Hatcher §3.2

Ex 3.2.1

Let $q\colon M_g \to \bigvee M_1$ be the quotient map of M_g to a wedge of g tori $M_1 = S^1 \times S^1$. We know [1, 3.13] that $\bigoplus \tilde{H}_*(M_1) \cong \tilde{H}_*(\bigvee M_1)$. The induced map $H_1(f)\colon H_1(M_g) \to H_1(\bigvee M_1)$ is an isomorphism and $H_2(f)\colon \mathbf{Z} \cong H_2(M_g) \to H_2(\bigvee M_1) \cong \bigoplus \mathbf{Z}$ is the diagonal map. (Use local degree [1, 2.30] to see that each summand $H_2(M_g) \to H_2(M_1)$ is an isomorphism.)

The induced ring map $H^*(f)\colon \prod \tilde{H}^*(M_1) \cong \tilde{H}^*(\bigvee M_1) \to H^*(M_g)$ in cohomology is an isomorphism in degree 1 and in degree 2 the restriction $H^2(M_1) \to H^2(M_g)$ to any of the g summands is an isomorphism.

Let $\alpha_i, \beta_i \in H^1(M_g)$, $1 \leq i \leq g$, and be the standard generators [1, 3.8]. We may also view α_i, β_i as elements of the ith summand $H^1(M_1) = H^1(S^1 \times S^1)$. Since $\alpha_i \cup \beta_i$ generate $H^2(M_1)$, this cup product taken in $H^1(M_g)$ will generate $H^2(M_g)$. In view of the general property that $\alpha^2 = -\alpha^2$ when α has odd degree [1, 3.14], ${\alpha_i}^2 = 0 = {\beta_i}^2$. In view of the ring isomorphism $\tilde{H}^*(\bigvee X_i) \cong \prod \tilde{H}^*(X_i)$ [1, 3.13], the cup products $\alpha_i \cup \alpha_j = 0 = \alpha_i \cup \beta_j$ when $i \neq j$.

Ex 3.2.3

(a) Use the relations in the cohomology algebras.
(b) Show that the induced map $\pi_1(\mathbf{R}P^{n-1}) \to \pi_1(\mathbf{R}P^n)$ is non-zero; look at the proof of [1, 3.20].

Ex 3.2.7

See [1, p 215]. The two spaces have isomorphic homology and cohomology groups but not isomorphic cohomology rings as

$$H^1(\mathbf{R}P^3;\mathbf{Z}/2) \cup H^2(\mathbf{R}P^3;\mathbf{Z}/2) \neq 0, \quad H^1(\mathbf{R}P^2 \vee S^3;\mathbf{Z}/2) \cup H^2(\mathbf{R}P^2 \vee S^3;\mathbf{Z}/2) = 0$$

Ex 3.2.14

The identification $\mathbf{R}^{2n+2} = \mathbf{C}^{n+1}$ induces a quotient map

$$\begin{gathered} \mathbf{R}P^{2n+1} = S^{2n+1}/\{\pm 1\} \xrightarrow{q} S^{2n+1}/S^1 = \mathbf{C}P^n \\ [x_0 : y_0 : \cdots : x_n : y_n] \to [x_0 + iy_0 : \cdots : x_n + iy_n] \end{gathered} \tag{1}$$

where we represent points of $\mathbf{R}P^{2n+1}$ and $\mathbf{C}P^n$ as equivalence classes of non-zero vectors of $\mathbf{R}^{2n+2} \ni (x_0, y_0, \ldots, x_n, y_n) = (x_0 + iy_0, \ldots, x_n + iy_n) \in \mathbf{C}^{n+1}$. We shall also write $q\colon \mathbf{R}P^{2n} \to \mathbf{C}P^n$ for the restriction of (1) to $\mathbf{R}P^{2n}$. This map is cellular since it takes the $(2n-1)$-skeleton, $\mathbf{R}P^{2n-1}$, of $\mathbf{R}P^{2n}$ into the $(2n-1)$ skeleton, $\mathbf{C}P^{n-1}$, of $\mathbf{C}P^n$. Since the composite

$$\begin{gathered} \mathbf{R}^{2n} \xrightarrow{\simeq} \mathbf{R}P^{2n} - \mathbf{R}P^{2n-1} \xrightarrow{q} \mathbf{C}P^n - \mathbf{C}P^{n-1} \xleftarrow{\simeq} \mathbf{C}^n \\ (x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \to [x_0 : y_0 : \cdots : x_{n-1} : y_{n-1} : 1 : 0] \to [x_0 + iy_0 : \cdots : x_{n-1} + iy_{n-1} : 1] \\ \leftarrow (x_0 + iy_0, \ldots, x_{n-1} + iy_{n-1}) \end{gathered}$$

is a homeorphism, the induced map $\mathbf{R}P^{2n}/\mathbf{R}P^{2n-1} = (\mathbf{R}P^{2n} - \mathbf{R}P^{2n-1})^+ \to (\mathbf{C}P^n - \mathbf{C}P^{n-1})^+ = \mathbf{C}P^n/\mathbf{C}P^{n-1}$ is also a homeomorphism. This describes the effect of q on the cellular (co)chain complexes.

In particular, $\mathbf{R}P^2/\mathbf{R}P^1$ is mapped homeomorphically to $\mathbf{C}P^1/\mathbf{C}P^0$ (where $\mathbf{C}P^0$ is a point) and since $R \cong H^2(\mathbf{R}P^2, \mathbf{R}P^1; \mathbf{R}) \to H^2(\mathbf{R}P^2; \mathbf{R}) \cong R/2R$ is surjective by cellular cohomology we see from

$$\begin{array}{ccc} H^2(\mathbf{R}P^2, \mathbf{R}P^1; \mathbf{R}) & \twoheadrightarrow & H^2(\mathbf{R}P^2; \mathbf{R}) \\ \cong \uparrow & & \uparrow \\ H^2(\mathbf{C}P^2, \mathbf{C}P^0; \mathbf{R}) & \xrightarrow{\cong} & H^2(\mathbf{C}P^2; R) \end{array}$$

that $R \cong H^2(\mathbf{C}P^1; R) \to H^2(\mathbf{R}P^2; R) \cong R/2R$ is surjective. The cup structure then implies that $q^*\colon H^*(\mathbf{C}P^n; R) \to H^*(\mathbf{R}P^{2n}; R)$ is surjective in even degrees.

Let $X = \mathbf{C}P^n \cup_q \mathbf{R}P^\infty$ be the adjunction space with data $\mathbf{C}P^n \xleftarrow{q} \mathbf{R}P^{2n} \xhookrightarrow{\iota} \mathbf{R}P^\infty$. In other words X is the push-out

(2)
$$\begin{array}{ccc} \mathbf{R}P^{2n} & \xhookrightarrow{\iota} & \mathbf{R}P^\infty \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \overline{q}} \\ \mathbf{C}P^n & \xhookrightarrow[\overline{\iota}]{} & X \end{array}$$

of the maps q and ι. Since the quotient spaces $X/\mathbf{C}P^n$ and $\mathbf{R}P^\infty/\mathbf{R}P^{2n}$ are homeomorphic, the induced map

$$H^*(X, \mathbf{C}P^n; R) \xrightarrow[\cong]{\overline{q}^*} H^*(\mathbf{R}P^\infty, \mathbf{R}P^{2n}; R)$$

is an isomorphism. Thus $H^*(X, \mathbf{C}P^n; R)$ is known [1, 3.12, 3.24].

The map $\overline{q}\colon (\mathbf{R}P^\infty, \mathbf{R}P^{2n}) \to (X, \mathbf{C}P^n)$ induces a morphism of the associated long exact sequences. Since $H^i(\mathbf{R}P^\infty, \mathbf{R}P^{2n}; R) \to H^i(\mathbf{R}P^\infty; R)$ is injective (use the cellular cochain complex for $\mathbf{R}P^\infty/\mathbf{R}P^{2n}$), we obtain the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & H^i(X, \mathbf{C}P^n; R) & \to & H^i(X; R) & \xrightarrow{\overline{\iota}^*} & H^i(\mathbf{C}P^n; R) & \to & 0 \\ & & \downarrow{\scriptstyle \cong} & & {\scriptstyle \overline{q}^*}\downarrow & & \downarrow & & \\ 0 & \to & H^i(\mathbf{R}P^\infty, \mathbf{R}P^{2n}; R) & \hookrightarrow & H^i(\mathbf{R}P^\infty; R) & \to & H^i(\mathbf{R}P^{2n}; R) & \to & 0 \end{array}$$

with exact rows. Since $H^i(X, \mathbf{C}P^n; R) = 0$ for $i \le 2n$ and $H^i(\mathbf{C}P^n; R) = 0 = H^i(\mathbf{R}P^{2n}; R)$ for $i > 2n$, it follows that

$$H^i(X; R) \cong \begin{cases} H^i(\mathbf{C}P^n; R) & i \le 2n \\ H^i(\mathbf{R}P^\infty; R) & i > 2n \end{cases}$$

where the isomorphisms are induced by the maps $\mathbf{C}P^n \xhookrightarrow{\overline{\iota}} X \xleftarrow{\overline{q}} \mathbf{R}P^\infty$ from (2).

Assume that $R = \mathbf{Z}$. The cohomology of X is concentrated in even dimensions and

$$H^{2i}(X; \mathbf{Z}) \cong \begin{cases} \mathbf{Z} & 0 \le i \le n \\ \mathbf{Z}/2 & i > n \end{cases}$$

is generated by α^i where $\alpha \in H^2(X) \cong H^2(\mathbf{C}P^n) \cong \mathbf{Z}$ is a generator. To see this, note that both $H^*(\mathbf{C}P^n; \mathbf{Z}) \cong \mathbf{Z}[\overline{\iota}^*\alpha]/(\overline{\iota}^*\alpha^{n+1})$ and $H^*(\mathbf{R}P^\infty; \mathbf{Z}) \cong \mathbf{Z}[\overline{q}^*\alpha]/(2\overline{q}^*\alpha)$ are generated by the images of α. In other words, we have an isomorphism

$$\mathbf{Z}[\alpha]/(2\alpha^{n+1}) \xrightarrow{\cong} H^*(X; \mathbf{Z})$$

of graded rings.

Assume that $R = \mathbf{Z}/2$. The cohomology of X is concentrated in even dimensions and in odd dimensions greater then $2n$. The even degree cohomology group $H^{2i}(X; \mathbf{Z}/2) \cong \mathbf{Z}/2$ is generated by α^i where $\alpha \in H^2(X; \mathbf{Z}/2) \cong H^2(\mathbf{C}P^n; \mathbf{Z}/2) \cong \mathbf{Z}/2$ is the non-zero element; the odd degree $> 2n$ cohomology group $H^{2n+1+2i}(X; \mathbf{Z}/2) \cong H^{2n+1+2i}(\mathbf{R}P^\infty; \mathbf{Z}/2) \cong \mathbf{Z}/2$ is generated by $\beta\alpha^i$ where $\beta \in H^{2n+1}(X; \mathbf{Z}/2) \cong H^{2n+1}(\mathbf{R}P^\infty; \mathbf{Z}/2) \cong \mathbf{Z}/2$ is the non-zero element, $i \ge 0$. We have $\alpha^{2n+1} = \beta^2 \in H^{4n+2}(X; \mathbf{Z}/2) \cong H^{4n+2}(\mathbf{R}P^\infty; \mathbf{Z}/2)$ for both classes are non-zero. In other words, we have an isomorphism

$$\mathbf{Z}/2[\alpha, \beta]/(\alpha^{2n+1} - \beta^2) \xrightarrow{\cong} H^*(X; \mathbf{Z}/2)$$

of graded rings.

Analogously, there is a quotient map $q\colon \mathbf{C}P^{2n} \to \mathbf{H}P^n$ inducing an isomorphism $q^*\colon H^4(\mathbf{H}P^1; \mathbf{Z}) \xrightarrow{\cong} H^4(\mathbf{C}P^2; \mathbf{Z})$. Form the adjunction space $Y = \mathbf{H}P^n \cup_q \mathbf{C}P^\infty$ as in the push-out diagram

(3)
$$\begin{array}{ccc} \mathbf{C}P^{2n} & \xhookrightarrow{\iota} & \mathbf{C}P^\infty \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \overline{q}} \\ \mathbf{H}P^n & \xhookrightarrow[\overline{\iota}]{} & Y \end{array}$$

As abovce, there are isomorphisms

$$H^i(X) \cong \begin{cases} H^i(\mathbf{H}P^n) & i \le 4n \\ H^i(\mathbf{C}P^\infty) & i > 4n \end{cases}$$

induced by the maps $\mathbf{H}P^n \overset{\bar{\iota}}{\hookrightarrow} Y \overset{\bar{q}}{\leftarrow} \mathbf{C}P^\infty$ from (3). The cohomology of Y is concentrated in degrees divisible by 4 and in even degrees greater then $4n$. The cohomology group $H^{4i}(Y;\mathbf{Z}) \cong \mathbf{Z}$ is generated by α^i where $\alpha \in H^4(Y;\mathbf{Z}) \cong H^4(\mathbf{H}P^n;\mathbf{Z})$ is a generator; the cohomology group $H^{4n+2+4i}(Y;\mathbf{Z}) \cong \mathbf{Z}$ is generated by $\beta\alpha^i$ where $\beta \in H^{4n+2}(Y;\mathbf{Z}) \cong H^{4n+2}(\mathbf{C}P^\infty;\mathbf{Z})$ is a generator. We have $\beta^2 = \pm\alpha^{2n+1}$ for the images of these two cohomology classes generate $H^{8n+4}(\mathbf{C}P^\infty;\mathbf{Z}) \cong \mathbf{Z}$. In other words, we have an isomorphism

$$\mathbf{Z}[\alpha,\beta]/(\alpha^{2n+1} - \beta^2) \xrightarrow{\cong} H^*(Y;\mathbf{Z})$$

of graded rings.

References

[1] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR **2002k:**55001