SINGULAR HOMOLOGY WITH CHAINS AS COEFFICIENTS

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Let X be a space and let $C(X)$ be the singular chain complex. Furthermore let K be a chain complex. The singular homology of X with coefficients in K is defined to be the graded group

$$H_*(X;K) = \bigoplus_{m\in\mathbb{Z}} H_m(C(X)\otimes_{\mathbb{Z}} K),$$

see Hatcher p. 273 for the definition of the tensor product chain complex.

Since $C_i(X)$ is a free abelian group for all i the homology groups $H_m(C(X)\otimes_{\mathbb{Z}} K)$ can be computed using Hatcher 3B.5, i.e.

$$(\dagger) \qquad H_m(C(X)\otimes_{\mathbb{Z}} K) = \left[\bigoplus_{i+j=m} (H_i(X)\otimes_{\mathbb{Z}} H_j(K))\right] \oplus \left[\bigoplus_{i+j=m} \mathrm{Tor}^1_{\mathbb{Z}}(H_i(X), H_{j-1}(K))\right].$$

Note that if K is the chain complex

$$0 \longrightarrow G \longrightarrow 0,$$

where G is in dimension 0, then $H_*(X;K) = H_*(X;G)$, i.e. the usual singular homology with coefficients in G.

If K is exact it is clear from $(\dagger)$ that $H_*(X;K) = 0$, the trivial case.

Homology of a point: $H_i(\{x\})$ is free for all i, hence the Tor terms in $(\dagger)$ vanishes. It follows that

$$\begin{aligned} H_m(C(\{x\})\otimes_{\mathbb{Z}} K) &= \bigoplus_{i+j=m} (H_i(\{x\})\otimes_{\mathbb{Z}} H_j(K)) \\ &= H_0(\{x\})\otimes_{\mathbb{Z}} H_m(K) \\ &= \mathbb{Z}\otimes_{\mathbb{Z}} H_m(K) \\ &= H_m(K), \end{aligned}$$

hence $H_*(\{x\};K) = \bigoplus_{m\in\mathbb{Z}} H_m(K)$.

Homology of a sphere: $H_i(S^n)$ is free for all i, hence the Tor terms in $(\dagger)$ vanishes. It follows that

$$\begin{aligned} H_m(C(S^n)\otimes_{\mathbb{Z}} K) &= \bigoplus_{i+j=m} (H_i(S^n)\otimes_{\mathbb{Z}} H_j(K)) \\ &= (H_0(S^n)\otimes_{\mathbb{Z}} H_m(K)) \oplus (H_n(S^n)\otimes_{\mathbb{Z}} H_{m-n}(K)) \\ &= (\mathbb{Z}\otimes_{\mathbb{Z}} H_m(K)) \oplus (\mathbb{Z}\otimes_{\mathbb{Z}} H_{m-n}(K)) \\ &= H_m(K)\oplus H_{m-n}(K), \end{aligned}$$

hence

$$H_*(S^n;K) = \bigoplus_{m\in\mathbb{Z}} (H_m(K)\oplus H_{m-n}(K)) = \bigoplus_{m\in\mathbb{Z}} H_m(K)^2.$$

So $H_*(S^n;K) = H_*(\{x\};K)$ iff $H_m(K) = 0$ for all m iff K is exact, i.e. the trivial case.