=\epsilon_{k+\ell}\rho^*(\psi\cup\phi)$$

and this proves the theorem. □

2. Orientation of manifolds

What does it mean that a manifold is orientable?

4.47. Local homology groups. Let R be a commutative domain with unit $\varepsilon\colon\mathbf{Z}\to R$. The most important examples will be $R=\mathbf{Z}$ and $R=\mathbf{F}_2$. Let also X be a space and $A\subset X$ a subspace. The *local homology group* is

$$H_k(X|A;R)=H_k(X,X-A;R)$$

and if $A_1\subset A_2$, the *restriction* homomorphism $r_{A_1}^{A_2}\colon H_k(X|A_2;R)\to H_k(X|A_1;R)$ is the homomorphism (in the opposite direction of the inclusion) induced from the inclusion $(X,X-A_2)\subset(X,X-A_1)$. In particular, there is a restriction homomorphism $r_A^X\colon H_k(X)=H_k(X|X;R)\to H_k(X|A;R)$ for all subspaces A of X. The most important coefficients rings will be $R=\mathbf{Z}$ and $R=\mathbf{F}_2$.

The groups are said to be *local* because they only depend on a neighborhood of A.

4.48. Lemma. Let $A\subset X$ be a pair of spaces and R a commutative ring.

(1) $H_k(U|A;R)\cong H_k(X|A;R)$ if $A\subset\operatorname{cl}A\subset\operatorname{int}U\subset U$.
(2) If $f\colon X\to Y$ is injective on some open neighborhood of $\operatorname{cl}A$ then there is an induced map $f_*\colon H_k(X|A;R)\to H_k(Y|B;$ where $f(A)=B$.

PROOF. (1) Since $X - \operatorname{int} U = \operatorname{cl}(X - U) \subset \operatorname{int}(X - A) = X - \operatorname{cl} A$ we can excise $X - \operatorname{int} U$ from $(X, X - A)$.
(2) Suppose that U is some open neighborhood of $\operatorname{cl} A$ such that the restriction of f to U is injective. Then $f|U$ takes A into $B = f(A)$ and $X - U$ to $Y - B$. Define $f_*\colon H_k(X|A) \to H_k(Y|B)$ as $H_k(X|A) \xleftarrow{\cong} H_k(U|A) \xrightarrow{(f|U)_*} H_k(Y|B)$. This might depend on the choice of U. Let U_1 and U_2 be open neighborhoods of $\operatorname{cl} A$ such that the restrictions of f to U_1 and U_2 are injective. From the commutative diagram

$$\begin{tikzcd}
H_k(U_1|A) \arrow[d, "\cong"'] \arrow[rrr, bend left=10, "(f|U_1)_*"] & & & \\
H_k(X|A) & & H_k(U_1 \cap U_2|A) \arrow[ll, "\cong"'] \arrow[r, "f|(U_1 \cap U_2)_*"] & H_k(Y|B) \\
H_k(U_2|A) \arrow[u, "\cong"] \arrow[rrru, bend right=10, "(f|U_2)_*"'] & & &
\end{tikzcd}$$

we see that the definition is unambiguous. $\square$

Let now M be a topological n-manifold. Since M is locally euclidian, $H_n(M|x; R) \cong H_n(\mathbf{R}^n|x; R) \cong \widetilde{H}_{n-1}(\mathbf{R}^n - x; R) \cong \widetilde{H}_{n-1}(S^{n-1}; R) \cong R$ for all points $x \in M$.

4.49. LEMMA (Local continuation). Suppose that $x \in B \subset \mathbf{R}^n \subset M$ where $\mathbf{R}^n$, $n > 0$, is coordinate neighborhood of x and B is an open disc (ball) in that coordinate neighborhood (so that $\operatorname{cl} B \subset \mathbf{R}^n$). Then the restriction homomorphism

$$r_x^B\colon H_n(M|B; R) \to H_n(M|x; R)$$

is an isomorphism and both homology groups are free R-modules of rank one.

PROOF. By locality we can assume that $M = \mathbf{R}^n$ for $H_n(M|B) \cong H_n(\mathbf{R}^n|B)$ and $H_n(M|x) \cong H_n(\mathbf{R}^n|x)$ (4.49). In that case, since $\mathbf{R}^n$ is contractible,

$$H_n(\mathbf{R}^n|B) = H_n(\mathbf{R}^n, \mathbf{R}^n - B) \cong \widetilde{H}_{n-1}(\mathbf{R}^n - B)$$
$$H_n(\mathbf{R}^n|x) = H_n(\mathbf{R}^n, \mathbf{R}^n - x) \cong \widetilde{H}_{n-1}(\mathbf{R}^n - x)$$

where $\mathbf{R}^n - x$ contains $\mathbf{R}^n - B$ contains S^{n-1} as a deformation retract. $\square$

4.50. The orientation covering. We construct a covering space of M by placing the local homology group above each point (similar to the construction of the tangent bundle of a smooth manifold). Define the set

$$M_R = \coprod_{x \in M} H_n(M|x; R),$$

to be the disjoint union of the local homology groups and define $p\colon M_R \to M$ to be the map that sends $H_n(M|x; R) \subset M_R$ to $x \in M$. The set M_R has a local product structure that we use to define a topology on M_R. For each open disc contained in a coordinate neighborhood $B \subset \mathbf{R}^n \subset M$ there is a bijection (4.49)

$$B \times H_n(M|B; R) \xrightarrow{r} p^{-1}(B) = M_R|B = \coprod_{y \in B} H_n(M|y; R), \qquad r(y, \mu) = (y, r_y^B \mu) \tag{4.51}$$

that we declare to be a homeomorphism (where $H_n(M|B; R)$ has the discrete topology). This makes $p\colon M_R \to M$ a covering map by design. It is clear that M_R is again a manifold since any local homeomorphism $\mathbf{R}^n \hookrightarrow M$ lifts to a unique local homeomorphism

$$\begin{tikzcd}
& M_R \arrow[d, "p"] \\
\mathbf{R}^n \arrow[ur, "\widetilde{h}"] \arrow[r, hook] & M
\end{tikzcd} \tag{4.52}$$

once the value $\widetilde{h}(x)$ is specified. (Any covering space of a manifold is a manifold.) Orientability is the question of whether M_R globally is a product.

For each path homotopy class $\omega\colon x_0 \to x_1$ let $\cdot\omega\colon H_n(M|x_0;R) \to H_n(M|x_1;R)$ be the map defined by unique path lifting. This means that $\mu_0 \cdot \omega = \mu_1$ if there exist $\mu_t \in H_n(M|\omega(t);R)$ that are consistent in the sense that $\mu_t = r^B_{\omega(t)}\mu_B$ for some $\mu_B \in H_n(M|B;R)$ for all t such that $\omega(t)$ lies in the ball B.

4.53. LEMMA. $\cdot\omega\colon H_n(M|x_0;R) \to H_n(M|x_1;R)$ is an isomorphism of right R-modules.

PROOF. We have $(\mu+\nu)\cdot\omega = \mu\cdot\omega + \nu\cdot\omega$ because fibrewise addition of two lifts of ω is a lift of ω, and we have $(\mu r)\cdot\omega = (\mu\cdot\omega)r$ since the scalar multiplication of a lift of ω is another lift of ω. □

The unit $\varepsilon\colon \mathbf{Z} \to R$ induces a natural group homomorphism $\varepsilon\colon H_n(M|B;\mathbf{Z}) \to H_n(M|B;R)$ of local homology groups and hence a morphism

$$\begin{tikzcd} M_{\mathbf{Z}} \arrow[rr, "\varepsilon_*"] \arrow[dr] & & M_R \arrow[dl] \\ & M & \end{tikzcd}$$

of covering spaces. The restriction to the fibre over x, $\varepsilon_*\colon H_n(M|x;\mathbf{Z}) \to H_n(M|x;R)$ is then a morphism of right $\pi_1(M,x)$-sets.

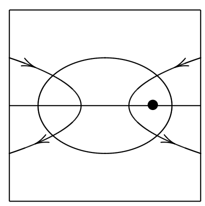


FIGURE 4. An orientation reversing loop on the Möbius band

Since $\mathrm{Aut}_{\mathbf{Z}}(H_n(M|x;\mathbf{Z})) = \{\pm 1\}$, the action of $\pi_1(M,x)$ on $H_n(M|x;\mathbf{Z})$ is given by $\nu\cdot\omega = \nu(\pm 1) = \nu\theta(\omega)$ for some group homomorphism $\theta\colon \pi_1(M,x) \to \{\pm 1\} = \mathbf{Z}^\times$. As the group homomorphism $\varepsilon_*\colon H_n(M|x;\mathbf{Z}) \to H_n(M|x;R)$ is a morphism of right $\pi_1(M,x)$-sets, we also have that $\nu\cdot\omega = \nu\theta(\omega)$ for all $\nu \in H_n(M|x;R)$. (The action of ω on $H_n(M|x;R)$ is multiplication by some unit of R so the action is known when it is known what it does to just one element, for instance an element coming form $H_n(M|x;\mathbf{Z})$.) Recall from covering space theory that if M is connected the covering map $M_R \to M$ is completely determined by the right $\pi_1(M,x)$-module $H_n(M|x;R)$.

4.54. Local and global orientations. An element μ of a right R-module H is a *generator* of H if the submodule μR is all of H. If H is free on the generator μ, then any element of H is of the form μr for a unique $r \in R$ and then $R \to \mathrm{End}_R(H)\colon r \to (h \to hr)$ is a ring isomorphism. In particular, $R^\times \cong \mathrm{Aut}_R(H)$.

4.55. DEFINITION. A local R-orientation at $x \in M$ is a generator μ_x for $H_n(M|x;R)$. An R-orientation for M is a section $\mu\colon M \to M_R$ such that $\mu(x)$ is a local R-orientation at all points $x \in M$. A manifold is R-orientable if it has an R-orientation.

By the definition of the topology on M_R this means that an R-orientation for M is a function $x \to \mu_x$ that assigns a local R-orientation to each point $x \in M$ such that for all open discs $B \subset \mathbf{R}^n \subset M$ there is a generator $\mu_B \in H_n(M|B;R)$ such that $r^B_x(\mu_B) = \mu_x$ for all points $x \in B$. (We say that the choice of local orientations is consistent.) When the ring is not specified it is understood that $R = \mathbf{Z}$: "M is orientable" means "M is $\mathbf{Z}$-orientable".

Can you compute the monodromy action for $M = \mathrm{MB}$, the (open) Möbius band (imagine a person walking head up along the core circle of the band), and $M = \mathbf{R}P^2$?

4.56. PROPOSITION. Assume that the manifold M is connected. Then the following are equivalent:

(1) M is R-orientable
(2) The monodromy action $\pi_1(M,x) \xrightarrow{\theta} \{\pm 1\} \to R^\times$ is the trivial homomorphism
(3) The orientation covering M_R is isomorphic to the trivial covering $M \times R \to M$

If the fundamental group $\pi_1(M)$ contains no index 2 subgroups (eg if M is simply connected) then M is orientable. If M is orientable then it has exactly two orientations. If M is orientable then M is R-orientable for all R, and if M is nonorientable then M is R orientable iff $-1 = 1$ in R. All connected manifolds are $\mathbf{F}_2$-orientable. Any open submanifold of an oriented manifold is oriented.

PROOF. If $\mu\colon M \to M_R$ is an orientation for M then the map $M \times R \to M_R$ given by $(x, r) \to \mu(x)r$ is a trivialization. This explains (1) $\iff$ (3). Let $\Gamma(M_R \to M)$ denote the R-module of sections of the covering map $M_R \to M$. According to Classification of covering maps evaluation at x is a bijection (and an R-module homomorphism)

$$\Gamma(M_R \to M) \xrightarrow{\cong} H_n(M|x; R)^{\pi_1(M,x)}$$

Using this we see that

$$\begin{aligned} M \text{ is } R\text{-orientable} \iff M_R = M \times R \to M \iff \\ H_n(M|x;R)^{\pi_1(M,x)} = \Gamma(M_R \to M) = H_n(M|x;R) \iff \\ \pi_1(M,x) \text{ acts trivially on } H_n(M|x;R) \iff \pi_1(M,x) \xrightarrow{\theta} \{\pm 1\} \to R^\times \text{ is trivial} \end{aligned}$$

If the fundamental group $\pi_1(M)$ contains no index 2 subgroups then θ is trivial since otherwise the kernel would be an index 2-subgroup. If $U \subset M$ is an open submanifold then $U_R = M_R|U$ is trivial if M_R is trivial. □

For instance, any surface that contains a Möbius band is nonorientable.

4.57. THEOREM (Orientations along compact subspaces). *Let M be a an n-manifold and $A \subset M$ a compact subspace. Then*

(1) $H_{>n}(M|A; R) = 0$
(2) *For any section $\alpha\colon M \to M_R$ there exists*

$$\begin{tikzcd} & M_R \arrow[d, shift right] \\ A \arrow[ur, dashed, "\alpha|A"] \arrow[r, hook] & M \arrow[u, shift right, bend right, "\alpha"'] \end{tikzcd}$$

a unique class $\alpha_A \in H_n(M|A; R)$ such that $(\alpha|A)(x) = r^A_x \alpha_A$ for all $x \in A$.

The proof of this important theorem is divided into several steps.

4.58. LEMMA. If Theorem 4.57 is true for the compact subsets A, B and $A \cap B$ of M then it also true for $A \cup B$.

PROOF. Use the relative Mayer–Vietoris sequence [10, p 237]

$$\begin{aligned} \cdots \to H_{i+1}(M|A\cap B) \to \\ H_i(M|A \cup B) \xrightarrow{(r^{A\cup B}_A, r^{A\cup B}_B)} H_i(M|A) \oplus H_i(M|B) \xrightarrow{r^A_{A\cap B} - r^B_{A\cap B}} H_i(M|A \cap B) \to \cdots \end{aligned}$$

for the pairs $(M, M - A)$ and $(M, M - B)$ where $(M, (M - A) \cap (M - B)) = (M, M - (A \cup B))$ and $(M, (M - A) \cup (M - B)) = (M, M - (A \cap B))$. □

4.59. LEMMA. Theorem 4.57 is true when $M = \mathbf{R}^n$.

PROOF. To get the existence part (2) place A inside a (large) ball B. By continuity of the section α of the covering $\mathbf{R}^n_R \to \mathbf{R}^n$, there is a class $\alpha_B \in H_n(\mathbf{R}^n|B)$ such that $r^B_x \alpha_B = \alpha_x$ for all $x \in B$. Let $\alpha_A = r^B_A \alpha_A$ be the restriction of α_B to A.

To prove the rest of the theorem, assume first that $A \subset \mathbf{R}^n$ is compact and convex. Let x be any point of A. Since both $\mathbf{R}^n - A$ and $\mathbf{R}^n - x$ deformation retracts onto a (large) sphere centered at x, restriction $H_i(\mathbf{R}^n|A) \to H_i(\mathbf{R}^n|x)$ is an isomorphism just as in 4.49. This implies (1) and the uniqueness part of (2). By 4.58 and induction, the theorem is true for any finite union of compact convex subsets.

Finally, let $A \subset \mathbf{R}^n$ be an arbitrary compact subset. Consider a local homology class $\zeta_A = [z] \in H_i(\mathbf{R}^n|A) = H_i(\mathbf{R}^n, \mathbf{R}^n - A)$ represented by a relative cycle $z \in \mathbf{C}_n(X)$ with $\partial z \in C_{i-1}(\mathbf{R}^n - A)$. The support $|\partial z|$ of ∂z is a compact set disjoint from the compact set A. There is a compact set B such that

B is a finite union of closed balls centered at points in A, B contains A, and B is disjoint from $|\partial z|$. Let $\zeta_B \in H_i(\mathbf{R}^n|B)$ be the homology class represented by the relative cycle z. Since ζ_B restricts to ζ_A and the theorem is true for B it is also true for A: If $i > n$, $\zeta_B = 0$ so also $\zeta_A = 0$. Assume that $i = n$ and that ζ_A restricts to 0 at all $x \in A$. Let y be any point in B. Then y lies in a closed ball centered at a point $x \in A$. Let S be the boundary sphere of the closed ball containing x and y. The commutative diagram

$$\begin{tikzcd}
H_i(\mathbf{R}^n|B) \arrow[r, "r_y^B"] \arrow[dd] & H_i(\mathbf{R}^n|y) \arrow[r, "\partial"] & \widetilde{H}_{i-1}(\mathbf{R}^n - y) \\
& & \widetilde{H}_{i-1}(S) \arrow[u, "\cong"'] \arrow[d, "\cong"] \\
H_i(\mathbf{R}^n|A) \arrow[r, "r_x^A"] & H_i(\mathbf{R}^n|x) \arrow[r, "\partial"] & \widetilde{H}_{i-1}(\mathbf{R}^n - x)
\end{tikzcd}$$

shows that ζ_B restrict to 0 at y. Thus $\zeta_B = 0$ and then also $\zeta_A = 0$. $\square$

Proof of Theorem 4.57. By 4.59, the theorem is true for any compact set $A \subset M$ contained in a coordinate neighborhood. By 4.58 it is also true for any finite union of such sets. But any compact subset of M has this form. $\square$

4.60. Corollary. Let M be a connected compact n-manifold and x a point in M. Then

(1) $H_{>n}(M;R) = 0$

(2) Restriction is an isomorphism

$$H_n(M;R) \xrightarrow[\cong]{r_x^M} H_n(M|x;R)^{\pi_1(M,x)} \cong \begin{cases} R & M \text{ is } R\text{-orientable} \\ {}_2R & M \text{ is not } R\text{-orientable} \end{cases}$$

Proof. (1) Take $K = M$ in 4.57 and remember that $H_k(M|M;R) = H_k(M;R)$.
(2) Recall that $\Gamma(M_R \to M)$ stands for the module of sections of the covering $M_R \to M$. Consider the homomorphisms

$$H_n(M;R) \to \Gamma(M_R \to M) \to H_n(M|x;R)^{\pi_1(M,x)}$$

where the first map takes $\alpha \in H_n(M;R)$ to the section $y \to r_y^M \alpha$ and the second map is evaluation at $x \in M$. The composition of these two maps is r_x^M. According to Theorem 4.57 with $K = M$ the first map is bijective. We already noted that also the second map is bijective by covering space theory. If M is R-orientable, $\pi_1(M,x)$ acts trivially; if not, $H_n(M|x;R)^{\pi_1(M,x)} = H_n(M|x;R)^{\{\pm1\}} = R^{\{\pm1\}} = \{r \in R \mid 2r = 0\}$. $\square$

The corollary says that in a compact manifold a local orientation extends to a global orientation if and only if it is invariant under all loops. Any such invariant local orientation $\mu_x \in H_n(M|x;R)$ extends uniquely to a global R-orientation class $[M] = \mu_M \in H_n(M;R)$ with $r_x^M[M] = \mu_x$.

For a noncompact R-oriented manifold there is not a single R-orientation class but rather a system of R-orientation classes $\mu_K \in H_n(M|K;R)$ along the compact subsets of M agreeing under restriction homomorphisms.

4.61. The oriented cover of a nonorientable manifold. All the nonorientable manifolds that we know have the form $M = \widetilde{M}/\{\pm1\}$ for some orientable manifold $\widetilde{M}$. This is no coincidence: All nonorientable manifolds have this form!

Let M be any manifold and let $\widetilde{M} \subset M_{\mathbf{Z}}$ be the double covering space of M consisting of the two generators in each fibre, $H_n(M|x;\mathbf{Z})$, of $M_{\mathbf{Z}} \to M$.

4.62. Proposition. The manifold $\widetilde{M}$ is orientable. If M is connected and orientable, $\widetilde{M} \to M$ is the trivial double covering space. If M is connected and nonorientable, $\widetilde{M} \to M$ is the unique double covering space with connected and orientable total space.

Proof. Let $\mu_x \in H_n(M|x;\mathbf{Z})$ be a generator. Using the isomorphism $H_n(\widetilde{M}|\mu_x) \to H_n(M|x)$ (4.48) induced by the covering map $\widetilde{M} \to M$ we get a trivialization

$$\widetilde{M} \times \mathbf{Z} \to \widetilde{M}_{\mathbf{Z}} \colon (\mu_x, z) \to \mu_x z \in H_n(M|x) \cong H_n(\widetilde{M}|\mu_x)$$

and therefore the manifold $\widetilde{M}$ is orientable (4.56).

If M connected and orientable, $M_{\mathbf{Z}} = M \times \mathbf{Z} \to M$ is the trivial covering space so $\widetilde{M} = M \times \{\pm 1\} \subset M_{\mathbf{Z}}$ is also trivial. If M connected and nonorientable, $\widetilde{M}$ is connected since the action of $\pi_1(M, x)$ on the generator set of $H_n(M|x; \mathbf{Z})$ is transitive and we know from covering space theory that the set of components of $\widetilde{M}$ is the set of orbits for the $\pi_1(M, x)$ action on the fibre. Suppose that $\widetilde{M} \to M$ is an orientable connected double cover. Then $\pi_1(\widetilde{M}) \subset \ker\theta \subset \pi_1(M)$ so that in fact $\pi_1(\widetilde{M}) = \ker\theta$ since both subgroups have index two. □

4.63. Example. The orientable surface M_g is orientable because $H_2(M_g; \mathbf{Z}) \cong \mathbf{Z}$ and the nonorientable surface N_g is nonorientable because $H_2(M_g; \mathbf{Z}) = 0$ (1.74). The orientation cover of the nonorientable surface N_{g+1} is M_g (because $2\chi(N_{g+1}) = 2(2 - (g + 1)) = 2 - 2g = \chi(M_g)$.

The orientation cover $M_g \to N_{g+1}$ arises just as $\mathbf{R}P^2$ comes from S^2: Embed M_g symmetrically around $(0, 0, 0)$ in $\mathbf{R}^3$ such that there is an antipodal ± 1-action on $M_g \subset \mathbf{R}^3$. The quotient space is a surface; it contains a Möbius band so it is nonorientable, and it has Euler characteristic $\frac{1}{2}\chi(M_g) = \chi(N_{g+1})$ so it must be $M_g/\{\pm 1\} = N_{g+1}$.

The antipodal map of S^n, $x \to -x$, is orientation preserving iff n is odd so that

$$\{\pm 1\}\backslash S^n = \mathbf{R}P^n \text{ is orientable } \iff n \text{ is odd}$$

The homeomorphism $(x, t) \to (-x, -t)$ of $S^n \times \mathbf{R}$ is orientation preserving iff n is even so that

$$\{\pm 1\}\backslash (S^n \times \mathbf{R}) \text{ is orientable } \iff n \text{ is even}$$

For $n = 1$ we obtain the nonorientable Möbius band and for odd $n > 1$ we obtain a higher dimensional analogs (aka the tautological line bundle over $\mathbf{R}P^n$). For compact versions one could replace $\mathbf{R}$ by S^1 (or S^m) and obtain higher dimensional anlogs of the Klein bottle.

The covering maps $S^{2n} \to S^{2n}/\{\pm 1\} = \mathbf{R}P^{2n}$ and of $S^{2n+1} \times \mathbf{R} \to \{\pm 1\}\backslash (S^{2n+1} \times \mathbf{R})$ are the orientation coverings.

4.64. Exercise. Show that any local homeomorphism $f\colon M \to N$ between manifolds lifts to a map of covering spaces $f_*\colon M_{\mathbf{Z}} \to N_{\mathbf{Z}}$ so that $f^* M_{\mathbf{Z}} = N_{\mathbf{Z}}$. Apply this to the double covering map $\widetilde{M} \to M$ and show again that $\widetilde{M}$ is orientable.

3. Poincaré duality for compact manifolds

We first state the Poincaré duality theorem for closed (compact with no boundary) manifolds. We later state and prove Poincaré duality for not necessarily compact manifolds.

4.65. Theorem. *Let M be a compact, connected R-oriented n-dimensional manifold. Cap product with the orientation class $[M] \in H_n(M; R)$*

$$\mathrm{PD}\colon H^k(M; R) \to H_{n-k}(M; R), \qquad \mathrm{PD}(\alpha) = [M] \cap \alpha$$

is an isomorphism.

4.66. Theorem. *The cohomology algebras for finite dimensional projective spaces are*

$$H^*(\mathbf{R}P^n; \mathbf{F}_2) \cong \mathbf{F}_2[\alpha]/(\alpha^{n+1}), \qquad |\alpha| = 1$$
$$H^*(\mathbf{C}P^n; \mathbf{Z}) \cong \mathbf{Z}[\alpha]/(\alpha^{n+1}), \qquad |\alpha| = 2$$
$$H^*(\mathbf{H}P^n; \mathbf{Z}) \cong \mathbf{Z}[\alpha]/(\alpha^{n+1}), \qquad |\alpha| = 4$$

The cohomology of the infinite projective spaces are the corresponding polynomial algebas.

Proof. The manifolds $\mathbf{C}P^n$ and $\mathbf{H}P^n$ (1.79) are orientable (they are simply connected) and the manifold $\mathbf{R}P^n$ (1.77) is $\mathbf{F}_2$-orientable. We shall here take the case of $\mathbf{C}P^n$ (the two other cases are similar). Let $\alpha \in H^2(\mathbf{C}P^n; \mathbf{Z}) \cong \mathbf{Z}$ be a generator. The claim is that α^i generates $H^{2i}(\mathbf{C}P^n; \mathbf{Z})$ when $1 \le i \le n$. It is enough to prove that α^n generates $H^{2n}(\mathbf{C}P^n; \mathbf{Z})$ (for if α^i is divisible by some natural number $k > 1$ then α^n is also divisible by k). This is certainly true when $n = 1$. Assume, inductively, that the claim holds for $\mathbf{C}P^{n-1}$. We evaluate α^n on the orientation class $[\mathbf{C}P^n]$ and find that

$$\langle [\mathbf{C}P^n], \alpha^n \rangle = \langle [\mathbf{C}P^n], \alpha \cup \alpha^{n-1} \rangle = \langle [\mathbf{C}P^n] \cap \alpha, \alpha^{n-1} \rangle = \pm 1$$

for, by Poincaré duality (4.65), $[\mathbf{C}P^n]\cap\alpha$ generates $H_{2n-2}(\mathbf{C}P^n)$ and by induction hypothesis α^{n-1} generates $H^{2n-2}(\mathbf{C}P^{n-1}) = H^{2n-2}(\mathbf{C}P^n) = \mathrm{Hom}(H_{2n-2}(\mathbf{C}P^n), \mathbf{Z})$. But then α^n must be a generator of $H^{2n}(\mathbf{C}P^n) = \mathrm{Hom}(H_{2n}(\mathbf{C}P^n), \mathbf{Z})$. □

4.67. Theorem. *The cohomology algebra for the lens space $L^{2n+1}(m)$ (1.81) with coefficients in the ring $\mathbf{Z}/m$ is*

$$H^*(L^{2n+1}(m); \mathbf{Z}/m) = \begin{cases} \mathbf{Z}/m[\alpha,\beta]/(\alpha^2, \beta^{n+1}) & m \text{ is odd} \\ \mathbf{Z}/m[\alpha,\beta]/(\alpha^2 - \frac{m}{2}\beta, \beta^{n+1}) & m \text{ is even} \end{cases}$$

where the generators have degrees $|\alpha| = 1$ and $|\beta| = 2$.

Proof. The cellular cochain complex with $\mathbf{Z}/m$-coefficients (4.34, 1.81) for the lense space tells us that $H^i(L^{2n+1}(m); \mathbf{Z}/m) = \mathbf{Z}/m$ for $0 \le i \le m$. Let $\alpha \in H^1(L^{2n+1}(m); \mathbf{Z}/m)$ be 1-dimensional and $\beta \in H^2(L^{2n+1}(m); \mathbf{Z}/m)$ a 2-dimensional generator. From simplicial computations (4.45) in the 2-skeleton $M(\mathbf{Z}/m, 1) = S^1 \cup_m D^2$ of $L^{2n+1}(m)$ we know that we may choose β such that $\alpha^2 = \frac{m}{2}\beta$ when m is even. When m is odd $\alpha^2 = 0$ by graded commutativity (4.46). If course, $\beta^{n+1} = 0$ for dimensional reasons. We claim that $\beta^i\alpha^j$ generates $H^{2i+j}(L^{2n+1}(m); \mathbf{Z}/m)$ when $2i+j \le 2n+1$ and $j = 0, 1$. It is enough to show that $\beta^n\alpha$ generates $H^{2n+1}(L^{2n+1}(m); \mathbf{Z}/m)$. The lense space $L^{2n+1}(m)$ is orientable (4.60), hence $\mathbf{Z}/m$-orientable (4.56), as $H_{2n+1}(L^{2n+1}(m)) = \mathbf{Z}$ (1.81). We evaluate this cohomology class on the orientation class and find that

$$\langle [L], \beta^n\alpha\rangle = \langle [L]\cap\beta, \beta^{n-1}\alpha\rangle$$

is a unit in $\mathbf{Z}/m$: By Poincaré duality (4.65), $[L]\cap\beta$ generates $H_{2n-1}(L^{2n+1}(m); \mathbf{Z}/m)$, and by induction hypothesis $\alpha\beta^{n-1}$ generates $H^{2n-1}(L^{2n-1}(m); \mathbf{Z}/m) \cong H^{2n-1}(L^{2n+1}(m); \mathbf{Z}/m)$ which is isomorphic to the dual group $\mathrm{Hom}(H_{2n-1}(L^{2n+1}(m); \mathbf{Z}/m), \mathbf{Z}/m)$. But then $\beta^n\alpha$ must generate $H^{2n+1}(L^{2n+1}(m); \mathbf{Z}/m)$. □

When p is a prime number

$$H^*(L^\infty(p); \mathbf{F}_p) = \begin{cases} \mathbf{F}_2[\alpha] & p = 2 \\ E(\alpha)\otimes\mathbf{F}_p[\beta] & p > 2 \end{cases}$$

where $|\alpha| = 1$ and $|\beta| = 2$.

4.68. Connection with cup product. When M is a compact, connected, R-oriented n-manifold there is a commutative diagram

$$\begin{tikzcd} H^i(M;R)\times H^{n-i}(M;R) \arrow[r, "\cup"] \arrow[d, "\mathrm{PD}\times\mathrm{id}"', "\cong"] & H^n(M;R) \arrow[d, "\cong"', "\mathrm{PD}"] \\ H_{n-i}(M;R)\times H^{n-i}(M;R) \arrow[r, "\cap"] & H_0(M;R) \end{tikzcd}$$

which says that Poincaré duality translates the cup product of two classes, $\alpha \in H^i(M;R)$ and $\beta \in H^{n-i}(M;R)$, in complementary dimensions, into the evaluation homomorphism

$$H_{n-i}(M;R)\times H^{n-i}(M;R) \xrightarrow{\cap} H_0(X;R) \xrightarrow[\cong]{\varepsilon} R$$

of the bottom line. So when this evaluation pairing is a duality pairing, eg if R is a field, also cup product is an evaluation pairing. The reason is simply that

$$\mathrm{PD}(\alpha\cup\beta) = [M]\cap(\alpha\cup\beta) = ([M]\cap\alpha)\cap\beta = \mathrm{PD}(\alpha)\cap\beta = \langle\mathrm{PD}(\alpha), \beta\rangle$$

because $H_*(M;R)$ is a right $H^*(M;R)$-module (4.29).

The *signature* of a compact, connected, R-oriented manifold of dimension $4k$ is the signature of the symmetric bilinear cup product form on $H^{2k}(M;\mathbf{R})$.

4. Colimits of modules

4.69. DEFINITION. A (right) *directed set* is a partially ordered nonempty set J with the property that for any pair of elements $i, j \in J$ there is a third element $k \in J$ such that both $i \leq k$ and $j \leq k$. A *directed system* of R-modules over the directed set J is a functor M from J into the category of R-modules.

The colimit of a directed system M of R-modules is an R-module L with R-module homomorphisms $f_i \colon M_i \to L$ such that

$$\begin{tikzcd} M_i \arrow[rr, "M_{i<j}"] \arrow[dr, "f_i"'] & & M_j \arrow[dl, "f_j"] \\ & L & \end{tikzcd}$$

commutes and such that for any other R-module A with homomorphisms $a_i \colon M_i \to A$ such that $a_j M_{i<j} = a_i$ there is a unique R-module homomorphism $L \to A$ such that

(4.70)
$$\begin{tikzcd} M_i \arrow[rr, "M_{i<j}"] \arrow[dr, "f_i"'] \arrow[ddr, bend right, "a_i"'] & & M_j \arrow[dl, "f_j"] \arrow[ddl, bend left, "a_j"] \\ & L \arrow[d, dotted] & \\ & A & \end{tikzcd}$$

commutes.

The colimit of the directed system M is the universal example of a constant system with a map from M. In particular, the colimit of the constant functor is the value of the functor.

4.71. THEOREM (Existence and uniqueness of colim). *Any directed system M of R-modules has a colimit, unique up to isomorphism.*

PROOF. Let L be the quotient of $\bigoplus_{i \in I} M_i$ by the the submodule generated by all elements of the form $x_i - M_{i<j}x_i$ for $i \leq j$ and $x_i \in M_i$. One can verify that this works. □

The colimit of the system M is denoted $\operatorname{colim} M$. From the above explicit construction of the colimit we get:

4.72. COROLLARY (Recognition principle). The map $\operatorname{colim} M \to A$ arising from the universal property (4.70) is an isomorphism if and only if

(1) Every $a \in A$ we can find $i \in I$ and $x_i \in M_i$ such that $a = f_i x_i$
(2) If $a_i x_i = 0$ for some $i \in I$ and $x_i \in M_i$ then $M_{i<j}x_i = 0$ for some $j > i$.

4.73. THEOREM (colim is an exact functor from the category of J-directed R-modules to R-modules). [2] *If A, B, and C are directed systems of R-modules and $A \to B \to C$ are morphisms of J-directed systems such that $0 \to A_j \to B_j \to C_j \to 0$ is exact for each $j \in J$ then the limit sequence $0 \to \operatorname{colim} A_j \to \operatorname{colim} B_j \to \operatorname{colim} C_j \to 0$ is also exact.*

PROOF. Verify the theorem using the explicit defintion of the colimit. □

4.74. EXAMPLE (Colimits over $\mathbf{N}$). If the directed set is the natural numbers $\mathbf{N}$, a directed set is a sequence of R-modules $M_1 \to M_2 \to \cdots \to M_n \to M_{n+1} \to \cdots$. If the maps are inclusions, then $\operatorname{colim}_{\mathbf{N}} M_n = \bigcup_{n \in \mathbf{N}} M_n$. If $M_n = \mathbf{Z} \xrightarrow{\cdot p} \mathbf{Z} = M_{n+1}$ for all n, then $\operatorname{colim}_{n \in \mathbf{N}} M_n = \mathbf{Z}[1/p]$ with maps $f_n \colon \mathbf{Z} \to \mathbf{Z}[1/p]$ given by $f_n(1) = p^{-n}$.

The commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{Z} \arrow[r, "p^n"] \arrow[d, equal] & \mathbf{Z} \arrow[r] \arrow[d, "p"] & \mathbf{Z}/p^n \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \mathbf{Z} \arrow[r, "p^{n+1}"] & \mathbf{Z} \arrow[r] & \mathbf{Z}/p^{n+1} \arrow[r] & 0 \end{tikzcd}$$

shows a short exact sequences of direct systems of $\mathbf{N}$. Let $\mathbf{Z}/p^\infty = \bigcup \mathbf{Z}/p^n$ denote the limit of the third system of inclusions. The short exact sequence

$$0 \to \mathbf{Z} \to \mathbf{Z}[1/p] \to \mathbf{Z}/p^\infty \to 0$$

shows that $\mathbf{Z}/p^\infty = \mathbf{Z}[1/p]/\mathbf{Z}$.

If $e\colon M \to M$ is an idempotent, $e \circ e = e$, then the colimit over $\mathbf{N}$ of the system $M \xrightarrow{e} M \xrightarrow{e} M \to \cdots$ is eM. This follows directly from the definition or from the short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & eM \arrow[r] \arrow[d, "1"'] & M \arrow[r] \arrow[d, "e"'] & M/eM \arrow[r] \arrow[d, "0"'] & 0 \\
0 \arrow[r] & eM \arrow[r] & M \arrow[r] & M/eM \arrow[r] & 0
\end{tikzcd}$$

of directed systems since colim is exact.

If I is a sub-directed set of J then the universal property for the colimit gives a map $\operatorname{colim}_I M|I \to \operatorname{colim}_J M$ from the colimit of the small set to the colimit over the large set. More generally, if $g\colon I \to J$ is a map of directed sets and M is a J-directed system of R-modules, g^*M, with $(g^*M)_i = M_{g(i)}$ is an I-directed system of R-modules and the universal property produces a map $\operatorname{colim}_I g^*M \to \operatorname{colim}_J M$.

4.75. Corollary. The map $\operatorname{colim}_I g^*M \to \operatorname{colim}_J M$ is an isomorphism of R-modules if the map g is *cofinal* in the sense that for every $j \in J$ there is an $i \in I$ such that $j \leq g(i)$.

In particular, if $I \subset J$ then $\operatorname{colim}_I M|I \xrightarrow{\cong} \operatorname{colim}_J M$ when I is cofinal in J. In the extreme case, if I has a largest element m (with $i \leq m$ for all $i \in I$), then $M_m \to \operatorname{colim}_I M$ is an isomorphism.

4.76. Lemma (Iterated colimits). Let $J = \bigcup_{I \in \mathcal{I}} I$ be a directed set that is the union of a collection $\mathcal{I} = \{I\}$, directed under inclusion, of sub-directed sets $I \subset J$. Then the map

$$\operatorname{colim}_{\mathcal{I}} \operatorname{colim}_I M|I \to \operatorname{colim}_J M$$

is an isomorphism for any J-directed system M of R-modules.

Proof. If I_1 and I_2 belong to the collection $\mathcal{I}$ and $I_1 \subset I_1 \subset J$ by the universal property for colimits (4.70) there are maps

$$\begin{tikzcd}
\operatorname{colim}_{I_1} M|I_1 \arrow[rr] \arrow[dr] & & \operatorname{colim}_{I_2} M|I_2 \arrow[dl] \\
& \operatorname{colim}_J M &
\end{tikzcd}$$

that induce the map of the lemma by the universal property again. This corresponds to the isomorphism $\bigoplus_{I \in \mathcal{I}} \bigoplus_{i \in I} M_i = \bigoplus_{j \in J} M_j$. The map is clearly surjective. Suppose that $x_i \in M_i$ where $i \in I \subset J$ and that $x_i = 0$ in $\operatorname{colim}_J M$. Then we can find $j \geq i$ such that $j \in I_2 \supset I_1$ and $M_{i<j} x_i = 0$. This means that the image of x_i in $\operatorname{colim} M|I_2$ is zero. This shows that the map is an isomorphism (4.72). $\square$

4.77. Colimits of chain complexes. A J-directed system of chain complexes is a functor from the directed set J to the category of chain complexes. If C^j, $j \in J$, is a directed system of chain complexes of R-modules, then $\operatorname{colim} C^j$ denotes the chain complex which in degree q is $(\operatorname{colim} C^j)_q = \operatorname{colim}(C^j_q)$.

4.78. Corollary (Homology commutes with direct limits of directed systems of chain complexes). If C^j is a directed system of chain complexes then the map $\operatorname{colim} H_q(C^j) \xrightarrow{\cong} H_q(\operatorname{colim} C^j)$, induced from $C^j \to \operatorname{colim} C^j$, is an isomorphism.

PROOF. Apply the exact functor colim to the commuative diagram of directed systems of R-modules

$$\begin{tikzcd}
& 0 \arrow[d] & & & \\
& B_q^j \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & Z_q^j \arrow[r] \arrow[d] & C_q^j \arrow[r] \arrow[dr, "\partial_i"'] & B_{q-1}^j \arrow[r] \arrow[d] & 0 \\
& H_q(C^j) \arrow[d] & & C_{q-1}^j & \\
& 0 & & &
\end{tikzcd}$$

with exact row and columns and obtain the commutative diagram of R-modules

$$\begin{tikzcd}
& 0 \arrow[d] & & & \\
& \operatorname{colim} B_q^j \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{colim} Z_q^j \arrow[r] \arrow[d] & \operatorname{colim} C_q^j \arrow[r] \arrow[dr, "\operatorname{colim} \partial^j"'] & \operatorname{colim} B^j[q-1] \arrow[r] \arrow[d] & 0 \\
& \operatorname{colim} H_q(C^j) \arrow[d] & & \operatorname{colim} C_{q-1}^j & \\
& 0 & & &
\end{tikzcd}$$

with exact row and columns. We read off that the kernel and the image of the boundary map $\operatorname{colim} \partial_i$ of the chain complex $\operatorname{colim} C^j$ are $\operatorname{colim} Z_i$ and $\operatorname{colim} B_i$ so that

$$H_q(\operatorname{colim} C^j) = \frac{\operatorname{colim} Z_q^j}{\operatorname{colim} B_q^j} = \operatorname{colim} H_q(C^j)$$

as claimed. □

Here is an immediate application to topology:

4.79. COROLLARY. Let X be a topological space that is the union of a collection $\mathcal{A}$ of subspaces of X. Assume that $(\mathcal{A}, \subset)$ is a directed set and that any compact subset of X is contained in a member of the collection. Then the map $\operatorname{colim}_{A \in \mathcal{A}} H_q(A) \xrightarrow{\cong} H_q(X)$ is an isomorphism.

PROOF. The assumption on compact subsets, applied to supports of singular chains in X, and the recognition principle (4.72), show that $\operatorname{colim}_{A \in \mathcal{A}} C_q(A) \to C_q(X)$ is an isomorphism. This isomorphism survives to homology by 4.78. □

4.80. EXAMPLE (Homology of mapping telescopes). The *mapping telescope* of a sequence

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \cdots X_{i-1} \xrightarrow{f_i} X_i \to \cdots$$

of maps between spaces is the union $T = \bigcup_{i=1}^{\infty} M_{f_i}$ of the mapping cylinders. The telescope is the union of its subspaces $T_1 \subset T_2 \subset \cdots T_j \subset T_{j+1} \subset \cdots$ where $T_j = \bigcup_{j=1}^{i} M_{f_i}$ is the union of the first j mapping cylinders. T_j contains X_j as a deformation retract. The homotopy commutative diagram

$$\begin{tikzcd} T_j \arrow[r, hook] & T_{j+1} \\ X_j \arrow[u, hook] \arrow[r, "f_{j+1}"] & X_{j+1} \arrow[u, hook] \end{tikzcd}$$

illustrates that the inclusion $T_j \subset T_{j+1}$ turns the map $f_{j+1}\colon X_j \to X_{j+1}$ into an inclusion. According to 4.79 there is an isomorphism

$$\operatorname{colim}_{i\in\mathbf{N}} H_*(X_i) \xrightarrow{\cong} \operatorname{colim}_{i\in\mathbf{N}} H_*(T_i) \xrightarrow{\cong} H_*(T)$$

on homology.

For any self-map $f\colon X \to X$ we write $\mathrm{Tel}(f)$ for the mapping telescope of $X \xrightarrow{f} X \xrightarrow{f} X \to \cdots$. For instance $M(\mathbf{Z}[1/p], n) = \mathrm{Tel}(p)$ is the telescope for the degree p-map $S^n \xrightarrow{p} S^n$ of the n-sphere (4.74). Try to visualize $M(\mathbf{Z}[1/2], 1)$.

If $e\colon X \to X$ induces an idempotent on homology, then $H_*(\mathrm{Tel}(e)) = e_* H_*(X)$ (4.74).

4.81. Cohomology with compact support. Let M be a manifold and R a ring. As a notational convention we write $H^q(M|A;R)$ for $H^q(M, M-A;R)$, just as we did for homology (4.47). When $A_1 \subset A_2$, there is an R-module extension homomorphism $e^{A_1}_{A_2}\colon H^q(M|A_1) \to H^q(M|A_2)$ (in the same direction as the inclusion) induced from the inclusion $(M, M-A_2) \subset (M, M-A_1)$.

Let $\mathcal{K}_M$ be the directed set of compact subspaces of M ordered by inclusion. *Cohomology with compact support* of M is the colimit

$$H^q_c(M;R) = \operatorname{colim}_{K\in\mathcal{K}_M} H^q(M|K;R)$$

of the $\mathcal{K}_M$-directed R-module $\mathcal{K}_M \ni K \to H^q(M|K;R)$. By the universal property for colimits (4.70) there is a unique map $H^q_c(M;R) \to H^q(M;R)$ such that the diagrams

$$\begin{tikzcd} H^q(M|K) \arrow[rr] \arrow[dr] \arrow[ddr] & & H^q(M|L) \arrow[dl] \arrow[ddl] \\ & H^q_c(M) \arrow[d, dotted] & \\ & H^q(M) & \end{tikzcd}$$

commute for all compact subsets $K \subset L$ of M. The singular cohomology group of M is an example of an R-module that receives a map from the directed system $\mathcal{K}_M \ni K \to H^q(M|K;R)$ but cohomology with compact support of M is the universal such example.

4.82. Remark. (1) If M is compact $H^q_c(M) = H^q(M)$ because M is the largest element in $\mathcal{K}$. More generally, $H^q_c(M)$ can be computed using any cofinal subdirected sets of compact subsets of M.
(2) Suppose that $f\colon M \to N$ is a *proper* map and let $f^{-1}\colon \mathcal{K}_N \to \mathcal{K}_M$ be the map af directed systems given by pre-images. There is an induced map

$$H^q_c(N) = \operatorname{colim}_{L\in\mathcal{K}_N} H^q(N|L) \xrightarrow{f^*} \operatorname{colim}_{L\in\mathcal{K}_N} H^q(M|f^{-1}(L)) \to \operatorname{colim}_{K\in\mathcal{K}_M} H^q(M|K) = H^q_c(M)$$

of cohomology groups with compact support. There is no such induced map for an arbitrary map between manifolds so cohomology with compact support is not functorial for arbitray continuous maps.
(3) The closed balls $D(0,r)$ of radius $r = 1, 2, \dots$ centered at $0 \in \mathbf{R}^n$ are cofinal in the directed set of compact subsets of $\mathbf{R}^n$. The inclusion maps $(\mathbf{R}^n, \mathbf{R}^n - 0) \subset (\mathbf{R}^n, \mathbf{R}^n - D(0,r))$ induce an isomorphism $H^q(\mathbf{R}^n, \mathbf{R}^n - 0) \to H^q(\mathbf{R}^n, \mathbf{R}^n - D(0,r))$ of directed systems so

$$H^q_c(\mathbf{R}^n) = \operatorname{colim}_{\mathbf{N}} H^q(\mathbf{R}^n, \mathbf{R}^n - D(0,r)) \xleftarrow{\cong} \operatorname{colim}_{\mathbf{N}} H^q(\mathbf{R}^n, \mathbf{R}^n - 0) = H^q(\mathbf{R}^n, \mathbf{R}^n - 0)$$

$$= \begin{cases} R & q = n \\ 0 & q \neq n \end{cases}$$

Note that $H^q_c(\mathbf{R}^n) \neq H^q_c(*)$ for $n > 0$ so $H^q_c(-)$ is *not* a homotopy invariant.

(4) For any open submanifold U of M define $H^q_c(U) \to H^q_c(M)$ to be the map

$$H^q_c(U) = \operatorname{colim}_{K\in\mathcal{K}_U} H^q(U|K) \xleftarrow[\text{exc}]{\cong} \operatorname{colim}_{K\in\mathcal{K}_U} H^q(M|K) \to \operatorname{colim}_{K\in\mathcal{K}_M} H^q(M|K) = H^q_c(M) \tag{4.83}$$

where an excision isomorphism occurs (cf 4.48). (Note the direction of the arrow.)

4.84. LEMMA. Let U and V be open submanifolds of the manifold M. Then there is an exact Mayer-Vietoris sequence for the pairs $(M, M-(K\cup L)) \subset (M, M-K), (M, M-L) \subset (M, M-(K\cap L))$

$$\cdots \to H^q_c(U\cap V) \to H^q_c(U)\oplus H^q_c(V) \to H^q_c(U\cup V) \to H^{q+1}_c(U\cap V) \to \cdots$$

in cohomology with compact support.

PROOF. We may assume that $M = U\cup V$. Suppose that $K\subset U$ and $L\subset V$ are compact subsets. Then there is relative Mayer-Vietoris sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H^q(M|K\cap L) \arrow[r] \arrow[d, "\cong"] & H^q(M|K)\oplus H^q(M|L) \arrow[r] \arrow[d, "\cong"] & H^q(M|K\cup L) \arrow[r] \arrow[d, equal] & \cdots \\
& H^q(U\cap V|K\cap L) & H^q(U|K)\oplus H^q(V|L) & H^q(U\cup V|K\cup L) &
\end{tikzcd}$$

where the vertical arrows are excision isomorphisms. Pass to the colimit over the directed set $\mathcal{K}_U\times\mathcal{K}_V$ with order relation $(K_1,L_1)\le(K_2,L_2) \overset{\text{def}}{\iff} K_1\le K_2, L_1\le L_2$. The maps $\mathcal{K}_U\times\mathcal{K}_V\to\mathcal{K}_{U\cup V}\colon (K,L)\to K\cup L$ is cofinal because local compactness implies that any compact subset of $U\cup V$ is of the form $K\cup L$. $\square$

5. Poincaré duality for noncompact oriented manifolds

Let M be an R-oriented manifold and let $\mu_x\in H_n(M|x;R)$ be the local R-orientation at $x\in M$. There is (4.57) a system of unique R-orientation classes $\mu_K\in H_n(M|K;R)$ such that $r^K_x\mu_K=\mu_x$ for all $x\in M$ and $r^L_K\mu_L=\mu_K$ for compact subsets $K\subset L\subset M$. Using the relative cap product $H_n(M|K;R)\times H^q(M|K;R)\to H_{n-q}(M;R)$ (4.33) we obtain a system of R-module homomorphisms $\mu_K\cap -\colon H^q(M|K;R)\to H_{n-q}(M;R)$ for $K\in\mathcal{K}_M$. Naturality of the cap product (4.31) implies two observations:

- Since $\mu_L\cap e^K_L\phi = r^L_K\mu_L\cap\phi = \mu_K\cap\phi$ for all $\phi\in H^q(M|K)$, this is a $\mathcal{K}_M$-directed system of R-module homomorphisms (4.69). By the universal property (4.70) there is a unique R-module homomorphism $\mathrm{PD}\colon H^q_c(M;R)\to H_{n-q}(M;R)$ such that the diagrams

$$\begin{tikzcd}
H^q(M|K;R) \arrow[rr, "e^K_L"] \arrow[dr] \arrow[ddr, "\mu_K\cap-"'] & & H^q(M|L;R) \arrow[dl] \arrow[ddl, "\mu_L\cap-"] \\
& H^q_c(M;R) \arrow[d, dotted, "\mathrm{PD}"] & \\
& H_{n-q}(M;R) &
\end{tikzcd}$$

  commute for all compact subsets $K\subset L$ of M.
- The Poincaré duality map PD is natural for open submanifolds of M: Let $i\colon U\to M$ be the inclusion of an open submanifold (with the induced orientation) and let K be a compact subset of U. Since $i_*(\mu_K\cap i^*\phi)=\mu_K\cap\phi$ for all $\phi\in H^q(M|K)\cong H^q(U|K)$ the diagram

(4.85)
$$\begin{tikzcd}
H^q_c(U) \arrow[d, "\mathrm{PD}"'] & \operatorname{colim}_{K\in\mathcal{K}_U} H^q(M|K) \arrow[l, "i^*"', "\cong"] \arrow[r] & H^q_c(M) \arrow[d, "\mathrm{PD}"] \\
H_{n-q}(U)) \arrow[rr, "i_*"] & & H_{n-q}(M)
\end{tikzcd}$$

  commutes.

4.86. THEOREM (Poincaré duality). *Let M be an R-oriented n-manifold. The R-module homomorphism*

$$\mathrm{PD}\colon H^q_c(M;R)\to H_{n-q}(M;R)$$

is an isomorphism.

The proof is divided into several steps.

4.87. LEMMA. If Theorem 4.86 is true for the open subsets U, V and $U \cap V$ of the oriented manifold M then it also true for $U \cup V$.

PROOF. Consider the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^q_c(U\cap V) \arrow[r] \arrow[d, "\mathrm{PD}"] & H^q_c(U) \oplus H^q_c(V) \arrow[r] \arrow[d, "\mathrm{PD}"] & H^q_c(U \cup V) \arrow[r, "\delta"] \arrow[d, "\mathrm{PD}"] & H^{q+1}_c(U \cap V) \arrow[r] \arrow[d, "\mathrm{PD}"] & \cdots \\
\cdots \arrow[r] & H_{n-q}(U\cap V) \arrow[r] & H_{n-q}(U) \oplus H_{n-q}(V) \arrow[r] & H_{n-q}(U \cap V) \arrow[r, "\partial"] & H_{n-q-1}(U \cap V) \arrow[r] & \cdots
\end{tikzcd}$$

where the top row is the exact sequence from (4.84), the bottom row is the Mayer-Vietoris sequence (1.39) for $(U \cup V, U, V)$, and the vertical maps are Poincaré duality maps. It is clear from (4.85) that the squares not involving conecting homomorphisms are commutative. A longer and nontrivial argument shows that also the connecting homomorphisms commute with PD up to sign. Now use the 5-lemma. $\square$

4.88. LEMMA. Let $\mathcal{U} = \{U\}$ be a directed collection of open subsets of M ordered by inclusion. If Theorem 4.86 is true for all $U \in \mathcal{U}$ then it is also true for $\bigcup_{U\in\mathcal{U}} U$.

PROOF. The colimit of the isomorphisms $H^q_c(U) \xrightarrow{\mathrm{PD}} H_{n-q}(U)$, $U \in \mathcal{U}$, is an isomorphism

$$H^q_c(\bigcup U) \overset{4.76}{\cong} \operatorname{colim}_{\mathcal{U}} \operatorname{colim}_{\mathcal{K}_U} H^q(U|K) \cong \operatorname{colim}_{\mathcal{U}} H^q_c(U) \xrightarrow{\cong} \operatorname{colim}_{\mathcal{U}} H_{n-q}(U) \overset{4.79}{\cong} H_{n-q}(\bigcup U)$$

where we use that $\bigcup_{U\in\mathcal{U}} \mathcal{K}_U = \mathcal{K}_{\bigcup_{U\in\mathcal{U}} U}$ is the directed set of compact subsets of $\bigcup_{U\in\mathcal{U}} U$. This is Poincaré duality for $\bigcup_{U\in\mathcal{U}} U$. $\square$

4.89. LEMMA. Theorem 4.86 is true for $\mathbf{R}^n$ and for any open subset of $\mathbf{R}^n$.

PROOF. By definition of PD the composite map

$$H^n(\mathbf{R}^n|0) \xrightarrow{\cong} H^n_c(\mathbf{R}^n) \xrightarrow{\mathrm{PD}} H_0(\mathbf{R}^n)$$

is cap product with the local orientation $\mu_0 \in H_n(\mathbf{R}^n|0)$. But the cap product

$$R \times R \overset{\mathrm{UCT}}{=} H_n(\mathbf{R}^n|0;\mathbf{Z}) \otimes R \times \operatorname{Hom}(H_n(\mathbf{R}^n|0;Z), R) = H_n(\mathbf{R}^n|0) \times H^n(\mathbf{R}^n|0) \xrightarrow{\cap} H_0(\mathbf{R}^n) \underset{\cong}{\xrightarrow{\varepsilon}} R$$

equals evaluation (4.27) or the ring multiplication in R and multiplication with a unit is an isomorphism. It is then also true for any open convex subset of $\mathbf{R}^n$ as any such is homeomorphic to $\mathbf{R}^n$. By Lemma 4.87 and induction it is true for any finite union of open convex subsets of $\mathbf{R}^n$. Let now U be an arbitry open subset of $\mathbf{R}^n$. U is the union of countably many open balls V_i and of the open sets $U_i = V_1 \cup \cdots \cup V_i$ that are directed, even linearly ordered, by inclusion. Each of these satisfy Poincaré duality and so does their union (4.88). $\square$

PROOF OF THEOREM 4.86. Consider the poset of all open subsets of M that enjoy Poincaré duality. By 4.89 this is a nonempty collection and by 4.88 any linearly ordered subset has an upper bound. Zorn's lemma now says that there are maximal elements. Such a maximal element must equal M for by 4.89 and 4.87 we can always enlarge any open proper open subset of M with Poincaré duality to a larger open subset with Poincaré duality. $\square$

6. Alexander duality

Let M be a manifold and A a closed subset of M. Let $\mathcal{U}_A$ denote the poset of open neighborhoods of A ordered by inclusion and $\mathcal{U}_A^{\mathrm{op}}$ the opposite poset (where $U_0 \le U_1$ if $U_0 \supset U_1$). The *Alexander–Čech* cohomology group of the embedding $A \subset M$ is the colimit

$$\check{H}^q(A;R) = \operatorname{colim}_{U\in\mathcal{U}_A^{\mathrm{op}}} H^q(U;R)$$

of the $\mathcal{U}_A^{\mathrm{op}}$-directed system $\mathcal{U}_A^{\mathrm{op}} \ni U \to H^q(U;R)$. By the universal property for colimits (4.70) there is a unique map $\check{H}^q(A;R) \to H^q(A;R)$ such that the diagrams

$$\begin{tikzcd}
H^q(U_0;R) \arrow[rr, "i^*"] \arrow[dr] \arrow[ddr] & & H^q(U_1;R) \arrow[dl] \arrow[ddl] \\
& \check{H}^q(A;R) \arrow[d, dotted] & \\
& H^q(A;R) &
\end{tikzcd}$$

commute for all $U_0, U_1 \in \mathcal{U}_A$ with $U_0 \supset U_1$. The singular cohomology group of A is an example of an R-module that receives a map from the directed system $\mathcal{U}_A^{\mathrm{op}} \ni U \to H^q(U;R)$ but the Alexander–Čech cohomology group of A is the universal such example.

4.90. PROPOSITION. Let M be a compact manifold and A a closed subset of M. Then there is a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^q(M,A) \arrow[r] & H^q(M) \arrow[r] & H^q(A) \arrow[r, "\delta"] & H^{q+1}(M,A) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^q_c(M-A) \arrow[r, "(4.83)"] \arrow[u] & H^q(M) \arrow[r] \arrow[u, equal] & \check{H}^q(A) \arrow[r, "\delta"] \arrow[u] & H^{q+1}_c(M-A) \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

with exact rows.

PROOF. Note that $U \to M - U$ is an isomorphism of directed sets $\mathcal{U}_A^{\mathrm{op}} \to \mathcal{K}_{M-A}$ and that $H^q(M-A|M-U) = H^q(M-A, U-A)$. Compare the long exact sequences for (M,A) and (M,U) and take the colimit over $\mathcal{U}_A^{\mathrm{op}} \cong \mathcal{K}_{M-A}$ to obtain the commutative diagram of the exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H^q(M,A) \arrow[r] & H^q(M) \arrow[r] & H^q(A) \arrow[r, "\delta"] & H^{q+1}(M,A) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^q_c(M-A) \arrow[r] \arrow[u, dotted] & H^q(M) \arrow[r] \arrow[u, equal] & \check{H}^q(A) \arrow[r, "\delta"] \arrow[u, dotted] & H^{q+1}_c(M-A) \arrow[r] \arrow[u, dotted] & \cdots \\
\cdots \arrow[r] & H^q(M,U) \arrow[r] \arrow[u] \arrow[d, "\mathrm{exc}"', "\cong"] & H^q(M) \arrow[r] \arrow[u, equal] & H^q(U) \arrow[r, "\delta"] \arrow[u] & H^{q+1}(M,U) \arrow[r] \arrow[u] \arrow[d, "\mathrm{exc}"', "\cong"] & \cdots \\
& H^q(M-A,U-A) & & & H^{q+1}(M-A,U-A) &
\end{tikzcd}$$

Since colim is an exact functor (4.73), the limit sequence in the middle is still exact. □

In some cases the value of $\check{H}^q(A;R)$ depends only on A and not on M nor the embedding of A into M.

4.91. PROPOSITION. If A and M are compact manifolds (more generally, compact ANRs[1] [**11**, pp 25–32] [**8**]) then $\check{H}^q(A;R) \to H^q(A;R)$ and $H^q_c(M-A) \to H^q(M,A)$ are isomorphisms.

Suppose that M is compact and R-oriented with orientation classes $\mu_A \in H_n(M|A;R)$ for all closed (compact) subsets A of M (4.57).

For any open neighborhood U of A, there is cap product (4.33)

$$H_n(U|A) \times H^q(U) \xrightarrow{\cap} H_{n-q}(U|A)$$

Under the excision isomorphism $H_*(U|A) \cong H_*(M|A)$ this is a cap product

$$H_n(M|A) \times H^q(U) \xrightarrow{\cap} H_{n-q}(M|A)$$

[1]A space Y is an ANR if every continuous map into Y from a closed subspace of a normal space extends to an open neighborhood of the closed subspace.

Cap product with with the orientation class μ_A in $H_n(M|A)$ gives a system of homomorphisms $\mu_A \cap -\colon H^q(U) \to H_{n-q}(M|A)$ compatible with inclusions: When $U_0 \supset U_1 \supset A$, naturality of the cap product (4.31) says that $\mu_A \cap i^*\phi = \mu_A \cap \phi$ for all $\phi \in H^q(U_0)$. The universal property for colimits (4.70) shows that there is a map $\mathrm{AD}\colon \check{H}^q(A;R) \to H_{n-q}(M|A;R)$ such that the diagrams

$$\begin{tikzcd}
H^q(U_0;R) \arrow[rr, "i^*"] \arrow[dr] \arrow[ddr, "\mu_A\cap-"'] & & H^q(U_1;R) \arrow[dl] \arrow[ddl, "\mu_A\cap-"] \\
& \check{H}^q(A;R) \arrow[d, dotted, "\mathrm{AD}"] & \\
& H_{n-q}(M|A;R) &
\end{tikzcd}$$

commute.

4.92. Theorem (Alexander duality). *Let M be a compact R-oriented n-manifold and $A \subset M$ a closed subset. Then the R-module homomorphism*

$$\mathrm{AD}\colon \check{H}^q(A;R) \to H_{n-q}(M|A;R)$$

is an isomorphism.

Proof. Let U be an open neighborhood of A. Cap products with the orientation classes produces a diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^q(M-A|M-U) \arrow[r] \arrow[d, "\mu_{M-U}\cap-"] & H^q(M) \arrow[r] \arrow[d, "\mu_M"] & H^q(U) \arrow[r] \arrow[d, "\mu_A\cap-"] & H^{q+1}(M-A|M-U) \arrow[r] \arrow[d, "\mu_{M-U}\cap-"] & \cdots \\
\cdots \arrow[r] & H_{n-q}(M-A) \arrow[r] & H_{n-q}(M) \arrow[r] & H_{n-q}(M,M-A) \arrow[r] & H_{n-q-1}(M-A) \arrow[r] & \cdots
\end{tikzcd}$$

connecting the long exact sequences for the pairs (M,U) and $(M,M-A)$. This diagram commutes up to sign. The limit diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^q_c(M-A) \arrow[r] \arrow[d, "\mathrm{PD}", "\cong"'] & H^q(M) \arrow[r] \arrow[d, "\mathrm{PD}", "\cong"'] & \check{H}^q(A;R) \arrow[r, "\delta"] \arrow[d, "\mathrm{AD}"] & H^{q+1}_c(M-A) \arrow[r] \arrow[d, "\mathrm{PD}", "\cong"'] & \cdots \\
\cdots \arrow[r] & H_{n-q}(M-A) \arrow[r] & H_{n-q}(M) \arrow[r] & H_{n-q}(M|A;R) \arrow[r, "\partial"] & H_{n-q-1}(M-A) \arrow[r] & \cdots
\end{tikzcd}$$

also commutes up to sign and the top row is still exact. The 5-lemma implies that AD is an isomorphism. □

4.93. Corollary. Let $A \subset \mathbf{R}^{n+1}$ be a compact n-manifold (or ANR) embedded in $\mathbf{R}^{n+1}$. Then

$$H^q(A;R) \cong \widetilde{H}_{n-q}(\mathbf{R}^{n+1} - A;R)$$

for all commutative rings R.

Proof. Since A is a compact manifold embedded in the orientable manifolds R^{n+1} or S^{n+1}, $\check{H}^q(A) \cong H^q(A)$ (4.91), and Alexander duality (4.92) applies to the embedding $A \subset \mathbf{R}^{n+1} \subset S^{n+1}$ and gives

$$\check{H}^q(A) \overset{\mathrm{AD}}{\cong} H_{n+1-q}(S^{n+1}|A) \overset{\mathrm{exc}}{\cong} H_{n+1-q}(\mathbf{R}^{n+1}|A) \cong \widetilde{H}_{n-q}(\mathbf{R}^{n+1}-A)$$

where we use that $\mathbf{R}^{n+1}$ is contractible for the last isomorphism. □

We can now show a vast generalization of (part of) the Jordan curve theorem.

4.94. Theorem (General Separation Theorem). *Let $A \subset \mathbf{R}^{n+1}$ be a compact n-manifold embedded in $\mathbf{R}^{n+1}$. If A has k components then the complement $\mathbf{R}^{n+1} - A$ has $k+1$ components.*

Proof. Using both Poincaré (4.86) and Alexander Duality (4.92) with $\mathbf{F}_2$-coefficients we get

$$H_0(A;\mathbf{F}_2) \overset{\mathrm{PD}}{\cong} H^n(A;\mathbf{F}_2) \overset{\mathrm{AD}}{\cong} \widetilde{H}_0(\mathbf{R}^{n+1}-A;\mathbf{F}_2)$$

We use that any manifold is $\mathbf{F}_2$-orientable. □

4.95. COROLLARY. A compact nonorientable n-manifold cannot embed in $\mathbf{R}^{n+1}$.

PROOF. Suppose that A embeds in $\mathbf{R}^{n+1}$. We compute $H^n(A;\mathbf{Z})$ in two different ways. First, Alexander duality 4.93 says that

$$H^n(A;\mathbf{Z}) \overset{\text{AD}}{\cong} \widetilde{H}_0(\mathbf{R}^{n+1} - A;\mathbf{Z})$$

since A is compact and nonorientable then $H_n(A;\mathbf{Z}) = 0$ by (4.60) so that UCT (4.12) says that

$$\operatorname{Ext}_{\mathbf{Z}}(H_{n-1}(A;\mathbf{Z}),\mathbf{Z}) \cong H^n(A;\mathbf{Z})$$

(where $H_{n-1}(A;\mathbf{Z})$ is a finitely generated abelian group). The first equality says that $H^n(A;\mathbf{Z})$ is a free nontrivial (for $A \neq \mathbf{R}^{n+1}$) abelian group, in particular infinite. The second equality says that $H^n(A;\mathbf{Z})$ is finite. $\square$

In particular, the nonorientable compact surfaces cannot embed in $\mathbf{R}^3$. Do they embed in $\mathbf{R}^4$?

4.96. Linking number. Suppose that A_1 and A_2 are disjoint compact submanifolds of $\mathbf{R}^n$. The *linking number* is the bilinear map

$$L\colon \widetilde{H}_p(A_1;\mathbf{Z}) \times H_{n-p-1}(A_2;\mathbf{Z}) \to \widetilde{H}_p(\mathbf{R}^n - A_2;\mathbf{Z}) \times H_{n-p-1}(A_2;\mathbf{Z})$$
$$\xleftarrow[\cong]{(4.93)\times \mathrm{id}} H^{n-p-1}(A_2;\mathbf{Z}) \times H_{n-p-1}(A_2;\mathbf{Z}) \xrightarrow{\langle\cdot,\cdot\rangle} \mathbf{Z}$$

For instance, A_1 and A_2 could be knots in $\mathbf{R}^3$.

4.97. Invariance of Domain. We can use Alexander duality to reprove 3.6 and 3.7.

4.98. LEMMA. Let d^r be a subspace of S^n that is homeomorphic to D^r for some $r \geq 0$. The homology groups of the complement are $\widetilde{H}_*(S^n - d^r) = 0$.

Let s^r be a subspace of S^n that is homeomorphic to S^r for some $r \geq 0$. Then $r \leq n$. If $r = n$, then $s^r = S^n$. If $0 \leq r < n$ then $H_*(S^n - s^{r-1}) = H_*(S^n - S^r) = H_*(S^{n-r-1})$.

PROOF. We may assume that $r > 0$ since $S^n - s^0 = \mathbf{R}^n - 0 \simeq S^{n-1}$ and the formula is true. Alexander duality says that

$$H_{n-q}(S^n, S^n - s^r) \cong \check{H}^q(s^r) \cong H^q(S^r) \cong H_{n-q}(S^n, S^n - S^r) = \begin{cases} \mathbf{Z} & n-q = n, n-r \\ 0 & n-q \neq n, n-r \end{cases}$$

Since the pair $(S^n, S^n - s^r)$ has nonzero homology in degree $n-r$, we have $n - r \geq 0$. If $r = n$, $H_0(S^n, S^n - s^n) = \mathbf{Z}$ so $S^n - s^n = \emptyset$ or $s^n = S^n$. If $0 < r < n$, $H_0(S^n, S^n - s^n) = 0$ so $S^n - s^r \neq \emptyset$. The long exact sequence (in reduced homology) for the pair $(S^n, S^n - s^r)$ contains the segments

$$\widetilde{H}_{n-r}(S^n) = 0 \to H_{n-r}(S^n, S^n - s^r) \to \widetilde{H}_{n-r-1}(S^n - s^r) \to \widetilde{H}_{n-r-1}(S^n) = 0$$

$$0 \to H_n(S^n - s^r) \to H_n(S^n) \xrightarrow{\cong} H_n(S^n, S^n - s^r) \to H_{n-1}(S^n - s^r) \to 0 = H_{n-1}(S^n)$$

which say that $\widetilde{H}_{n-r-1}(S^n - s^r) = \mathbf{Z}$ and $H_n(S^n - s^r) = 0 = H_{n-1}(S^n - s^r)$, and for $i \neq n-r, n, n+1$ it contains the segment

$$H_i(S^n, S^n - s^r) = 0 \to \widetilde{H}_{i-1}(S^n - s^r) \to \widetilde{H}_{i-1}(S^n) = 0$$

which says that $\widetilde{H}_{i-1}(S^n - s^r) = 0$. The map $H_n(S^n) \to H_n(S^n, S^n - s^r)$ is an isomorphism because $H_n(S^n) \to H_n(S^n, S^n - s^r) \to H_n(S^n, S^n - x) = H_n(S^n|x)$, $x \in s^r$, is an isomorphism as S^n is orientable. $\square$

CHAPTER 5

Cohomology operations

A cohomology operation of type $(m, K; n, G)$ is a natural transformation from the functor $H^m(-;K)$ to the functor $H^n(-;G)$. We shall first look at an operation of type $(n, \mathbf{Z}/p; n+1, \mathbf{Z}/p)$.

1. The Bockstein homomorphism

Let C be a chain complex of free abelian groups and $0 \to G \to H \to K \to 0$ a short exact sequence of abelian groups. Map the chain complex into the short exact sequence and obtain a short exact sequence of chain complexes

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \mathrm{Hom}(C_{n+1}, G) \arrow[r] \arrow[u] & \mathrm{Hom}(C_{n+1}, H) \arrow[r] \arrow[u] & \mathrm{Hom}(C_{n+1}, K) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathrm{Hom}(C_n, G) \arrow[r] \arrow[u, "\delta"] & \mathrm{Hom}(C_n, H) \arrow[r] \arrow[u, "\delta"] & \mathrm{Hom}(C_n, K) \arrow[r] \arrow[u, "\delta"] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

The Bockstein homomorphism is the connecting homomorphism β in the associated long exact sequence

$$\cdots \longrightarrow H^n(C;H) \longrightarrow H^n(C;K) \xrightarrow{\beta} H^{n+1}(C;G) \longrightarrow H^{n+1}(C;H) \longrightarrow \cdots$$

of cohomology groups. (There is a similar Bockstein in homology.) The Bockststein homomorphism is natural in C and in morphisms of short exact sequences. We shall be interested mostly in the Bockstein

$$H^n(C;\mathbf{Z}/p) \xrightarrow{\beta} H^{n+1}(C;\mathbf{Z}/p)$$

for the short exact sequence $0 \to \mathbf{Z}/p \xrightarrow{\cdot p} \mathbf{Z}/p^2 \to \mathbf{Z}/p \to 0$ where p is a prime number.

5.1. Example. The Bockstein homomorphism $H^n(C;\mathbf{Z}/p) \xrightarrow{\beta} H^{n+1}(C;\mathbf{Z}/p)$ for the elementary chain complexes

$$\cdots \longleftarrow 0 \longleftarrow \mathbf{Z} \longleftarrow 0 \longleftarrow \cdots$$

$$\cdots \longleftarrow 0 \longleftarrow \mathbf{Z} \xleftarrow{p} \mathbf{Z} \longleftarrow \cdots$$

$$\cdots \longleftarrow 0 \longleftarrow \mathbf{Z} \xleftarrow{p^r} \mathbf{Z} \longleftarrow \cdots$$

$$\cdots \longleftarrow 0 \longleftarrow \mathbf{Z} \xleftarrow{q^r} \mathbf{Z} \longleftarrow \cdots$$

with $\mathbf{Z}$ in degrees n or $n+1$ are

$$\mathbf{Z} \longrightarrow 0 \qquad \mathbf{Z}/p \xrightarrow{\cong} \mathbf{Z}/p \qquad \mathbf{Z}/p \xrightarrow{0} \mathbf{Z}/p \qquad 0 \longrightarrow 0$$

where the exponent $r \geq 2$ in p^r and q is some prime $\neq p$.

The example in fact computes the Bockstein homomorphism in most cases of interest.

5.2. PROPOSITION. Any chain complex C of free abelian groups with $H_n(C)$ finitely generated for all n is quasi-isomorphic to a direct sum of elementary chain complexes.

PROOF. Write each homology group $H_n(C)$ as a direct sum of infinite cyclic groups, $\mathbf{Z}$, and finite cyclic groups, $\mathbf{Z}/p^r$ or $\mathbf{Z}/q^r$, of prime power order [**5**, §4.2, §16.10]. There are chain maps from the corresponding elementary chain complexes to C. □

5.3. PROPOSITION.
$$\begin{tikzcd} H^{n-1}(C;\mathbf{Z}/p) \arrow[r, "\beta"] \arrow[rr, bend right, "0"'] & H^n(C;\mathbf{Z}/p) \arrow[r, "\beta"] & H^{n+1}(C;\mathbf{Z}/p) \end{tikzcd}$$

PROOF. The morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{Z} \arrow[r, "p"] \arrow[d, "\rho"] & \mathbf{Z} \arrow[r] \arrow[d] & \mathbf{Z}/p \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbf{Z}/p \arrow[r, "p"] & \mathbf{Z}/p^2 \arrow[r] & \mathbf{Z}/p \arrow[r] & 0 \end{tikzcd}$$

induces a morphism

$$\begin{tikzcd} H^n(C;\mathbf{Z}) \arrow[r, "\rho"] \arrow[d] & H^n(C;\mathbf{Z}/p) \arrow[r, "\widetilde{\beta}"] \arrow[d, equal] & H^n(C;\mathbf{Z}) \arrow[r] \arrow[d, "\rho"] & H^n(C;\mathbf{Z}) \arrow[d] \\ H^n(C;\mathbf{Z}/p^2) \arrow[r] & H^n(C;\mathbf{Z}/p) \arrow[r, "\beta"] & H^{n+1}(C;\mathbf{Z}/p) \arrow[r] & H^{n+1}(C;\mathbf{Z}/p^2) \end{tikzcd}$$

of long exact sequences which shows that $\beta = \rho\widetilde{\beta}$. Thus $\beta\beta = (\rho\widetilde{\beta})(\rho\widetilde{\beta}) = \rho(\widetilde{\beta}\rho)\widetilde{\beta} = 0$ by exactness of the upper sequence. □

5.4. PROPOSITION. Suppose that C is the chain complex of a Δ-set so that the cup product in $H^*(C;\mathbf{Z}/p)$ is defined. Then β is a derivation in the sense that

$$\beta(x \cup y) = \beta x \cup y + (-1)^{|x|} x \cup \beta y$$

when x and y are homogeneous elements of $H^*(C;\mathbf{Z}/p)$.

PROOF. The Bockstein is defined by a zig-zag in the commutative diagram

$$\begin{tikzcd} & \vdots & \vdots & \vdots & \\ 0 \arrow[r] & \mathrm{Hom}(C_{n+1},\mathbf{Z}/p) \arrow[r, "i"] \arrow[u] & \mathrm{Hom}(C_{n+1},\mathbf{Z}/p^2) \arrow[r, "\rho"] \arrow[u] & \mathrm{Hom}(C_{n+1},\mathbf{Z}/p) \arrow[r] \arrow[u] & 0 \\ 0 \arrow[r] & \mathrm{Hom}(C_n,\mathbf{Z}/p) \arrow[r, "i"] \arrow[u, "\delta"] & \mathrm{Hom}(C_n,\mathbf{Z}/p^2) \arrow[r, "\rho"] \arrow[u, "\delta"] & \mathrm{Hom}(C_n,\mathbf{Z}/p) \arrow[r] \arrow[u, "\delta"] & 0 \\ & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \end{tikzcd}$$

induced from the short exact sequence $0 \to \mathbf{Z}/p \xrightarrow{i} \mathbf{Z}/p^2 \xrightarrow{\rho} \mathbf{Z}/p \to 0$. Let (also) $x \in \mathrm{Hom}(C_m,\mathbf{Z}/p)$ and $y \in \mathrm{Hom}(C_n,\mathbf{Z}/p)$ be cocycles representing the cohomology classes x and y. Since ρ is surjective, $x = \rho\overline{x}$ and $y = \rho\overline{y}$ for some cochains $\overline{x} \in \mathrm{Hom}(C_m,\mathbf{Z}/p^2)$ and $\overline{y} \in \mathrm{Hom}(C_n,\mathbf{Z}/p^2)$. By definition, $i\beta x = \delta\overline{x}$ and $i\beta y = \delta\overline{y}$.

Since $\rho(ab) = \rho(a)\rho(b)$ for $\rho\colon \mathbf{Z}/p^2 \to \mathbf{Z}/p$ we have $\rho(\overline{x} \cup \overline{y}) = \rho\overline{x} \cup \rho\overline{y} = x \cup y$ and since $i(a\rho(b)) = i(a)b$ for $i\colon \mathbf{Z}/p \to \mathbf{Z}/p^2$ we have $i(\beta x \cup y) = i(\beta x \cup \rho\overline{y}) = i\beta x \cup \overline{y} = \delta\overline{x} \cup \overline{y}$. The equations

$$i(\beta x \cup y + (-1)^{|x|} x \cup \beta y) = \delta\overline{x} \cup \overline{y} + (-1)^{|x|}\overline{x} \cup \delta\overline{y} = \delta(\overline{x} \cup \overline{y}), \quad \rho(\overline{x} \cup \overline{y}) = x \cup y$$

mean that $\beta(x \cup y) = \beta x \cup y + (-1)^{|x|} x \cup \beta y$. □

5.5. The Bockstein spectral sequence. Let C be a chain complex of free abelian groups with $H_n(C)$ finitely generated for all n. When we map C into the short exact sequence $0 \to \mathbf{Z} \xrightarrow{\cdot p} \mathbf{Z} \to \mathbf{Z}/p \to 0$ we obtain a short exact sequence of chain complexes and the long exact Bockstein sequence

(5.6) $$H^{n-1}(C;\mathbf{Z}/p) \xrightarrow{\beta} H^n(C) \xrightarrow{\cdot p} H^n(C) \xrightarrow{\rho} H^n(C;\mathbf{Z}/p) \xrightarrow{\beta} H^{n+1}(C) \xrightarrow{\cdot p} H^{n+1}(C) \xrightarrow{\rho} H^{n+1}(C;\mathbf{Z}/p) \to H^{n+2}(C;\mathbf{Z}/p)$$

in cohomology. This exact sequence is often more concisely depicted as the exact Bockstein triangle

$$\begin{tikzcd} H^*(C) \arrow[rr, "\cdot p"] & & H^*(C) \arrow[dl, "\rho"] \\ & H^*(C;\mathbf{Z}/p) \arrow[ul, "\beta"] & \end{tikzcd}$$

or as

$$\begin{tikzcd} H^*(C) \arrow[rr, "\cdot p"] & & H^*(C) \arrow[dl, "\rho_1"] \\ & E_1^*(C) \arrow[ul, "\beta_1"] & \end{tikzcd}$$

where we write $E_1^*(C)$ for $H^*(C;\mathbf{Z}/p)$ and ρ_1 for ρ, β_1 for β. Now, $E_1^*(C)$ is (5.3) a chain complex with differential $d_1 = \rho_1\beta_1 = \beta$, the Bockstein homomorphism for the exact sequence $0 \to \mathbf{Z}/p \to \mathbf{Z}/p^2 \to \mathbf{Z}/p \to 0$. The *derived triangle* (or sequence) is

$$\begin{tikzcd} pH^*(C) \arrow[rr, "\cdot p"] & & pH^*(C) \arrow[dl, "\rho_2"] \\ & E_2^*(C) \arrow[ul, "\beta_2"] & \end{tikzcd}$$

where $E_2(C)$ is the homology of the chain complex $(E_1(C), d_1)$, the map β_2 is induced by β, and $\rho_2(px) = \rho_1(x)$.

5.7. Example. If C is the elementary chain complex $0 \leftarrow \mathbf{Z} \leftarrow 0$ concentrated in degree n, the Bockstein long exact sequence (5.6) is

$$0 \xrightarrow{\beta} \mathbf{Z} \xrightarrow{\cdot p} \mathbf{Z} \xrightarrow{\rho} \mathbf{Z}/p \xrightarrow{\beta} 0 \xrightarrow{\cdot p} 0 \xrightarrow{\rho} 0 \longrightarrow 0$$

where the nonzero groups are in degree n. The chain complex $E_1^*(C)$ is $0 \longrightarrow \mathbf{Z}/p \longrightarrow 0$ concentrated in degree n, so $E_1^*(C) = E_2^*(C)$. The derived triangle is identical to the first triangle.

If C is the elementary chain complex $0 \leftarrow \mathbf{Z} \xleftarrow{p} \mathbf{Z} \leftarrow 0$ concentrated in degrees n and $n+1$, the Bockstein long exact sequence (5.6) is

$$0 \xrightarrow{\beta} 0 \xrightarrow{\cdot p} 0 \xrightarrow{\rho} \mathbf{Z}/p \xrightarrow{\beta} \mathbf{Z}/p \xrightarrow{\cdot p=0} \mathbf{Z}/p \xrightarrow{\rho} \mathbf{Z}/p \longrightarrow 0$$

where the first nonzero group is $\mathbf{Z}/p = H^n(C;\mathbf{Z}/p)$. The chain complex $E_1^*(C)$ is

$$0 \longrightarrow \mathbf{Z}/p \xrightarrow{\cong} \mathbf{Z}/p \longrightarrow 0$$

concentrated in degrees n and $n+1$, so $E_2^*(C) = 0$. The derived triangle consists of 0s.

If C is the elementary chain complex $0 \longleftarrow \mathbf{Z} \xleftarrow{p^r} \mathbf{Z} \longleftarrow 0$, $r \geq 2$, concentrated in degrees n and $n+1$, the Bockstein long exact sequence (5.6) is

$$0 \xrightarrow{\beta} 0 \xrightarrow{\cdot p} 0 \xrightarrow{\rho} \mathbf{Z}/p \xrightarrow{\beta} \mathbf{Z}/p^r \xrightarrow{\cdot p} \mathbf{Z}/p^r \xrightarrow{\rho} \mathbf{Z}/p \longrightarrow 0$$

where the first nonzero group is $\mathbf{Z}/p = H^n(C;\mathbf{Z}/p)$. The chain complex $E_1^*(C)$ is

$$0 \longrightarrow \mathbf{Z}/p \xrightarrow{0} \mathbf{Z}/p \longrightarrow 0$$

concentrated in degrees n and $n+1$, so $E_2^*(C) = E_1^*(C)$. The derived triangle

$$0 \xrightarrow{\beta} 0 \xrightarrow{\cdot p} 0 \xrightarrow{\rho} \mathbf{Z}/p \xrightarrow{\beta} \mathbf{Z}/p^{r-1} \xrightarrow{\cdot p} \mathbf{Z}/p^{r-1} \xrightarrow{\rho} \mathbf{Z}/p \longrightarrow 0$$

is the Bockstein long exact sequence for the elementary chain complex $0 \longleftarrow \mathbf{Z} \xleftarrow{p^{r-1}} \mathbf{Z} \longleftarrow 0$.

5.8. Lemma. The derived triangle of an exact triangle is exact.

Proof. This follows from Example 5.7 because C is quasi-isomorphic to direct sum elementary chain complexes and these constructions preserve direct sum. Alternatively, this is proved by a diagram chase in the exact triangle. $\square$

We can therefore continue to derive the exact triangles and obtain a sequence of chain complexes

$$(E_1^*(C), d_1), (E_2^*(C), d_2), \dots, (E_r^*(C), d_r), (E_{r+1}^*(C), d_{r+1}), \dots$$

where $E_{r+1}(C) = H(E_r^*(C), d_r)$. Such a sequence of chain complexes is called a *spectral sequence* and this particular one is the mod p *Bockstein spectral sequence*.

Since each homology group $H_n(C)$ is finitely generated, Example 5.7 implies that for each n there is an r such that $E_{r+1}^n(C) = E_{r+2}^n(C) = \cdots$. Let $E_\infty(C)$, the *limit* of the Bockstein spectral sequence, be the graded abelian group which in degree n is $E_\infty^n(C) = E_r^n(C)$ for $r \gg 0$. Table 1 shows the Bockstein spectral sequences for the elementary chain complexes and we conclude that

- Each $\mathbf{Z}$-summand in $H^n(C)$ contributes one $\mathbf{Z}/p$-summand to $E_1^n(C), E_2^n(C), \dots, E_\infty^n(C)$.
- Each $\mathbf{Z}/p$-summand in $H^{n+1}(C)$ contributes one $\mathbf{Z}/p$-summand to $E_1^n(C) = H^n(C;\mathbf{Z}/p)$ and one $\mathbf{Z}/p$-summand to $E_1^{n+1}(C) = H^{n+1}(C;\mathbf{Z}/p)$ that are connected by a nonzero $d_1 = \beta$-differential so that they disappear in $E_2^*(C)$ and $E_\infty^*(C)$.
- Each $\mathbf{Z}/p^r$-summand in $H^{n+1}(C)$ contributes one $\mathbf{Z}/p$-summand to $E_1^n(C), \dots, E_r^n(C)$ and one $\mathbf{Z}/p$-summand to $E_1^{n+1}(C), \dots, E_r^{n+1}(C)$ that are connected by a nonzero d_r-differential so that they disappear in $E_{r+1}^*(C)$ and $E_\infty^*(C)$.

C and $H^*(C)$	$E_r(C)$
$0 \longleftarrow \mathbf{Z} \longleftarrow 0 \longleftarrow 0$ $0 \quad \mathbf{Z} \quad 0 \quad 0$	$E_1^*(C) = (\ 0 \longrightarrow \mathbf{Z}/p \longrightarrow 0 \longrightarrow 0\ )$ $E_1^*(C) = E_2^*(C) = \cdots$ $E_\infty^*(C) = (\ 0 \quad \mathbf{Z}/p \quad 0 \quad 0\ )$
$0 \longleftarrow \mathbf{Z} \xleftarrow{p} \mathbf{Z} \longleftarrow 0$ $0 \quad 0 \quad \mathbf{Z}/p \quad 0$	$E_1^*(C) = (\ 0 \longrightarrow \mathbf{Z}/p \xrightarrow{\cong} \mathbf{Z}/p \longrightarrow 0$ $0 = E_2^*(C) = E_3^*(C) = \cdots$ $E_\infty^*(C) = (\ 0 \quad 0 \quad 0 \quad 0\ )$
$0 \longleftarrow \mathbf{Z} \xleftarrow{p^r} \mathbf{Z} \longleftarrow 0$ $0 \quad 0 \quad \mathbf{Z}/p^r \quad 0$	$E_1^*(C) = (\ 0 \longrightarrow \mathbf{Z}/p \xrightarrow{0} \mathbf{Z}/p \longrightarrow 0\ ) = E_2^*(C) = \cdots = E_{r-1}^*(C)$ $E_r^*(C) = (\ 0 \longrightarrow \mathbf{Z}/p \xrightarrow{\cong} \mathbf{Z}/p \longrightarrow 0\ )$ $0 = E_{r+1}^*(C) = E_{r+2}^*(C) = \cdots$ $E_\infty^*(C) = (\ 0 \quad 0 \quad 0 \quad 0\ )$
$0 \longleftarrow \mathbf{Z} \xleftarrow{q^r} \mathbf{Z} \longleftarrow 0$ $0 \quad 0 \quad \mathbf{Z}/q^r \quad 0$	$0 = E_1^*(C) = E_2^*(C) = \cdots = E_\infty^*(C)$ $E_\infty^*(C) = (\ 0 \quad 0 \quad 0 \quad 0\ )$

Table 1. Bockstein spectral sequences for elementary chain complexes

- $\mathbf{Z}/q^r$-summands in $H^n(C)$ with $q \neq p$ do not contribute to the (mod p) Bockstein spectral sequence.

5.9. Theorem (The mod p Bockstein spectral sequence). *Let C be a chain complex of free abelian groups with finitely generated homology $H_n(C)$ in each degree. There is a spectral sequence of $\mathbf{Z}/p$-vector spaces*

$$H^*(C;\mathbf{Z}/p) = E_1^*(C) \Longrightarrow E_\infty^*(C) = \mathbf{Z}/p \otimes \big(H^*(C)/\text{torsion}\big)$$

If $E_{r+1}^*(C) = E_\infty^*(C)$ then the p-torsion summands in $H^*(C)$ are among $\mathbf{Z}/p, \ldots, \mathbf{Z}/p^r$. The differential d_r of the chain complex $E_r(C)$ points to the $\mathbf{Z}/p^r$-summands of $H^*(C)$ in that the number of $\mathbf{Z}/p^r$-summands of $H^{n+1}(C)$ equals the number of $\mathbf{Z}/p$ summands of $d_r(E_r^n(C)) \subset E_r^{n+1}(C)$.

Proof. Since $\rho_{(1)} = \rho\colon H^*(C) \to E_1^*(C)$ factors through the subgroup $\ker\beta_1 = \ker\beta$ of $E_1^*(C) = H^*(C;\mathbf{Z}/p)$ there is an induced map

$$\begin{tikzcd}
H^*(C) \arrow[r, "\rho_1"] \arrow[ddr, dotted, "\rho_{(2)}"'] & \ker\beta_1 \arrow[r, hook] \arrow[d, hook] & E_1^*(C) \\
& \ker d_1 \arrow[d] & \\
& E_2^*(C) &
\end{tikzcd}$$

of $H^*(C)$ into $E_2^*(C)$. This homomorphism, $\rho_{(2)}\colon H^*(C) \to E_2^*(C)$ factors through the subgroup $\ker\beta_2$ of $E_2^*(C)$ because β_2 is induced by from β. In this way we obtain maps $\rho_{(r)}\colon H^*(C) \to E_r^*(C)$ and $\rho_{(\infty)}\colon H^*(C) \to E_\infty^*(C)$. The map $\rho_{(r)}\colon H^*(C) \to E_r^*(C)$ vanishes on p-torsion summands $\mathbf{Z}/p, \ldots, \mathbf{Z}/p^{r-1}$ and is nonzero on $\mathbf{Z}/p^r, \mathbf{Z}/p^{r+1}, \ldots, \mathbf{Z}$. The surjective map $\rho_{(\infty)}\colon H^*(C) \to E_\infty^*(C)$ vanishes on all torsion and is nonzero on the free $\mathbf{Z}$-sumands. $\square$

5.10. Example. $H^*(\mathbf{R}P^\infty;\mathbf{Z}/2) = \mathbf{Z}/2[x]$ with $x \in H^1(\mathbf{R}P^\infty;\mathbf{Z}/2)$ and $\beta x = x^2$. (That $\beta x = x^2$ follows from Table 1 because $H^2(\mathbf{R}P^\infty;\mathbf{Z}) = \mathbf{Z}/2$.) Therefore (5.3), $\beta x^{2k+1} = x^{2k+2}$ and $\beta x^{2k} = 0$. The chain complexes E_1^* and $E_2^* = E_3^* = \cdots$ are

$$1 \xrightarrow{0} x \xrightarrow{1} x^2 \xrightarrow{0} x^3 \xrightarrow{1} \cdots \longrightarrow x^{2k-1} \xrightarrow{1} x^{2k} \xrightarrow{0} x^{2k+1} \longrightarrow \cdots$$

$$1 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

This shows that $\widetilde{H}^*(\mathbf{R}P^\infty;\mathbf{Z})$ is $\mathbf{Z}/2$ in every even degree and 0 in every odd degree so that reduction mod 2 takes $\widetilde{H}^*(\mathbf{R}P^\infty;\mathbf{Z})$ isomorphically to $d_1\widetilde{H}^*(\mathbf{R}P^\infty;\mathbf{Z}/2) = \{x^2, x^4, \ldots\}$. We conclude that $H^*(\mathbf{R}P^\infty;\mathbf{Z}) = \mathbf{Z}[y]/(2y)$ where $y \in H^2(\mathbf{R}P^\infty;\mathbf{Z})$ is the 2-dimensional class with $\rho y = x$. (In fact, $H^*(L^\infty(p);\mathbf{Z}) = \mathbf{Z}[y]/(py)$, $|y| = 2$, for any prime p.)

The E_1-page of the Bockstein spectral sequence for $\mathbf{R}P^\infty \times \mathbf{R}P^\infty$ is $E_1^*(\mathbf{R}P^\infty) \otimes E_1^*(\mathbf{R}P^\infty)$ so that again the Bockstein spectral sequence collapses at E_2^*. The image of d_1 is generated by $d_1E_1^1 = \{x_1^2, x_2^2, x_1^2x_2 + x_1x_2^2\}$.

5.11. Example. The mod 2 cohomology of the compact Lie group G_2, the automorphism group of the Cayley algebra, is $H^*(\mathrm{G}_2;\mathbf{Z}/2) = \mathbf{Z}/2[x_3,x_5]/(x_3^4, x_5^2) = \mathbf{Z}/2[x_3]/(x_3^4) \otimes E(x_5)$ with Bockstein $d_1 = \beta$ given by $d_1x_5 = x_3^2$ [**13**, Appendix A]. The Bockstein chain complex $E_1^*(\mathrm{G}_2)$ is

$$1 \longrightarrow 0 \longrightarrow 0 \longrightarrow x_3 \longrightarrow 0 \longrightarrow x_5 \longrightarrow x_3^2 \longrightarrow 0 \longrightarrow x_3x_5 \longrightarrow x_3^3 \longrightarrow 0 \longrightarrow x_3^2x_5 \longrightarrow 0 \longrightarrow 0 \longrightarrow x_3^3x_5$$

Since $E_2^*(\mathrm{G}_2) = \mathbf{Z}/2\{x_3, x_3^2x_5, x_3^3x_5\}$ the differential $d_2 = 0$ for dimensional reasons so $E_2^*(\mathrm{G}_2) = E_\infty^*(\mathrm{G}_2)$. Thus $H^*(\mathrm{G}_2;\mathbf{Z})$ contains $\mathbf{Z}/2$-torsion, in degrees 6 and 9, but no $\mathbf{Z}/4$-torsion. The rational cohomology algebra $H^*(\mathrm{G}_2;\mathbf{Q})$ is concentrated in degrees 0, 3, 11 and 14. Since G_2 is a compact orientable manifold, its rational cohomology algebra must must satisfy Poincaré duality (4.68) and so we conclude that $H^*(\mathrm{G}_2;\mathbf{Q}) = E(y_3, y_{11})$ is an exterior algebra generated by a class in degree 3 and a class in degree 11. It is possible to determine the cohomology algebra $H^*(\mathrm{G}_2;\mathbf{Z}_{(2)})$ with coefficients in $\mathbf{Z}$ localized at the prime ideal (2).

5.12. Proposition. If an element of $H^*(C;\mathbf{Z}/p)$ is nontrivial in $E_r^*(C)$ then it lies in the image of $H^*(C;\mathbf{Z}/p^r) \to H^*(C;\mathbf{Z}/p)$. The differential d_r of $E_r^*(C)$ and the Bockstein of the short exact sequence

$$0 \longrightarrow \mathbf{Z}/p \longrightarrow \mathbf{Z}/p^{r+1} \longrightarrow \mathbf{Z}/p^r \longrightarrow 0$$

are related.

PROOF. The maps of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z}/p \arrow[r, "p^r"] & \mathbf{Z}/p^{r+1} \arrow[r] & \mathbf{Z}/p^r \arrow[r] & 0 \\
0 \arrow[r] & \mathbf{Z} \arrow[r, "p^r"] \arrow[u] \arrow[d, "p^{r-1}"'] & \mathbf{Z} \arrow[r] \arrow[u] \arrow[d, equal] & \mathbf{Z}/p^r \arrow[r] \arrow[u, equal] \arrow[d] & 0 \\
0 \arrow[r] & \mathbf{Z} \arrow[r, "p"] & \mathbf{Z} \arrow[r] & \mathbf{Z}/p \arrow[r] & 0
\end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd}
H^n(C;\mathbf{Z}/p^r) \arrow[r, "\beta"] & H^{n+1}(C;\mathbf{Z}/p) \\
H^n(C;\mathbf{Z}/p^r) \arrow[r, "\beta"] \arrow[u, equal] \arrow[d] & H^{n+1}(C;\mathbf{Z}) \arrow[u] \arrow[d, "p^{r-1}"] \\
H^n(C;\mathbf{Z}/p) \arrow[r, "\beta"] & H^{n+1}(C;\mathbf{Z})
\end{tikzcd}$$

of Bockstein homomorphisms. If $x \in H^n(C;\mathbf{Z}/p)$ is nontrivial in $E_r^n(C)$ then x comes from $y \in H^n(C;\mathbf{Z}/p^r)$ and $d_r x = \beta y \in H^{n+1}(C;\mathbf{Z}/p)$. $\square$

Why is it called a spectral sequence?

Probably the best account of the Bockstein spectral sequence can be found here.

5.13. EXAMPLE. When p is odd, the lense spaces $L^{2n+1}(p)$ and $L^{2n+1}(p^2)$ have isomorphic modulo p cohomology algebras but they have different Bockstein homomorphisms as $\beta \neq 0$ on $H^1(L^{2n+1}(p);\mathbf{Z}/p)$ and $\beta = 0$ on $H^1(L^{2n+1}(p^2);\mathbf{Z}/p)$.

5.14. Reduction mod p^m. In fact, $H^*(\mathbf{R}P^\infty;\mathbf{Z}/2) \cong H^*(\mathbf{R}P^\infty;\mathbf{Z}/2^m)$ for all $m \geq 1$. This follows from Table 2 which shows that if $\widetilde{H}_*(C)$ or $\widetilde{H}^*(C)$ consist of p-torsion of type $\mathbf{Z}/p,\ldots,\mathbf{Z}/p^m$ only, then $H^*(C;\mathbf{Z}/p^m) \cong H^*(C;\mathbf{Z}/p^{m+1})$. The results of the table are easily verified using the exact sequences

$$0 \longrightarrow \mathbf{Z}/p^{\min\{m,r\}} \longrightarrow \mathbf{Z}/p^m \xrightarrow{\cdot p^r} \mathbf{Z}/p^m \longrightarrow \mathbf{Z}/p^{\min\{m,r\}} \longrightarrow 0$$

of cyclic groups.

$H_n(C)$	$H^n(C;\mathbf{Z}/p^m)$ and $H^{n+1}(C;\mathbf{Z}/p^m)$, $m<r$	$H^n(C;\mathbf{Z}/p^m)$ and $H^{n+1}(C;\mathbf{Z}/p^m)$, $m \geq r$
$\mathbf{Z}$	$\mathbf{Z}/p^m$ $\quad 0$	$\mathbf{Z}/p^m$ $\quad 0$
$\mathbf{Z}/p^r$	$\mathbf{Z}/p^m$ $\quad \mathbf{Z}/p^m$	$\mathbf{Z}/p^r$ $\quad \mathbf{Z}/p^r$
$\mathbf{Z}/q^r$, $q \neq p$	0 $\quad$ 0	0 $\quad$ 0

TABLE 2. Reduction modulo different powers of p

2. Steenrod operations

The Steenrod square Sq^i is a cohomology operation of type $(n,\mathbf{Z}/2;n+i,\mathbf{Z}/2)$ and the Steenrod power operation P^i is a cohomology operation of type $(n,\mathbf{Z}/p;n+2i(p-1),\mathbf{Z}/p)$ where p is an odd prime. We shall here consider a few applications of Steenrod operations. See [**20**] for more information.

5.15. LEMMA. Let $u \in H^1(X;\mathbf{Z}/2)$ be an element of degree 1. Then

$$\mathrm{Sq}^i u^n = \binom{n}{i} u^{n+i}$$

Let p be an odd prime and $v \in H^2(X;\mathbf{Z}/p)$ an element of degree 2. Then

$$\mathrm{P}^i v^n = \binom{n}{i} v^{n+i(p-1)}$$

PROOF. Use induction and the Cartan formula. □

If we introduce the total operations $\mathrm{Sq} = \sum_{i=0}^{\infty} \mathrm{Sq}^i$ and $\mathrm{P} = \sum_{i=0}^{\infty} \mathrm{P}^i$, the above formulas read

$$\mathrm{Sq}(u^n) = \sum_{i=0}^{n} \binom{n}{i} u^{n+i}, \qquad \mathrm{P}(v^n) = \sum_{i=0}^{n} \binom{n}{i} v^{n+i(p-1)}$$

when u has degree 1 and v degree 2.

The binomial coefficients are to be evaluated modulo p. They are best computed by the formula

$$\binom{\sum n_k p^k}{\sum i_k p^k} \equiv \prod \binom{n_k}{i_k} \bmod p$$

using the p-adic expansions of n and i. By convention, $\binom{n}{0} = 1$ for all $n \geq 0$ and $\binom{n}{i} = 0$ if i is negative.

5.16. EXAMPLE (Steenrod operations in $H^*(\mathbf{R}P^\infty;\mathbf{Z}/2)$ and $H^*(L^\infty(p);\mathbf{Z}/p)$). For u in the first cohomology group $H^1(\mathbf{R}P^\infty;\mathbf{Z}/2)$ we find that

$$\begin{array}{llll} \mathrm{Sq}(u) = u + u^2, & \mathrm{Sq}(u^3) = u^3 + u^4 + u^5 + u^6, & \mathrm{Sq}(u^7) = u^7 + u^8 + \cdots + u^{14}, & \ldots \\ \mathrm{Sq}(u) = u + u^2, & \mathrm{Sq}(u^2) = u^2 + u^4, & \mathrm{Sq}(u^4) = u^4 + u^8, & \ldots \end{array}$$

and, in general,

$$\mathrm{Sq}(u^{2^k-1}) = u^{2^k-1} + u^{2^k} + \cdots + u^{2(2^k-1)}, \qquad \mathrm{Sq}(u^{2^k}) = u^{2^k} + u^{2^{k+1}}$$

This shows that all powers of u are connected by Steenrod squares, $H^*(\mathbf{R}P^\infty;\mathbf{Z}/2) = \mathcal{A}_2 u$.

The situation is different when p is odd. For $v = \beta u$ in the second cohomology group $H^2(L^\infty(p);\mathbf{Z}/p)$ we find that

$$\mathrm{P}(v^{p^k}) = v^{p^k} + v^{p^{k+1}}$$

and that the even part of $H^*(L^\infty(p);\mathbf{Z}/p)$ is the sum of the $p-1$ $\mathcal{A}_p$-modules generated by $v, v^2, \ldots, v^{p-1}$. Does there exist a space realizing these modules?

5.17. Stable homotopy groups of spheres. $\pi_k^s = \operatorname{colim}_n \pi_{n+k}(S^n)$. Steenrod operations imply that $\pi_1^s, \pi_3^s, \pi_7^s \neq 0$.

5.18. Splittings of modules and spaces. Let M be a module over some ring R. The existence of a direct summand M_1 of M is equivalent to the existence of an R-endomorphism e_1 of M such that $e_1^2 = e_1$ and $e_1 M = M_1$. The idempotent e_1 is called the projection of M onto M_1. (The idempotent $1 - e_1$ is the projection of M onto a complement to M_1.)

5.19. LEMMA (Realizing direct summands of $\widetilde{H}_*(X)$). Let X be a space. Suppose that $\widetilde{H}_*(X)$ admits a direct summand, M_1, and that the projection of $\widetilde{H}_*(X)$ onto M_1 is induced by a self-map e_1 of X. Then there exists a space X_1 such that $\widetilde{H}_*(X_1) = M_1$ and a map $X \to X_1$ inducing the projection $\widetilde{H}(X) \to M_1$.

PROOF. $(e_1)_* H_*(X) = H_*(\mathrm{Tel}(e_1))$ (4.80, 4.74). □

The existence of a direct sum decomposition

$$M = M_1 \oplus \cdots \oplus M_t$$

is equivalent to [**5**, §11.B] the existence of a set of R-endomorphisms, $e_1, \ldots, e_t$, such that

$$1 = e_1 + \cdots + e_t, \qquad e_i^2 = e_i, \quad 1 \leq i \leq t, \qquad e_i e_j = 0, \quad i \neq j, \tag{5.20}$$

and

$$M_i = e_i M, \quad 1 \leq i \leq k$$

The R-endomorphism $e_i \in \mathrm{End}_R(M)$ is called the *projection* of M onto M_i. Equation (5.20) says that that the identity $1 \in \mathrm{End}_R(M)$ is a sum of orthogonal idempotents.

5.21. Lemma (Realizing direct sum decompositions of $\widetilde{H}_*(\Sigma X)$). Let ΣX be the suspension of a based space X. Suppose that $\widetilde{H}_*(\Sigma X)$ admits a direct sum decomposition

$$\widetilde{H}_*(\Sigma X) = M_1 \oplus \cdots \oplus M_t$$

and that the projection of $\widetilde{H}_*(\Sigma X)$ onto M_i is induced by some based self-map e_i of ΣX, $1 \leq i \leq t$. Then there exist spaces $X_1, \ldots, X_t$ such that $\widetilde{H}_*(X_i) = M_i$ and a homology isomorphism

$$\Sigma X \to X_1 \vee \ldots \vee X_s$$

inducing the direct sum decomposition of $\widetilde{H}_*(\Sigma X)$.

Proof. Let X_i be the telescope of the self-map e_i of ΣX so that $\Sigma X \to X_i$ induces the projection $\widetilde{H}_*(X) \twoheadrightarrow \widetilde{H}_*(X_i) = M_i$ (5.19). The map

$$\Sigma X \to \Sigma X \vee \ldots \vee \Sigma X \to X_1 \vee \ldots \vee X_s$$

is an isomorphism on reduced homology because $\widetilde{H}_*(\Sigma X) = \bigoplus M_i = \bigoplus \widetilde{H}_*(X_i) = \widetilde{H}_*(\bigvee X_i)$. □

5.22. Corollary. Let p be a prime and $\Sigma L^\infty(p)$ the suspension of the infinite lense space. There exists an $H_*\mathbf{Z}$-equivalence

$$\Sigma L^\infty(p) \to X_1 \vee \ldots \vee X_{p-1}$$

where X_j, $1 \leq j \leq p-1$, is a connected space such that

$$\widetilde{H}_{2j}(X_i; \mathbf{Z}) = \begin{cases} \mathbf{Z}/p & j \equiv i \bmod p-1 \\ 0 & \text{otherwise} \end{cases}$$

It is not possible to split X_j further.

Proof. The reduced homology groups of $\Sigma L^\infty(p)$ are concentrated in even degrees and $\widetilde{H}_{2i}(\Sigma L^\infty(p); \mathbf{Z}) = \mathbf{Z}/p = \widetilde{H}_{2i}(\Sigma L^\infty(p); \mathbf{Z}/p)$ for all $i = 1, 2, \ldots$.

Let A be a self-map of $L^\infty(p)$ and d an integer such that A_* is multiplication by d on $H_1(L^\infty(p); \mathbf{Z}/p) = \mathbf{Z}/p$. Such an A exists for any integer d. Then A^* is multiplication by d^i on $H^{2i}(L^\infty(p); \mathbf{Z}/p) = \mathrm{Ext}(H_{2i-1}(L^\infty(p); \mathbf{Z}), \mathbf{Z}/p)$, A_* is multiplication by d^i on $H_{2i-1}(L^\infty(p); \mathbf{Z})$, and $(\Sigma A)_*$ is multiplication by d^i on $H_{2i}(\Sigma L^\infty(p); \mathbf{Z})$. In particular, $H_{2i}(A; \mathbf{Z})\colon H_{2i}(\Sigma L^\infty(p); \mathbf{Z}) \to H_{2i}(\Sigma L^\infty(p); \mathbf{Z})$ only depends on the value of $i \bmod p-1$.

The group $\mathrm{Aut}(\mathbf{Z}/p)$ of automorphisms of $\mathbf{Z}/p$ acts on $\widetilde{H}_*(\Sigma L^\infty(p); \mathbf{Z}/p))$: Let a be a generator of the cylic group $\mathrm{Aut}(\mathbf{Z}/p) = \langle a \rangle$ of order $p-1$. The action of a is $\widetilde{H}_*(\Sigma A)$ where A is a self-map of $L^\infty(p)$ such that $A_* = a$ on $H_1(L^\infty(p); \mathbf{Z}/p) = \mathbf{Z}/p$. We can express this by saying that $\widetilde{H}_*(\Sigma L; \mathbf{Z}/p)$ is an $\mathbf{Z}/p[\mathrm{Aut}(\mathbf{Z}/p)]$-module. We have just seen that the 1-dimensional $\mathbf{Z}/p$-representations of $\mathrm{Aut}(\mathbf{Z}/p)$ on $\widetilde{H}_{2i}(L^\infty(p); \mathbf{Z})$ only depend on the value of i modulo $p-1$ and that all $p-1$ irreducible $\mathbf{Z}/p$-representations of $\mathrm{Aut}(\mathbf{Z}/p)$ occur on $\widetilde{H}_{2i}(L^\infty(p); \mathbf{Z})$, $1 \leq i \leq p-1$. Let e_i be the idempotent element of the group ring associated to to the irreducible representation on $\widetilde{H}_{2i}(L^\infty(p); \mathbf{Z})$. (In this case, multiplication by a on the group ring is diagonalizable and e_i an eigenvector with eigenvalue a^i. Consult [**5**, §25-§27] for representation theory.) The e_i satisfy (5.20) and the action of e_i on $\widetilde{H}_*(\Sigma L^\infty(p); \mathbf{Z})$ equals $(E_i)_*$ for a self-map E_i of $L^\infty(p)$ (if $e_i = \sum r_i a^i$, take $E_i = \sum r_i \Sigma A^i$). According to 5.21, the suspension $\Sigma L^\infty(p)$ is $H_*\mathbf{Z}$-isomorphic to wedge sum of $p-1$ spaces $X_1, \ldots, X_{p-1}$ such that $\widetilde{H}_*(X_i)$ is concentrated in even degrees $2j$ for $j \equiv i \bmod p-1$ all carrying the same $\mathrm{Aut}(\mathbf{Z}/p)$-representation.

The space X_i does not split further because all elements of $H^*(X_i; \mathbf{Z}/p)$, which is concentrated in degrees $2j-1$ and $2j$ for $j \equiv i \bmod p-1$, are connected by Steenrod operations. For instance, the Bockstein is an isomorphism from $H^{2j-1}(X_i; \mathbf{Z}/p)$ to $H^{2j}(X_i; \mathbf{Z}/p)$ because $H_{2j}(X_i; \mathbf{Z}) = \mathbf{Z}/p$ (Table 1). □

The $\mathcal{A}_p$-module $H^*(L^\infty(p); \mathbf{Z}/p)$ is injective.

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