Riemannian Geometry

Contents

CHAPTER 1

Smooth manifolds

1. Tangent vectors, cotangent vectors and tensors

1.1. Lemma. *Let $F\colon M^m \to N^n$ be a smooth map. Suppose that $(x^1,\dots,x^m)$ are local coordinates on M and $(y^1,\dots,y^n)$ local coordinates on N. Then*

$$F_*(\tfrac{\partial}{\partial x^j}) = \tfrac{\partial(y^iF)}{\partial x^j}\tfrac{\partial}{\partial y^i}, \qquad 1\le j\le m, \tag{1.2}$$

$$F^*(dy^i) = \tfrac{\partial(y^iF)}{\partial x^j}dx^j = d(y^iF), \qquad 1\le i\le n \tag{1.3}$$

The $(n\times m)$-matrix $\left(\frac{\partial(y^iF)}{\partial x^j}\right)$ is the matrix for F_* and the $(m\times n)$-matrix $\left(\frac{\partial(y^iF)}{\partial x^j}\right)^t$ is the matrix for F^*. In other words,

$$\begin{pmatrix} F_*\frac{\partial}{\partial x^1} & \dots & F_*\frac{\partial}{\partial x^m}\end{pmatrix} = \begin{pmatrix}\frac{\partial}{\partial y^1} & \dots & \frac{\partial}{\partial y^n}\end{pmatrix}\begin{pmatrix}\frac{\partial(y^iF)}{\partial x^j}\end{pmatrix}$$

$$\begin{pmatrix} F^*dy^1\\ \vdots \\ F^*dy^n\end{pmatrix} = \begin{pmatrix}\frac{\partial(y^iF)}{\partial x^j}\end{pmatrix}\begin{pmatrix} dx^1\\ \vdots\\ dx^m\end{pmatrix}$$

Proof. $F^*(\mathrm{d}y^i)(\frac{\partial}{\partial x^j}) = \mathrm{d}y^i(F_*(\frac{\partial}{\partial x^j})) = F_*\left(\frac{\partial}{\partial x^j}\right)(y^i) = \frac{\partial y^iF}{\partial x^j}$. □

2. The tangent bundle of a smooth manifold

3. Vector fields, covector fields, tensor fields, n-forms

1.4. Proposition. *The differential $d\colon C^\infty(M)\to\mathcal{T}^1(M)$ is an $\mathbf{R}$-linear map satisfying the derivational rules*

$$d(uv) = (du)v + u(dv), \qquad d\left(\tfrac{u}{v}\right) = \frac{du}{v} - \frac{u\,dv}{v^2} = \frac{v\,du - u\,dv}{v^2}$$

for all smooth functions $u,v\in C^\infty(M)$ (where $v(p)\neq 0$ for all $p\in M$ in the last formula).

If $F\colon M\to N$ is a smooth map, then the diagram

$$\begin{tikzcd} C^\infty(M) \arrow[d, "d"'] & C^\infty(N) \arrow[l, "F^*"'] \arrow[d, "d"] \\ \mathcal{T}^1(M) & \mathcal{T}^1(N) \arrow[l, "F^*"'] \end{tikzcd}$$

commutes meaning that $F^*(du) = d(uF)$ for all $u\in C^\infty(N)$.

1.5. Example. Let $s\colon \mathbf{R}^{n+1}\to\mathbf{R}$ be the smooth map $s(x) = |x|^2 = \sum_{i=1}^{n+1}(x^i)^2$. Then $s^{-1}(1) = S^n \subset \mathbf{R}^n$ is the sphere of radius R. The differential $\mathrm{d}s = \sum 2x^i\mathrm{d}x^i$ is the linear map $\mathbf{R}^{n+1} = T_p\mathbf{R}^{n+1}\to\mathbf{R}$ given by $\mathrm{d}s_p(v) = \sum 2p^iv^i = 2\langle p,v\rangle$ with kernel $\ker \mathrm{d}s_p = p^\perp$ at any point $p\neq 0$. Let p be any point of S^n and $\iota_*\colon T_pS^n\to T_p\mathbf{R}^{n+1} = \mathbf{R}^{n+1}$ the linear map induced by the inclusion, ι. For any tangent vector $X\in T_pS^n$, $\mathrm{d}s(\iota_*X) = X(s\iota) = X(1) = 0$. Hence the tangent space at p is the kernel of $\mathrm{d}s_p$,

$$T_pS^n = p^\perp \subset \mathbf{R}^{n+1} = T_p\mathbf{R}^{n+1}$$

and the tangent bundle of S^n,

$$TS^n = \{(p,v)\subset S^n\times\mathbf{R}^{n+1} \mid \langle v,p\rangle = 0\}\subset S^n\times\mathbf{R}^{n+1}$$

is the vector bundle whose fibre over any $p \in S^n$ is $p^\perp$. A smooth vector field on S^n is a smooth map $v\colon S^n \to \mathbf{R}^{n+1}$ such that $v(p) \perp p$ for all $p \in S^n$. Show that any odd sphere has a vector field without zeros. Does S^2 admit a smooth vector field with no zeros? Can you describe $T\mathbf{R}P^n$? Can you describe TM if $M = f^{-1}(0)$ consists of the manifold solutions to the equation $f(x) = 0$ for some smooth map $f\colon \mathbf{R}^{n+1} \to \mathbf{R}$?

1.6. Definition. *A smooth* $\binom{k}{\ell}$*-tensor field on* M *is a smooth section of the tensor bundle* $T^k_\ell(M) \to M$.

Particular cases are

- $\mathcal{T}^0_0(M) = C^\infty(M)$
- $\mathcal{T}^0_1(M)$ consists of vector fields on M
- $\mathcal{T}^1_0(M)$ consists of 1-forms on M

Tensor fields admit

- $\mathcal{T}^{k_1}_{\ell_1}(M) \times \mathcal{T}^{k_1}_{\ell_1}(M) \xrightarrow{\otimes} \mathcal{T}^{k_1+k_2}_{\ell_1+\ell_2}(M)$ (tensor product of tensor fields)
- $\mathcal{T}^{k+1}_{\ell+1}(M) \xrightarrow{\mathrm{tr}} \mathcal{T}^k_\ell(M)$ (contraction of tensor fields)

1.7. Example. Let $\omega \in \mathcal{T}^1_0(M)$ be a 1-form and $X \in \mathcal{T}^0_1(M)$ a vector field on M. Then $X \otimes \omega \in \mathcal{T}^1_1(M)$ is $\binom{1}{1}$-tensor field with contraction $\mathrm{tr}(X \otimes \omega) = \omega(X)$ (5.7).

In a coordinate patch any $\binom{k}{\ell}$-tensor field A is (5.5) a $C^\infty(M)$-linear combination

$$A = A^{j_1\cdots j_\ell}_{i_1\cdots i_k} \partial_{j_1} \otimes \cdots \partial_{j_\ell} \otimes \mathrm{d}x^{i_1} \otimes \cdots \mathrm{d}x^{i_k} \tag{1.8}$$

of tensor products of the basis tensor fields and basis 1-forms. The smooth functions $A^{j_1\cdots j_\ell}_{i_1\cdots i_k}$ are called the *components* of the tensor field A.

The tensor algebra of M is the graded algebra $\mathcal{T}^*(M) = \sum_{k=0}^\infty \mathcal{T}^k(M)$ equipped with the tensor product $\mathcal{T}^r(M) \times \mathcal{T}^s(M) \xrightarrow{\otimes} \mathcal{T}^{r+s}(M)$. If $F\colon M \to N$ is a smooth map, $F^*\colon \mathcal{T}^k(N) \to \mathcal{T}^k(M)$ is the linear map given by $F^*(A)(X_1, \ldots, X_k) = A(F_*X_1, \ldots, F_*X_k)$ for all $A \in \mathcal{T}^k(N)$ and all smooth vector fields $X_1, \ldots, X_k$ on M.

1.9. Lemma. $\mathcal{T}^*(M)$ *is a graded algebra.* $F^*\colon \mathcal{T}^*(N) \to \mathcal{T}^*(M)$ *is a homomorphism of* $C^\infty(N)$*-algebras:* $F^*(a\omega \otimes \eta) = F^*(a)F^*(\omega) \otimes F^*(\eta)$.

CHAPTER 2

Riemannian manifolds

Riemann's idea was that in the infinitely small, on a scale much smaller than the the smallest particle, we do not know if Euclidean geometry is still in force. Therefore we better not assume that this is the case and instead open up for the possibility that in the infinitely small there may be other length functions, there may be other inner products on the tangent space! A Riemannian manifold is a smooth manifold equipped with inner product, which may or may not be the Euclidean inner product, on each tangent space.

1. Riemannian metric

2.1. Definition. *A Riemannian metric on a smooth manifold M is a symmetric, positive definite $\binom{2}{0}$-tensor $g \in \mathcal{T}_0^2(M)$.*

In a coordinate frame we may write

$$g = g_{ij} dx^i \otimes dx^j, \qquad g_{ij} = g(\partial_i, \partial_j)$$

This means that $g(U^i\partial_i, V^j\partial_j) = g_{ij}U^iV^j$ and in particular that

$$\langle \partial_i, \partial_i \rangle = g_{ii}, \qquad \langle \partial_i, \partial_j \rangle = g_{ij} \tag{2.1}$$

Note that there are only $\frac{1}{2}n(n+1)$ different functions here as $g_{ij} = g_{ji}$ by symmetry.

2.2. Remark. Since the metric tensor is symmetric, it is traditional to write it in a basis of symmetric tensors. The symmetrization of $\omega \otimes \eta$ is the tensor

$$\omega\eta = \frac{1}{2}(\omega \otimes \eta + \eta \otimes \omega)$$

Note that $\omega\eta = \eta\omega$ and that $\omega^2 = \omega\omega = \omega \otimes \omega$. Observe that

$$g = g_{ij}\mathrm{d}x^i\mathrm{d}x^j = 2\sum_{i=1}^n g_{ii}(\mathrm{d}x^i)^2 + 2\sum_{1\le i<j\le n} g_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

2.3. Lemma. *Let $F\colon M \to N$ be an immersion and g a Riemannian metric on N.*

1. *F^*g is a Riemannian metric on M.*
2. *If $g = g_{ij}\, dy^i\, dy^j$ in a coordinate frame on N, then*

$$F^*(g)|F^{-1}(U) = g_{ij}\, dy^i F\, dy^j F$$

Proof. It is a general fact that $F^*(g)$ is a smooth 2-form on M (1.9). $F^*(g)$ is symmetric because g is symmetric and it is positive definite because g is positive definite and F_* is injective on each fibre. $F^*(g_{ij}\mathrm{d}y^i\mathrm{d}y^j) \overset{(1.9)}{=} g_{ij}F^*\mathrm{d}y^iF^*\mathrm{d}y^j \overset{(1.4)}{=} g_{ij}\mathrm{d}y^iF\mathrm{d}y^jF$. □

For instance if $F\colon U \to M$ is a parameterization (an inverse chart) of an open subset of $M \subset \mathbf{R}^m$, then the pull-back of the induced metric on M is

$$F^*(\overline{g}) = F^*(\delta_{ij}dx^idx^j) = \delta_{ij}dF^idF^j = \sum_{i=1}^m (dF^i)^2 \tag{2.4}$$

These expressions are tensor fields living in the tensor algebra $\mathcal{T}^*(M)$ of M.

2.5. Example. (Graphs) Let $M \subset \mathbf{R}^n \times \mathbf{R}$ be the graph of the smooth function $f\colon M \to \mathbf{R}$. Then $s(x) = (x, f(x))$ is a diffeomorphism so that the Riemannian manifold $(M, \iota^*\overline{g}^{n+1})$ is isometric to $(\mathbf{R}^n, s^*\overline{g}^{n+1})$ where the metric $s^*\overline{g}^{n+1}$ is

$$s^*(\sum_{i=1}^{n+1}(dx^i)^2) = \sum_{i=1}^{n}(dx^i)^2 + \big(\frac{\partial f}{\partial x^i}dx^i\big)^2 = \sum_{i=1}^{n}(dx^i)^2 + \frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}dx^i dx^j$$
$$= \big(\delta_{ij} + \frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}\big)dx^i dx^j = \sum_{i=1}^{n}\big(1 + (\frac{\partial f}{\partial x^i})^2\big)(dx^i)^2 + 2\sum_{1\le i<j\le n}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}dx^i dx^j$$

2.6. Example. let S^2_+ be the upper hemisphere on $S^2 \subset \mathbf{R}^3$ considered as the graph of the function $f(x,y) = \sqrt{1-x^2-y^2}$ defined on the unit ball $B^2 \subset \mathbf{R}^2$. Then $(S^2_+, \iota^*\overline{g}^3)$ is isometric to $(B^2, s^*\overline{g}^3)$ where

$$s^*\overline{g}^3 = (dx)^2 + (dy)^2 + \left(\frac{-x}{\sqrt{1-x^2-y^2}}dx + \frac{-y}{\sqrt{1-x^2-y^2}}dy\right)^2$$
$$= \frac{1-y^2}{1-x^2-y^2}(dx)^2 + \frac{1-x^2}{1-x^2-y^2}(dy)^2 + \frac{2xy}{1-x^2-y^2}dxdy$$

This means (2.1) that

$$\langle \partial_x, \partial_x\rangle = \frac{1-y^2}{1-x^2-y^2}, \quad \langle \partial_y, \partial_y\rangle = \frac{1-x^2}{1-x^2-y^2}, \quad \langle \partial_x, \partial_y\rangle = \frac{xy}{1-x^2-y^2}$$

at the point $(x,y) \in B^2$. In this metric, the basis tangent vectors, ∂_x and ∂_y, are not orthogonal at any point of the unit ball away from the axes. If we consider the curve $\gamma(t) = (t,0)$, $-1 \le t \le 1$, then the tangent vector $\gamma_*(\frac{d}{dt}) = \frac{d(x\gamma)}{dt}\partial_x + \frac{d(y\gamma)}{dt}\partial_y = \partial_x$ so that the length of this curve is

$$L_{s^*(g)}(\gamma) = \int_{-1}^{+1} |\gamma_*(\frac{d}{dt})| dt = \int_{-1}^{+1}\sqrt{\frac{1}{1-t^2}}dt = \pi$$

What is the distance between $(0,-1/2)$ and $(1/2,0)$? What is the curve of shortest length between these two points?

2.7. Example. (Surface of revolution in $\mathbf{R}^3$)

2.8. Definition. *A smooth map $F\colon (M,g) \to (N,h)$ between two Riemannian manifolds is an* isometry *if $g = F^*h$; if $g(X,Y) = h(F_*X, F_*Y)$ for all tangent vectors $X, Y \in T_pM$, $p \in M$.*

Two Riemannian manifolds are isometric if we can deform one into the other by bending but not stretching. Is the upper unit hemisphere $S^2_+ \subset \mathbf{R}^3$ isometric to the open unit ball $B^2 \subset \mathbf{R}^2$? Certainly, the diffeomorphism $s\colon B^2 \to S^2_+$ from Example 2.6 is not an isometry as for instance $\langle \partial_x, \partial_x\rangle \neq 1$ or because the tangent vectors ∂_x and ∂_y are not orthogonal throughout any open (nonempty) subspace of B^2. But there are many other diffeomorphisms and maybe we could find one that preserves the metrics? To decide if this is the case we need to find invariants of Riemannian metrics. Is S^2_+ curved? And what does that mean? We need to develop some theory to answer these questions.

In order to decide if two given Riemannian manifolds are isometric we have to know have the metric tensor transforms under change of coordinate system? (Remember that the coordinate expression for a metric is an artefact of the coordinate system and not an intrinsic property of the metric.)

2.9. Lemma. *Let $\phi\colon \mathbf{R}^n \to M$ and $\psi\colon \mathbf{R}^n \to M$ be parameterizations of the same open subspace of M. If*

$$\phi^*(g) = a_{ij}\,dx^i dx^j, \qquad \psi^*(g) = b_{ij}\,dy^i dy^j$$

then

$$(a_{ij}) = \left(\frac{\partial y^i}{\partial x^j}\right)^t (b_{ij}) \left(\frac{\partial y^i}{\partial x^j}\right)$$

where $y = \psi\phi^{-1}$.

PROOF. Put $P = \left(\frac{\partial y^i}{\partial x^j}\right)$. Then (1.1)

$$y^* \begin{pmatrix} \mathrm{d}y^1 \\ \vdots \\ \mathrm{d}y^n \end{pmatrix} = P \begin{pmatrix} \mathrm{d}x^1 \\ \vdots \\ \mathrm{d}x^n \end{pmatrix}$$

Therefore

$$\begin{aligned} a_{ij}\mathrm{d}x^i\mathrm{d}x^j &= \phi^* g = (\psi y)^* g = y^*(\psi^* g) = y^*(b_{ij}\mathrm{d}y^i\mathrm{d}y^j) \\ &= y^*(\begin{pmatrix} \mathrm{d}y^1 & \dots & \mathrm{d}y^n \end{pmatrix}(b_{ij})\begin{pmatrix} \mathrm{d}y^1 \\ \vdots \\ \mathrm{d}y^n \end{pmatrix}) \\ &= \begin{pmatrix} \mathrm{d}x^1 & \dots & \mathrm{d}x^n \end{pmatrix} P^t (b_{ij}) P \begin{pmatrix} \mathrm{d}x^1 \\ \vdots \\ \mathrm{d}x^n \end{pmatrix} \end{aligned}$$

so that $(a_{ij}) = P^t(b_{ij})P$. □

Suppose that $\phi^* g = a_{ij}\mathrm{d}x^i\mathrm{d}x^j$ for some parameterization ϕ. Is M locally flat? In other words, does there exist a re-parameterization $\psi = \phi y$ of M such that $\psi^* g = \delta_{ij}\mathrm{d}y^i\mathrm{d}y^j$? Such a re-parameterization exists if and only if the set of $\frac{1}{2}n(n-1)$ PDEs

$$(g_{ij}) = \left(\tfrac{\partial y^i}{\partial x^j}\right)^t \left(\tfrac{\partial y^i}{\partial x^j}\right),$$

or equivalently,

$$g_{ij} = \sum_{k=1}^{n} \tfrac{\partial y^k}{\partial x^i}\tfrac{\partial y^k}{\partial x^j}, \qquad 1 \le i \le j \le n, \tag{2.10}$$

has a solution $y = (y^1, \dots, y^n)$. Riemann showed (in an essay that was never properly recognized) that (2.10) is equivalent to

$$\frac{\partial^2 y^\ell}{\partial x^i \partial x^j} = \Gamma^k_{ij}\tfrac{\partial y^\ell}{\partial x^k}, \qquad 1 \le i, j, \ell \le n,$$

and thereby that (2.10) has a solution if and only if

$$R^m_{jk\ell} = 0, \quad 1 \le j, k, \ell, m \le n,$$

where the Γ^k_{ij} (the Christoffel symbols) and the $R^m_{jk\ell}$ are certain functions defined in terms of the functions g_{ij}. This was the birth of the Riemann curvature tensor $R^m_{jk\ell}$! This direct approach, however, is not the one used today since it is conceptually simpler first to introduce a device called a connection that will enable us to work in a coordinate-free way on M.

2.11. EXAMPLE. (Cylinders are flat.) Let $\gamma(s) = (x(s), y(s))$, $a < s < b$, be a smooth curve in $\mathbf{R}^2$ such that the tangent $\gamma_*(\frac{\mathrm{d}}{\mathrm{d}s}) = x'(s)\partial_x + y'(s)\partial_y \neq 0$ for all t. Let $M \subset \mathbf{R}^3$ be the cylinder over γ, the surface with parameterization $\phi(s,t) = (x(s), y(s), z(t))$ where $z(t) = t$. Then (2.3),

$$\phi^*(\overline{g}^3) = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2 = (x'(s)\mathrm{d}s)^2 + (y'(s)\mathrm{d}s)^2 + (\mathrm{d}t)^2 = |\gamma_*(\frac{\mathrm{d}}{\mathrm{d}s})|^2(\mathrm{d}s)^2 + (\mathrm{d}t)^2$$

What are Riemann's equations in this case? Is there a solution?

2. The three model geometries

The model geometries are Euclidean geometry, spherical geometry, and hyperbolic geometry.

2.1. Euclidean geometry. Euclidean geometry is the geometry of the Riemannian manifold $(\mathbf{R}^n, \overline{g}^n)$ where

$$\overline{g}^n = \delta_{ij}dx^i dx^j = \sum_{i=1}^n (dx^i)^2$$

meaning that $\overline{g}^n(U^i\partial_i, V^j\partial_j) = \sum_{i=1}^n U^iV^i$. (The straight line over the g is to remind you of Euclidean geometry.)

The isometry group of Euclidean geometry

$$\mathrm{Isom}(\mathbf{R}^n, \overline{g}^n) = \mathrm{Aff}(n) = \mathbf{R}^n \rtimes \mathrm{O}(n)$$

acts transitively on $\mathbf{R}^n$ by the rule $(v, A)(x) = v + Ax$. The isotropy subgroup at $0 \in \mathbf{R}^n$ is $\mathrm{O}(n)$ and the projection

$$\mathrm{O}(n) \to \mathrm{Aff}(n) = \mathrm{O}(T\mathbf{R}^n) \to \mathrm{Aff}(\mathbf{R}^n)/\mathrm{O}(n) = \mathbf{R}^n$$

is the unit n-frame bundle of Euclidean geometry $\mathbf{R}^n$.

2.2. Spherical geometry. Spherical geometry is the geometry of the Riemannian manifold (S^n_R, g^n_R) where

$$S^n_R = \{(\xi, \tau) \in \mathbf{R}^n \times \mathbf{R} \mid |\xi|^2 + \tau^2 = R^2\}$$

is the n-sphere of radius R and $g^n_R = \iota^*\big(\sum_{i=1}^n (d\xi_i)^2 + (d\tau)^2\big)$ is the restriction of the Euclidean metric on $\mathbf{R}^n \times \mathbf{R}$.

The isometry group of spherical geometry

$$\mathrm{Isom}(S^n_R, g_R) = \mathrm{O}(n+1)$$

(The smooth action of $\mathrm{O}(n+1) \subset \mathrm{Aff}(n+1)$ on $\mathbf{R}^{n+1}$ restricts to a smooth action on S^n – and in fact these are all isometries.) $\mathrm{O}(n+1)$ acts transitively on S^n. The isotropy subgroup at $N = (0, \ldots, 0, 1) \in S^n$ is $\mathrm{O}(n)$ and the projection

$$\mathrm{O}(n) \to \mathrm{O}(TS^n) = \mathrm{O}(n+1) \to \mathrm{O}(n+1)/\mathrm{O}(n) = S^n$$

is the unit n-frame bundle of spherical geometry S^n.

2.12. PROPOSITION. *Stereographic projection* $\sigma\colon S^n_R - N \to \mathbf{R}^n$ *is given by*

$$\sigma(\xi, \tau) = \frac{R}{R - \tau}\xi$$

and the inverse is given by

$$\sigma^{-1}(u) = (\xi(u), \tau(u)), \qquad \xi(u) = \frac{2R^2 u}{|u|^2 + R^2}, \qquad \tau(u) = R\frac{|u|^2 - R^2}{|u|^2 + R^2}$$

Stereographic projection is a diffeomorphism.

PROOF. Elementary. □

2.13. PROPOSITION. *The Riemannian manifold* $(S^n_R - N, g^n_R)$ *is isometric to the Riemannian manifold* $(\mathbf{R}^n, (\sigma^{-1})^* g^n_R)$ *where*

$$(\sigma^{-1})^* g^n_R = \frac{4R^4}{(R^2 + |u|^2)^2}\overline{g}^n$$

is conformally equivalent to the Euclidean metric.

PROOF. By 2.3

$$(\sigma^{-1})^* g_R = \sum_{j=1}^n (d\xi^j)^2 + (d\tau)^2$$

We will now use the derivation rules 1.4. The denominator of ξ and τ is $|u|^2+R^2$ and $d(|u|^2+R^2) = d(|u|^2) = \sum 2u^j du^j$ which we shall write as $2\langle u, du\rangle$. Because

$$\begin{aligned} d\xi^j &= d\left(\frac{2R^2u}{|u|^2+R^2}\right) = \frac{2R^2du^j}{|u|^2+R^2} - \frac{4R^2u^j\langle u,du\rangle}{(|u|^2+R^2)^2} \\ d\tau &= d\left(R\frac{|u|^2-R^2}{|u|^2+R^2}\right) = R\frac{2\langle u,du\rangle}{(|u|^2+R^2)^2} - R\frac{(|u|^2-R^2)2\langle u,du\rangle}{(|u|^2+R^2)^2} \\ &= \frac{2R\langle u,du\rangle(|u|^2+R^2) - 2R\langle u,du\rangle(|u|^2-R^2)}{(|u|^2+R^2)^2} \\ &= \frac{4R^3\langle u,du\rangle}{(|u|^2+R^2)^2} \end{aligned}$$

we get

$$\begin{aligned} \sum_{j=1}^n (d\xi^j)^2 &= \sum_{j=1}^n \left(\frac{4R^4(du^j)^2}{|u|^2+R^2} + \frac{16R^4(u^j)^2\langle u,du\rangle}{(|u|^2+R^2)^4} - \frac{16R^4u^jdu^j\langle u,du\rangle}{(|u|^2+R^2)^3}\right) \\ &= \frac{4R^4(du^j)^2}{(|u|^2+R^2)^2}\sum_{j=1}^n (du^j)^2 + \frac{16R^4|u|^2\langle u,du\rangle^2}{(|u|^2+R^2)^4} - \frac{16R^4\langle u,du\rangle^2}{(|u|^2+R^2)^3} \\ &= \frac{4R^4(du^j)^2}{(|u|^2+R^2)^2}\overline{g}^n + \frac{16R^4|u|^2\langle u,du\rangle - 16R^4|u|^2\langle u,du\rangle^2 - 16R^6\langle u,du\rangle^2}{(|u|^2+R^2)^4} \\ &= \frac{4R^4(du^j)^2}{(|u|^2+R^2)^2}\overline{g}^n - \frac{16R^6\langle u,du\rangle^2}{(|u|^2+R^2)^4} \\ &= \frac{4R^4(du^j)^2}{(|u|^2+R^2)^2}\overline{g}^n - (d\tau)^2 \end{aligned}$$

□

2.14. Hyperbolic geometry. Hyperbolic geometry is the geometry of the Riemannian manifold (H^n_R, h^n_R) where

$$H^n_R = \{(\xi,\tau) \in \mathbf{R}^n \times \mathbf{R}_+ \mid |\xi|^2 - \tau^2 = -R^2\}$$

id the hyperbolic space n-space of radius R and $h^n_R = \iota^*\big(\sum_{i=1}^n (d\xi^i)^2 - (d\tau)^2\big)$ is the restriction of the Minkowski metric on $\mathbf{R}^n \times \mathbf{R}$. Let $N = (0,\dots,0,R) \in H^n_R$ be the north pole.

2.15. Remark. (Minkowski metric) Let m be the inner product on $\mathbf{R}^{n+1}$ with matrix $D = \mathrm{diag}(1,\dots,1,-1)$. We can view m both as an inner product on the vector space $\mathbf{R}^n$ ($\langle X,Y\rangle = m(X,Y) = X^tDY$) or as a Minkowski metric on the manifold $\mathbf{R}^{n+1}$ ($\langle X^i\partial_i, Y^i\partial_i\rangle = m(X,Y)$). For each $p \in H^n_R$, the tangent space

$$T_pH^n_R = p^\perp \subset \mathbf{R}^{n+1} = T_p\mathbf{R}^{n+1}$$

exactly as in 1.5.

Let

$$\mathrm{O}(n,1) = \{A \in \mathrm{GL}(n,\mathbf{R}) \mid A^tDA = D\}$$

be the group of linear automorphisms of $\mathbf{R}^{n+1}$ that preserve the inner product. The columns (or rows) of each $\mathbf{A} \in \mathrm{O}(n,1)$ form an orthogonal basis for $\mathbf{R}^{n+1}$ of vectors of length $1,\dots,1,-1$. The elements of the Lie group $O(n,1)$ preserve the subspace $\{(\xi,\tau) \in \mathbf{R}^{n+1} \mid |(\xi,\tau)|^2 = -R^2\}$ of vectors of square length $-R^2$. This subspace has two connected components. Let $\mathrm{O}_+(n,1)$ be the subgroup of $\mathrm{O}(n,1)$ consisting of the elements that take the connected component H^n_R to itself. The Lie group $\mathrm{O}_+(n,1)$ acts transitively on H^n_R; given any $v \in H^n_R$, we can find an $A \in \mathrm{O}_+(n,1)$ whose last column is $\frac{1}{R}v$ so that $AN = v$ where N is the north pole. The isotropy subgroup at N consists of the $A \in \mathrm{O}_+(n,1)$ whose last column is $(0,\dots,0,1)$. The isotropy subgroup, isomorphic to $\mathrm{O}(n)$, act transitively on the tangent space $T_NH^n_R$. The projection

$$\mathrm{O}(n) \to \mathrm{O}(TH^n_R) = \mathrm{O}_+(n,1) \to \mathrm{O}_+(n,1)/\mathrm{O}(n) = H^n_R$$

is the unit n-frame bundle of H^n_R.

2.16. PROPOSITION. *Hyperbolic stereographic projection* $\pi\colon H^n_R \to B^n_R$ *is given by*

$$\pi(\xi,\tau) = \frac{R\xi}{R+\tau}$$

and its inverse is given by

$$\pi^{-1}(u) = (\xi(u),\tau(u)), \qquad \xi(u) = \frac{2R^2u}{R^2-|u|^2}, \quad \tau(u) = R\frac{R^2+|u|^2}{R^2-|u|^2}$$

Hyperbolic stereographic projection is a diffeomorphism.

PROOF. Elementary. □

2.17. PROPOSITION. *The Riemannian manifold* (H^n_R, h^n_R) *is isometric to the Riemannian manifold* $(B^n(R), (\pi^{-1})^*h^n_R)$ *where*

$$(\pi^{-1})^*(h^n_R) = \frac{4R^4}{(R^2-|u|^2)^2}\overline{g}^n$$

is conformally equivalent to the Euclidean metric.

PROOF. By 2.3 (applied to the Minkowski metric)

$$(\sigma^{-1})^*h_R = \sum_{j=1}^n (d\xi^j)^2 - (d\tau)^2$$

Using that $d(R^2-|u|^2) = -2\langle u, du\rangle$ and 1.4 we get

$$\begin{aligned}
d\xi^j &= \frac{2R^2du^j}{R^2-|u|^2} + \frac{2R^2u^j2\langle u,du\rangle}{(R^2-|u|^2)^2}\\
d\tau &= R\frac{2\langle u,du\rangle}{R^2-|u|^2} + R\frac{(R^2+|u|^2)2\langle u,du\rangle}{(R^2-|u|^2)^2}\\
&= \frac{2R^3\langle u,du\rangle - 2R|u|^2\langle u,du\rangle + 2R^3\langle u,du\rangle + 2R|u|^2\langle u,du\rangle}{(R^2-|u|^2)^2}\\
&= \frac{4R^3\langle u,du\rangle}{(R^2-|u|^2)^2}
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{j=1}^n (d\xi^j)^2 &= \sum_{j=1}^n\left(\frac{2R^2du^j}{R^2-|u|^2} + \frac{4R^2u^j\langle u,du\rangle}{(R^2-|u|^2)^2}\right)^2\\
&= \frac{4R^4}{(R^2-|u|^2)^2}\sum_{j=1}^n (du^j)^2 + \frac{16R^4|u|^2\langle u,du\rangle}{(R^2-|u|^2)^4} + \frac{16R^4\langle u,du\rangle^2}{(R^2-|u|^2)^3}\\
&= \frac{4R^4}{(R^2-|u|^2)^2}\overline{g}^n + \frac{16R^4|u|^2\langle u,du\rangle + 16R^6\langle u,du\rangle^2 - 16R^4|u|^2\langle u,du\rangle^2}{(R^2-|u|^2)^4}\\
&= \frac{4R^4}{(R^2-|u|^2)^2}\overline{g}^n + \frac{16R^6\langle u,du\rangle^2}{(R^2-|u|^2)^4}\\
&= \frac{4R^4}{(R^2-|u|^2)^2}\overline{g}^n + (d\tau)^2
\end{aligned}$$

□

Let $U^n = \{(x,y)\in R^{n-1}\times\mathbf{R} \mid y>0\}$ be the upper half plane in $\mathbf{R}^n$.

2.18. PROPOSITION. *The Riemannian manifold* (H^n_R, h^n_R) *is isometric to the smooth manifold* U^n *equipped with the Riemannian metric* $R^2\frac{1}{y^2}\overline{g}^n$.

PROOF. A computation. □

Does the hyperbolic plane H^2 embed isometrically in $\mathbf{R}^3$? Any Riemannian ma embeds isometrically into some Euclidean space [**6**].

3. Connections

Let $E \to M$ be a smooth vector bundle over M and $\mathcal{E}(M)$ the $C^\infty(M)$-module of smooth sections. A connection on E is a recipe for how to differentiate a section of E along a vector field.

2.19. Definition. *A* connection *on E is a map*

$$\begin{aligned} \mathcal{T}(M) \times \mathcal{E}(M) &\xrightarrow{\nabla} \mathcal{E}(M) \\ (X, Y) &\to \nabla_X Y \end{aligned}$$

which is $C^\infty(M)$-linear in X and $\mathbf{R}$-linear in Y and satisfies the product rule

$$\nabla_X(fY) = (Xf)Y + f\nabla_X Y$$

for all $f \in C^\infty(M)$.

There always are connections, for instance the 0-connection given by $\nabla_X Y = 0$.

2.20. Lemma. *The value $\nabla_X Y(p)$ at the point $p \in M$ only depends on Y in a neighborhood of p and X at p.*

Proof. Let us first focus on Y-variable. By linearity, it is enough to show that if $Y = 0$ in a neighborhood U of p, then $\nabla_X Y(p) = 0$. Choose a smooth bump function ϕ such that $\phi(p) = 1$ and $\phi = 0$ outside U. Then ϕY is the zero section so that

$$0 = \nabla_X 0 = \nabla_X(\phi Y) = (X\phi)Y + \phi\nabla_X Y$$

Evaluating at p, we get $0 = \nabla_X Y(p)$ since $Y(p) = 0$ and $\phi(p) = 1$.

Next, we focus on the X-variable. By an argument similar to the one just given, we first show that if X is 0 in a neighborhood of p, then $0 = \nabla_X Y(p)$. Suppose now that we only know that X vanishes at the point p, $X(p) = 0$. Choose a moving frame E_i in a neighborhood of p. Extend the locally defined vector fields E_i to globally defined smooth vector fields. There are smooth functions X^i such that $X = X^i E_i$ in a neighborhood of p. Then $\nabla_X Y(p) = \nabla_{X^i E_i} Y(p)$ since X and $X^i E_i$ are equal in a neighborhood of p. By $C^\infty(M)$-linearity in X,

$$\nabla_{X^i E_i} Y = X^i \nabla_{E_i} Y$$

which evaluated at p is 0 since $X^i(p) = 0$ for all i. □

We are particularly interested in connections on the tangent bundle of M.

2.21. Definition. *A* linear connection *is a connection $\mathcal{T}(M) \times \mathcal{T}(M) \xrightarrow{\nabla} \mathcal{T}(M)$ on the tangent bundle of M.*

Are there any linear connections, apart from the 0-connection, on M?

2.22. Example. The *Euclidean connection* $\overline{\nabla}_X(Y^j\partial_j) = X(Y^j)\partial_j$ is a nonzero linear connection on Euclidean space $M = \mathbf{R}^n$. Note that $\overline{\nabla}_{\partial_i}\partial_j = 0$, $1 \leq i, j \leq n$. A smooth vector field $Y = Y^i\partial_i$ on $\mathbf{R}^n$ is the same thing as a smooth map $Y = Y^i E_i \colon \mathbf{R}^n \to R^n$. The derivative in the direction of (tangent) vector X of the map $Y = Y^i E_i$ is $X(Y) = X(Y^i)E_i$. For consistency sake we better declare the derivative of the vector field $Y = Y^i\partial_i$ to be $\overline{\nabla}_X Y = X(Y^i)\partial_i$ to ensure that the diagram

$$\begin{tikzcd} C^\infty(\mathbf{R}^n, \mathbf{R}^n) \arrow[r, "X"] \arrow[d, equal] & C^\infty(\mathbf{R}^n, \mathbf{R}^n) \arrow[d, equal] \\ \mathcal{T}(\mathbf{R}^n) \arrow[r, "\overline{\nabla}_X"] & \mathcal{T}(\mathbf{R}^n) \end{tikzcd}$$

commutes.

If we can construct connections on $\mathbf{R}^n$, maybe we can define a connection in each coordinate patch on M and then put them together? What does a connection look like locally?

2.23. LEMMA. *Suppose that U is an open subspace of M that admit a moving frame E_i. The linear connection ∇ on M restricts to linear connection on U. If $X = X^i E_i$ and $Y = Y^j E_j$ are vector fields on U then*

$$\nabla_X Y = (XY^k + X^i Y^j \Gamma^k_{ij}) E_k$$

where the n^3 smooth functions Γ^k_{ij} are the Christoffel symbols *given by $\nabla_{E_i} E_j = \Gamma^k_{ij} E_k$.*

PROOF. We compute

$$\begin{aligned}\nabla_X Y = \nabla_X (Y^j E_j) &= XY^j E_j + Y^j \nabla_X E_j XY^j E_j + Y^j \nabla_{X^i E_i} E_j = XY^j E_j + X^i Y^j \nabla_{E_i} E_j \\ &= XY^j E_j + X^i Y^j \Gamma^k_{ij} E_k = XY^k E_k + X^i Y^j \Gamma^k_{ij} E_k = (XY^k + X^i Y^j \Gamma^k_{ij}) E_k\end{aligned}$$

where we use the Christoffel symbols. □

Thus we see that we can express $\nabla_X Y$ by means of the n^3 smooth functions Γ^k_{ij}. Conversely, for any choice of n^3 smooth functions Γ^k_{ij} we can define $\nabla_X Y$, $X, Y \in \mathcal{T}(U)$ by the above formula and that will be a connection. So a linear connection on U is the same thing as a collection of n^3 smooth functions on U.

In particular if M is a smooth manifold and $\{U_\alpha\}$ a smooth atlas on M then can find a linear connection ∇^α on each U_α. In order to construct a linear connection on M, let $\{\phi_\alpha\}$ be a smooth partition of unity subordinate to $\{U_\alpha\}$.

THEOREM 2.24. *(Existence of connections) Any smooth manifold M has many linear connections:*

$$\nabla_X Y = \sum_\alpha \phi_\alpha \nabla^\alpha_X Y$$

is a linear connection on M.

Armed with a linear connection we know how to differentiate vector fields along vector fields. But as a bonus we can even differentiate arbitrary tensor fields along vector fields!

2.25. LEMMA. *(Existence and uniqueness of the covariant derivative of a tensor field) Let $\mathcal{T}(M) \times \mathcal{T}(M) \xrightarrow{\nabla} \mathcal{T}(M)$ be a linear connection on M (2.21). Then there are unique connections*

$$\mathcal{T}(M) \times \mathcal{T}^k_\ell(M) \xrightarrow{\nabla} \mathcal{T}^k_\ell(M)$$

on all tensor bundles $T^k_\ell(M) \to M$ such that

(1) $\nabla_X f = Xf$ *for all* $f \in \mathcal{T}^0_0(M) = C^\infty(M)$
(2) $\nabla_X Y$ *is the given linear connection for all vector fields* $Y \in \mathcal{T}^1_0(M)$
(3) $\nabla_X (A \otimes B) = \nabla_X A \otimes B + A \otimes \nabla_X B$
(4) $\nabla_X \operatorname{tr} A = \operatorname{tr} \nabla_X A$ *for all* $A \in \mathcal{T}^{k+1}_{\ell+1}(M)$

Namely, for any 1-form ω, $\nabla_X \omega$ is the 1-form given by

$$(\nabla_X \omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y) \tag{2.26}$$

and, in general, for any $\binom{k}{\ell}$-tensor $A \in \mathcal{T}^k_\ell(M)$, $\nabla_X A$ is the $\binom{k}{\ell}$-tensor given by

$$\begin{aligned}(\nabla_X A)(\omega^1, \dots, \omega^\ell, Y_1, \dots, Y_k) &= X(A(\omega^1, \dots, \omega^\ell, Y_1, \dots, Y_k)) \\ - \sum_{i=1}^k A(\omega^1, \dots, \omega^\ell, Y_1, \dots, \nabla_X Y_i, \dots, Y_k) &- \sum_{j=1}^\ell A(\omega^1, \dots, \nabla_X \omega_j, \dots, \omega^\ell, Y_1, \dots, Y_k)\end{aligned} \tag{2.27}$$

for any choice of k vector fields $Y_1, \dots, Y_k$ and ℓ 1-forms $\omega^1, \dots, \omega^\ell$ on M.

PROOF. Let's assume that we have connections that satisfy items (1)–(4). What is the covariant derivative of a 1-form ω? For any two vector fields X and Y,

$$\begin{aligned}X(\omega(Y)) \overset{(1)}{=} \nabla_X(\omega(Y)) \overset{(1.7)}{=} \nabla_X(\operatorname{tr}(\omega \otimes Y)) \overset{(4)}{=} \operatorname{tr} \nabla_X(\omega \otimes Y) \overset{(3)}{=} \operatorname{tr}(\nabla_X \omega \otimes Y + \omega \otimes \nabla_X Y) \\ \overset{(1.7)}{=} (\nabla_X \omega)(Y) + \omega(\nabla_X Y)\end{aligned}$$

which is (2.26). So we are forced to define $\nabla_X\omega$ as in (2.26). But then there is at most one possibility for $\nabla_X A$ since any tensor is a sum of a smooth function times tensor products of vector fields and 1-forms (combine the local expression (1.8) for a tensor with a smooth partition of unity). This shows uniqueness.

To show existence, use (2.26) to define the covariant derivative of a 1-form and then use (2.27) to define $\nabla_X A$ in general. Check that this definition satisfies (1)–(4). $\square$

2.28. Definition. *The* total covariant derivative $\nabla\colon \mathcal{T}^k_\ell(M) \to \mathcal{T}^{k+1}_\ell(M)$ *is given by*

$$\nabla A(\omega^1,\dots,\omega^\ell,Y_1,\dots,Y_k,X) = (\nabla_X A)(\omega^1,\dots,\omega^\ell,Y_1,\dots,Y_k)$$

for all $A \in \mathcal{T}^k_\ell(M)$.

Note that the total covariant derivative of the tensor field $A \in \mathcal{T}^k_\ell(M)$ is zero if and only if the covariant derivative of A along all vector fields is zero: $\nabla A = 0 \iff \forall X \in \mathcal{T}(M)\colon \nabla_X A = 0$.

2.29. Example. There are total covariant derivatives

$$C^\infty(M) = \mathcal{T}^0_0(M) \xrightarrow{\nabla} \mathcal{T}^1_0(M) \xrightarrow{\nabla} \mathcal{T}^2_0(M) \xrightarrow{\nabla} \mathcal{T}^3_0(M) \xrightarrow{\nabla} \cdots$$

If $u \in C^\infty(M)$ is a smooth function and X a vector field, then

$$(\nabla u)(X) = \nabla_X(u) = X(u) = \mathrm{d}u(X)$$

so that $\nabla u = \mathrm{d}u$.

The 2-form $\nabla^2 u = \nabla\nabla u$ is called the *covariant Hessian of* u. If X and Y a vector fields, then

$$\begin{aligned}(\nabla^2 u)(Y,X) = (\nabla\nabla u)(Y,X) \overset{(2.28)}{=} (\nabla_X\nabla u)(Y) &\overset{(2.26)}{=} X((\nabla u)(Y)) - \nabla u(\nabla_X Y) \\ &\overset{\nabla u = \mathrm{d}u}{=} X(Y(u)) - (\nabla_X Y)(u)\end{aligned}$$

Consequently,

$$\nabla^2 u = 0 \iff \forall X,Y \in \mathcal{T}(M)\colon Y(X(u)) = (\nabla_Y X)(u)$$

2.30. Example. If $g \in \mathcal{T}^2_0(M)$ is the Riemannian metric, then ∇g is the 3-form given by

$$(\nabla g)(X,Y,Z) \overset{(2.28)}{=} (\nabla_Z g)(X,Y) \overset{(2.27)}{=} Zg(X,Y) - g(\nabla_Z X,Y) - g(X,\nabla_Z Y)$$

for any three vector fields X,Y,Z. Consequently

$$\nabla g = 0 \iff \forall X,Y,Z \in \mathcal{T}(M)\colon Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$$

for all vector fields X,Y,Z.

2.31. Example. There are total covariant derivatives

$$\mathcal{T}(M) = \mathcal{T}^0_1(M) \xrightarrow{\nabla} \mathcal{T}^1_1(M) \xrightarrow{\nabla} \mathcal{T}^2_1(M) \xrightarrow{\nabla} \cdots$$

If $V \in \mathcal{T}^0_1(M)$ is a vector field then ∇V is the $\binom{1}{1}$-tensor given by

$$(\nabla V)(\omega,X) = (\nabla_X V)(\omega) = \omega(\nabla_X V)$$

If $V = V^i\partial_i$ in local coordinates, then

$$(\nabla V)(dx^i,\partial_j) = dx^i(\nabla_{\partial_j}V) = dx^i(\nabla_{\partial_j}V^k\partial_k) = dx^i(\partial_j V^k\partial_k + V^k\Gamma^\ell_{jk}\partial_\ell) = \partial_j V^i + V^k\Gamma^i_{jk}$$

and therefore

$$\nabla V = (\partial_j V^i + V^k\Gamma^i_{jk})\partial_i \otimes dx^j$$

We say that the vector field V is *parallel* if $\nabla V = 0$. Since

$$\nabla V = 0 \iff \forall X \in \mathcal{T}(M)\colon \nabla_X V = 0,$$

V is parallel iff the covariant derivative of V along any vector field is zero. What tensor is $\nabla^2 V$? What does it mean if $\nabla^2 V = 0$?

4. Geodesics and parallel translation along curves

Let $\gamma\colon I \to M$ be a smooth curve on M.

2.32. DEFINITION. *For any $t \in I$, the tangent vector*

$$\overset{\bullet}{\gamma}(t) = \gamma_*(\frac{d}{dt}(t)) \in T_{\gamma(t)}M, \qquad \overset{\bullet}{\gamma}(t)f = \frac{d(f\circ\gamma)}{dt}(t), \quad f \in C^\infty(M)$$

is called the velocity vector of γ *at the point* $\gamma(t)$.

If x is a coordinate system around $\gamma(t_0)$ and $x\gamma = (\gamma^1, \ldots, \gamma^n)$ then

$$\overset{\bullet}{\gamma}(t) = \frac{\mathrm{d}\gamma^i}{\mathrm{dt}}(t)\partial_i(\gamma(t)) \tag{2.33}$$

as a special case of (1.2).

2.34. REMARK. We say that the curve γ, defined in an open neighborhood of 0, *represents* the tangent vector $V \in T_pM$ if $\gamma(0) = p$ and $\overset{\bullet}{\gamma}(0) = V$. Then $Vf = \overset{\bullet}{\gamma}(0)f = \frac{\mathrm{d}}{\mathrm{dt}}(f\gamma)(0)$ for any smooth function $f \in C^\infty(M)$ and F_*V is represented by the image curve $F\gamma$ for any smooth map $F\colon M \to N$.

2.35. DEFINITION. *A vector field along γ is a smooth map $V\colon I \to TM$ such that the diagram*

$$\begin{tikzcd} & TM \arrow[d] \\ I \arrow[ur, "V"] \arrow[r, "\gamma"'] & M \end{tikzcd}$$

commutes. A vector field V along is extendible if $V(t) = \overline{V}(\gamma(t))$ for some vector field $\overline{V}$ on a neighborhood of $\gamma(I)$. The $C^\infty(I)$-module of all vector fields along γ is denoted $\mathcal{T}(\gamma)$.

The velocity field $\overset{\bullet}{\gamma}(t)$ is an example of a vector field along γ.
The formulation of the lemma below makes use of 2.20.

2.36. LEMMA (Covariant differentiation along a curve). *Let ∇ be a connection on M. There exists precisely one* $\mathbf{R}$*-linear map $D_t\colon \mathcal{T}(\gamma) \to \mathcal{T}(\gamma)$ such that*

$$\begin{aligned} D_t(fV) &= \frac{df}{dt}V + fD_tV, && f \in C^\infty(I), \quad V \in \mathcal{T}(\gamma) \\ D_tV &= \nabla_{\overset{\bullet}{\gamma}(t)}\overline{V} && V \text{ extendible} \end{aligned}$$

- *If E_i is a local frame around $\gamma(t_0)$ and $V = V^jE_j$, then*

  $$D_tV = \frac{dV^j}{dt}E_j + V^j\nabla_{\overset{\bullet}{\gamma}(t)}E_j$$

  for t near t_0.
- *If x is a coordinate system around $\gamma(t_0)$ and $V = V^j\partial_j$, then*

  $$D_tV = (\frac{d\gamma^k}{dt} + \Gamma^k_{ij}\frac{d\gamma^i}{dt}V^j)\partial_k$$

  for t near t_0.

A curve on M is as close as possible to being a straight line if the curve at all times just continues in the direction of $\overset{\bullet}{\gamma}(t)$, if its velocity $\overset{\bullet}{\gamma}(t)$ does not change. No change means zero covariant derivative.

2.37. DEFINITION. *A smooth curve γ is a* geodesic *(with respect to the connection ∇) if the covariant derivative of its velocity field vanishes, $D_t\overset{\bullet}{\gamma} = 0$.*

A geodesic is a curve that follows its own nose. A geodesic is a curve with constant velocity. Light rays follow geodesics in space-time.

If x is a coordinate system around some point $\gamma(t_0)$ of γ, then γ is a geodesic iff

$$\frac{\mathrm{d}^2\gamma^k}{\mathrm{dt}^2} + \Gamma^k_{ij}\frac{\mathrm{d}\gamma^i}{\mathrm{dt}}\frac{\mathrm{d}\gamma^j}{\mathrm{dt}} = 0 \tag{2.38}$$

near t_0. In principle, we could determine geodesics by solving this equation. In practice, this is close to impossible. Instead we try to identify some properties that a geodesic must have (2.54, 2.58).

2.39. Proposition (Existence and uniqueness of geodesics). *Let p be a point on M and $V \in T_pM$ a tangent vector at p There exists a unique maximal geodesic $\gamma_V : I \to M$ defined on an open interval containing* 0 *such that $\gamma_V(0) = p$ and $\dot{\gamma}_V(0) = V$.*

2.40. Lemma (Rescaling lemma). *Let $V \in T_pM$ be a tangent vector at $p \in M$ and let $c \in \mathbf{R}$ be a real number. The geodesic γ_{cV} is defined at t iff the geodesic γ_V is defined at ct and then $\gamma_{cV}(t) = \gamma_V(ct)$.*

Proof. Consider the maximal geodesic $\gamma_V : I \to M$. Put $\gamma(t) = \gamma_V(ct)$ for all $t \in c^{-1}I$. Then $\gamma(t)$ is a geodesic because it satisfies (2.38) in any coordinate system and it is defined at $c^{-1}t$. In fact, $\gamma(t) = \gamma_{cV}(t)$ as $\gamma(0) = \gamma_V(0) = p$ and $\dot{\gamma}(0) = cV$. □

A vector field along a curve is parallel if it doesn't change; no change meaning zero covariant derivative. A curve is a geodesic if it has a parallel velocity field.

2.41. Definition. *A vector field V along γ is parallel if it does not change along γ, $D_tV = 0$.*

2.42. Proposition. *Let γ be a curve on M. Suppose that $p_0 = \gamma(t_0)$ is a point on γ and $V_0 \in T_{p_0}M$ a tangent vector at that point. There exists precisely one parallel vector field V along γ such that $V(t_0) = V_0$.*

5. The Riemannian connection

On a Riemannian manifold there is a preferred connection.

Theorem 2.43 (The fundamental theorem of Riemannian geometry). *A Riemannian manifold admits precisely one symmetric connection compatible with the metric.*

This particular connection is called the Riemannian connection or the Levi–Civitta connection.

2.44. Definition. *The connection ∇ is symmetric if*

$$\nabla_XY - \nabla_YX = [X, Y]$$

for all vector fields $X, Y \in \mathcal{T}(M)$.

2.45. Lemma. *Let E_i be a local moving frame such that $[E_i, E_j] = 0$, $1 \le i, j \le n$ (for instance $E_i = \partial_i$ could be a coordinate frame). Then ∇ is symmetric if and only if*

$$\Gamma^k_{ij} = \Gamma^k_{ji}, \quad 1 \le i, j, k \le n$$

Proof. $\nabla_{E_i}E_j - \nabla_{E_j}E_i = (\Gamma^k_{ij} - \Gamma^k_{ji})E_k$. □

2.46. Definition. *The connection ∇ is compatible with the metric g if*

$$X\langle Y, Z\rangle = \langle \nabla_XY, Z\rangle + \langle X, \nabla_XZ\rangle$$

for any three vector fields $X, Y, Z \in \mathcal{T}(M)$.

2.47. Lemma. *Let E_i be a local moving frame. Then ∇ is compatible with the metric g if and only if*

$$E_kg_{ij} = \Gamma^\ell_{ki}g_{j\ell} + \Gamma^\ell_{kj}g_{i\ell}, \qquad 1 \le i, j, k, \ell \le n$$

Proof. $E_k\langle E_i, E_j\rangle - \langle \nabla_{E_k}E_i, E_j\rangle - \langle E_i, \nabla_{E_k}E_j\rangle = E_kg_{ij} - \Gamma^\ell_{ki}g_{j\ell} - \Gamma^\ell_{kj}g_{i\ell}$. □

PROOF OF THEOREM 2.43. We first show uniqueness. Assume that ∇ is a symmetric connection that is compatible with g. Then

$$X\langle Y,Z\rangle \overset{2.44}{=} \langle \nabla_X Y,Z\rangle + \langle Y,\nabla_X Z\rangle \overset{2.46}{=} \langle \nabla_X Y,Z\rangle + \langle Y,\nabla_Z X\rangle + \langle Y,[X,Z]\rangle$$

for any three vector fields $X,Y,Z \in \mathcal{T}(M)$. Permute X,Y,Z cyclically, obtain

$$X\langle Y,Z\rangle - Z\langle X,Y\rangle + Y\langle X,Z\rangle = 2\langle \nabla_X Y,Z\rangle + \langle Y,[X,Z]\rangle - \langle X,[Z,Y]\rangle + \langle Z,[Y,X]\rangle$$

and conclude that

$$(2.48)\quad \langle \nabla_X Y,Z\rangle = \frac{1}{2}\left(X\langle Y,Z\rangle - Z\langle X,Y\rangle + Y\langle X,Z\rangle - \langle Y,[X,Z]\rangle + \langle X,[Z,Y]\rangle - \langle Z,[Y,X]\rangle\right)$$

This equation shows that ∇, if it exists, is determined by the metric.

Next, we show existence. We will define ∇ in any open submanifold where we have a moving frame E_i with $[E_i,E_j]=0$ (for instance in a chart domain). The only possibility is to put

$$(2.49)\qquad \Gamma_{ij}^k = \frac{1}{2}g^{k\ell}\left(E_i g_{j\ell} - E_\ell g_{ij} + E_j g_{i\ell}\right)$$

because

$$(2.50)\qquad \Gamma_{ij}^k g_{k\ell} = \langle \nabla_{E_i}E_j, E_\ell\rangle = \frac{1}{2}\left(E_i g_{j\ell} - E_\ell g_{ij} + E_j g_{i\ell}\right)$$

by equation (2.48). Since $\Gamma_{ij}^k = \Gamma_{ji}^k$ and

$$\Gamma_{ki}^\ell g_{j\ell} + \Gamma_{i\ell}^\ell g_{i\ell} = \Gamma_{ki}^\ell g_{\ell j} + \Gamma_{i\ell}^\ell g_{\ell i} \overset{2.50}{=} \frac{1}{2}\left(E_k g_{ij} - E_j g_{ik} + E_i g_{jk}\right) + \frac{1}{2}\left(E_k g_{ij} - E_i g_{jk} + E_j g_{ik}\right) = E_k g_{ij}$$

this connection ∇ is symmetric and compatible with g (2.45, 2.47). $\square$

2.51. EXAMPLE. The Euclidean connection $\overline{\nabla}$ (2.22) is the Riemannian connection on Euclidean space $\mathbf{R}^n$ for it is symmetric and compatible with the Euclidean metric $\overline{g}$ (2.1) since $\Gamma_{ij}^k = 0$.

What is the Riemannian connection in spherical and hyperbolic space?

2.52. LEMMA. *Let ∇ be a connection and g a metric on M. The following conditions are equivalent:*

(1) *∇ and g are compatible (2.46)*
(2) $\nabla g = 0$
(3) $\frac{d}{dt}\langle V,W\rangle = \langle D_t V,W\rangle + \langle V,D_t W\rangle$ *whenever V,W are vector fields along a smooth curve*
(4) *$\langle V,W\rangle$ is constant whenever V,W are parallel vector fields along a smooth curve*
(5) *Parallel transport is an isometry*

PROOF. (1) $\iff$ (2): 2.30
(3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) : Obvious.
(5) $\Longrightarrow$ (3) : Let $P_1,\ldots,P_n$ be parallel vector fields that are orthonormal at one point of the curve and hence orthonormal at any point. Write $V = V^iP_i$ and $W = W^iP_i$. Then $\langle V,W\rangle = \sum V^iW^i$ and (2.36) $D_tV = \frac{\mathrm{d}V^i}{\mathrm{dt}}P_i$ and $D_tW = \frac{\mathrm{d}W^i}{\mathrm{dt}}P_i$. Hence

$$\langle D_tV,W\rangle + \langle V,D_tW\rangle = \sum \frac{\mathrm{d}V^i}{\mathrm{dt}}W^i + \sum V^i\frac{\mathrm{d}W^i}{\mathrm{dt}} = \frac{\mathrm{d}}{\mathrm{dt}}\sum V^iW^i = \frac{\mathrm{d}}{\mathrm{dt}}\langle V,W\rangle$$

(3) $\Longrightarrow$ (1) : Let Y and Z be smooth vector fields on M and let γ be a smooth curve, $\gamma(0)=p$, $\dot\gamma(0) = X(p)$. Then

$$X_p\langle Y,Z\rangle = \dot\gamma(0)\langle Y,Z\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle Y,Z\rangle = \langle D_tY,Z\rangle + \langle Y,D_tZ\rangle = \left\langle \nabla_{X_p}Y,Z\right\rangle + \left\langle Y,\nabla_{X_p}Z\right\rangle$$

(1) $\Longrightarrow$ (3) : Let $p=\gamma(0)$, choose a vector field X with $X(p) = \dot\gamma(0)$, and choose an orthonormal moving frame, E_i, around p. Then

$$0 = X\langle E_i,E_j\rangle = \langle \nabla_X E_i,E_j\rangle + \langle E_i,\nabla_X E_j\rangle$$

since ∇ and g are compatible Write $V = V^i E_i$ and $W = W^j E_j$. Then $D_t V = \overset{\bullet}{V^i} E_i + V^i \nabla_X E_i$, and similarly for W, at the point p (2.36). Therefore,

$$\begin{aligned}\langle D_t V, W\rangle + \langle V, D_t W\rangle &= (\overset{\bullet}{V^i} W^j + V^i \overset{\bullet}{W^j}) \langle E_i, E_j\rangle + V^i W^j (\langle \nabla_X E_i, E_j\rangle + \langle E_i, \nabla_X E_j\rangle) \\ &= \sum (\overset{\bullet}{V^i} W^i + V^i \overset{\bullet}{W^i}) = \frac{\mathrm{d}}{\mathrm{dt}} \sum V^i W^i = \frac{\mathrm{d}}{\mathrm{dt}} \langle V, W\rangle\end{aligned}$$

□

2.53. Lemma (Covariant differentiation commutes with lowering and raising of indices (5.2)). *If the connection ∇ is compatible with the metric then the diagram*

$$\begin{tikzcd} \mathcal{T}^{k+1}_{\ell}(M) \arrow[r, shift left, "\sharp"] \arrow[d, "\nabla_X"'] & \mathcal{T}^{k}_{\ell+1}(M) \arrow[l, shift left, "\flat"] \arrow[d, "\nabla_X"] \\ \mathcal{T}^{k+1}_{\ell}(M) \arrow[r, shift left, "\sharp"] & \mathcal{T}^{k}_{\ell+1}(M) \arrow[l, shift left, "\flat"] \end{tikzcd}$$

commutes for any vector field X.

Proof. It will be enough to prove this for vector fields ($k = 0 = \ell$)). Suppose that X, U, V are vector fields. The claim is $\nabla_X(V^\flat) = (\nabla_X V)^\flat$. We compute

$$\begin{aligned}(\nabla_X(V^\sharp)(U) = X(V^\sharp) - V^\sharp(\nabla_X U) = X\langle U, V\rangle - \langle \nabla_X U, V\rangle &= \langle \nabla_X U, V\rangle + \langle U, \nabla_X V\rangle - \langle \nabla_X U, V\rangle \\ &= \langle U, \nabla_X V\rangle = (\nabla_X V)^\sharp(U)\end{aligned}$$

using (2.26) and (2.46). □

2.54. Corollary. *Riemannian geodesics have constant speed.*

Proof. $\frac{\mathrm{d}}{\mathrm{dt}} |\overset{\bullet}{\gamma}(t)|^2 = \frac{\mathrm{d}}{\mathrm{dt}} \langle \overset{\bullet}{\gamma}(t), \overset{\bullet}{\gamma}(t)\rangle = 2\langle D_t \overset{\bullet}{\gamma}(t), \overset{\bullet}{\gamma}(t)\rangle = 0$. □

6. Connections on submanifolds and pull-back connections

Let $(\overline{M}, \overline{g})$ be a Riemannian manifold and $M \subset \overline{M}$ be an embedded submanifold. Suppose that we have a connection ∇ on $\overline{M}$. How can we obtain a connection on the submanifold M?

For any tangent vector $X_p \in T_p\overline{M}$, let $X_p^T \in T_pM$ denote the orthogonal projection of X_p.

2.55. Proposition (Existence of uniqueness of tangential connections). *There exists precisely one connection ∇^T on M such that*

$$\nabla^T_X Y = \left(\nabla_{\overline{X}} \overline{Y}\right)^T$$

whenever $\overline{X}, \overline{Y}$ are vector fields on $\overline{M}$ and X, Y their restrictions to M. If ∇ is the Riemannian connection on $\overline{M}$, then ∇^T is the Riemannian connection on M.

2.56. Lemma. *Let ∇ be a connection and let $X, Y \in \mathcal{T}(M)$ be vector fields on M. Then $\nabla_X Y(p)$ only depends on $X(p)$ and Y along a curve tangent to $X(p)$.*

Proof. $\nabla_X Y(p) = \nabla_{\overset{\bullet}{\gamma}(0)} Y(p) = D_t Y(0)$ for any curve $\gamma\colon (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$ and $\overset{\bullet}{\gamma}(0) = X(p)$. □

2.57. Proposition (Pull-back connections). *Let $\phi\colon M \to \overline{M}$ be a diffeomorphism and ∇ a connection on $\overline{M}$.*

Let (also) ∇ be the map that makes the diagram

$$\begin{tikzcd} \mathcal{T}(M) \times \mathcal{T}(M) \arrow[r, dashed, "\nabla"] \arrow[d, "\phi_* \times \phi_*"', "\cong"] & \mathcal{T}(M) \arrow[d, "\phi_*", "\cong"'] \\ \mathcal{T}(\overline{M}) \times \mathcal{T}(\overline{M}) \arrow[r, "\nabla"] & \mathcal{T}(\overline{M}) \end{tikzcd}$$

commutative. Thus $\phi_* \nabla_X Y = \nabla_{\phi_* X} \phi_* Y$ for all $X, Y \in \mathcal{T}(M)$.

(1) ∇ *is a connection on* M.
(2) *Covariant differentiation wrt* ∇ *makes the diagram*

$$\begin{tikzcd} \mathcal{T}(\gamma) \arrow[r, "D_t"] \arrow[d, "\phi_*"', "\cong"] & \mathcal{T}(\gamma) \arrow[d, "\phi_*"', "\cong"] \\ \mathcal{T}(\phi\gamma) \arrow[r, "D_t"'] & \mathcal{T}(\phi\gamma) \end{tikzcd}$$

commutative. Thus $\phi_* D_t V = D_t \phi_* V$ *for any vector field* V *along the curve* γ *in* M.
(3) *Assume that* M *and* $\overline{M}$ *are Riemannian manifolds and that* ϕ *is an isometry. If* $\overline{\nabla}$ *the Riemannian connection on* $\overline{M}$, *then* ∇ *is the Riemannian connection on* M. *Thus* $\phi_*\nabla_X Y = \overline{\nabla}_{\phi_* X}\phi_*$ *for all* $X, Y \in \mathcal{T}(M)$.

2.58. Corollary. *Isometries of Riemannian manifolds take Riemannian geodesics to Riemannian geodesics.*

Proof. Let $\phi\colon (M,g) \to (\overline{M},\overline{g})$ be an isometry of Riemannian manifolds. Then the connection on M is the pull-back of the connection on $\overline{M}$ (2.57.(3)). Let γ_V, $V \in T_pM$, be a geodesic on M. Then

$$D_t((\phi\gamma_V)^\bullet) = D_t(\phi_* \overset{\bullet}{\gamma}) \overset{2.57.(2)}{=} \phi_* D_t(\overset{\bullet}{\gamma}) = \phi_* 0 = 0$$

so that $\phi\gamma_V = \gamma_{\phi_* V}$. □

7. Geodesics in the three geometries

We determine the geodesics in Euclidean, spherical, and hyperbolic geometry.

2.59. Euclidean geometry. The Riemannian connection on $\mathbf{R}^n$ is the Euclidean connection (2.22).

Let $p = (0,\dots,0) \in \mathbf{R}^n$ and $V = (1,0,\dots,0) \in T_p\mathbf{R}^n$. We know that there is a unique maximal geodesic γ_V running through p with velocity V. We also know that $\phi\gamma_V = \gamma_V$ for any isometry $\phi \in \mathrm{O}(n)$ of $\mathbf{R}^n$ preserving (p,V). The map $\phi(\xi_1,\xi_2,\xi_3,\dots,\xi_n) = (\xi_1,-\xi_2,\xi_3,\dots,\xi_n)$ is such an isometry (it is a diffeomorphism and it preserves $|\xi|$). Thus γ_V must have $\xi^2\gamma_V = 0$. Similarly, $\xi^3\gamma_V = 0,\dots,\xi^n\gamma_V = 0$. Thus γ_V must run along the ξ^1-axis. Since it has constant speed and $\overset{\bullet}{\gamma_V}(0) = V$, we must have

$$\gamma_V(t) = (t,0,\dots,0)$$

This was just one geodesic! But since $\mathbf{R}^n$ is homogeneous and isotropic, we have in fact determined all geodesics: The geodesics in Euclidean geometry are the straight lines. For any point not on a geodesic there is a unique geodesic passing through that point parallel to the given geodesic.

2.60. Spherical geometry. The Riemannian connection on

$$S^n_R = \{(\xi,\tau) \in \mathbf{R}^n \times \mathbf{R} \mid |\xi|^2 + \tau^2 = R^2\} \subset \mathbf{R}^{n+1}$$

is the tangential connection (2.55) arising from the Euclidean connection on ambient $\mathbf{R}^{n+1}$.

Let $N = (0,\dots,R) \in S^n_R$ be the North Pole and $V = (1,0,\dots,0) \in T_N\S^n_R$. What is γ_V, the geodesic running through N with velocity V? Using the isometries that change sign on ξ_i for $2 \le i \le n$ we see, as above, that γ_V must run in the intersection of S^n_R and the $\xi^1\tau$-plane. Thus

$$\gamma_V(t) = (R\sin(t/R),0,\dots,0,R\cos(t/R)) \tag{2.61}$$

We conclude that the geodesics in spherical geometry are great circles, the intersection of S^n_R with planes through the origin. For any point not on a geodesic there is a no geodesic passing through that point parallel to the given geodesic.

2.62. Hyperbolic geometry. The Riemannian connection on

$$H^n_R = \{(\xi,\tau) \in \mathbf{R}^n \times \mathbf{R}_+ \mid |\xi|^2 - \tau^2 = -R^2\} \subset \mathbf{R}^{n+1}$$

is the tangential connection (2.55) arising from the Euclidean connection, considered as the Riemannian connection of Minkowski metric $|(\xi,\tau)|^2 = |\xi|^2 - \tau^2$, on ambient $\mathbf{R}^{n+1}$.

Let $N = (0,\dots,R) \in S^n_R$ be the North Pole and $V = (1,0,\dots,0) \in T_N\S^n_R$. What is γ_V, the geodesic running through N with velocity V? Using the isometries that change sign on ξ_i for $2 \le i \le n$ we see, as above, that γ_V must run in the intersection of H^n_R and the $\xi^1\tau$-plane. Thus

$$\gamma_V(t) = (R\sinh(t/R), 0, \dots, 0, R\cosh(t/R)) \tag{2.63}$$

We conclude that the geodesics in hyperbolic geometry are great hyperbolas, the intersection of H^n_R with planes through the origin. (The isometry group $\mathrm{O}(n,1)_+$ takes planes through the origin, $u^\perp$, to planes through the origin.) For any point not on a geodesic there are uncountably many geodesic passing through that point parallel to the given geodesic.

8. The exponential map and normal coordinates

Let M be a Riemannian manifold with Riemannian connection ∇ (2.43). Put

$$\mathcal{E} = \{V \in TM \mid \gamma_V \text{ is defined at } 1\}$$

where γ_V, $V \in T_pM$, is the geodesic through $\gamma(0) = p$ with velocity $\overset{\bullet}{\gamma}(0) = V$ (2.39). We define

$$\exp\colon \mathcal{E} \to M \quad \text{by} \quad \exp(V) = \gamma_V(1) \tag{2.64}$$

meaning that $\exp(V)$ is obtained by following the geodesic with initial velocity vector V for one time unit. We let $\exp_p$ denote the restriction of exp to $\mathcal{E}_p = \mathcal{E} \cap T_pM$.

2.65. Proposition (Properties of the exponential map). *Let* $\exp\colon \mathcal{E} \to M$ *be the exponential map on the manifold* M*.*

(1) $\mathcal{E}$ *is an open subspace of* TM *and* $\exp\colon \mathcal{E} \to M$ *is a smooth map.*
(2) $\mathcal{E}_p$ *is star-shaped around* 0 *for each* $p \in M$*.*
(3) $\exp_p\colon \mathcal{E}_p \to M$ *takes straight lines through* $0 \in T_pM$ *to geodesics through* p*:* $\exp_p(tV) = \gamma_V(t)$ *(where both functions are defined for the same set of ts).*
(4) $(\exp_p)_*\colon T_0T_pM = T_pM \to T_pM$ *is the identity map.*
(5) *The exponential map commutes with isometries: The diagram*

$$\begin{tikzcd} T_pM \arrow[r, "\phi_*"] \arrow[d, "\exp_p"'] & T_p\overline{M} \arrow[d, "\exp_{\phi(p)}"] \\ M \arrow[r, "\phi"'] & \overline{M} \end{tikzcd}$$

commutes for any isometry $\phi\colon M \to \overline{M}$*.*

Proof. We defer the proof of (1). Let $V \in T_pM$ and $t \in \mathbf{R}$. Then

$$tV \in \mathcal{E}_p \iff \exp_p \text{ is defined at } tV \iff \gamma_{tV} \text{ is defined at } 1 \iff \gamma_V \text{ is defined at } t$$

and for such a t, $\exp_p(tV) = \gamma_{tV}(1) = \gamma_V(t)$ by the Rescaling lemma (2.40). In particular,

$$V \in \mathcal{E}_p \iff \gamma_V \text{ is defined at } 1 \Longrightarrow \gamma_V \text{ is defined at } s \iff sV \in \mathcal{E}_p$$

when $0 < s \le 1$. Thus $\mathcal{E}_p$ is star-shaped around 0. Assuming that $\mathcal{E}_p$ is open in T_pM and that $\exp_p$ is smooth we now compute the differential of $\exp_p$. Since the tangent vector $V \in T_0T_pM = T_pM$ is (2.34) represented by the image curve $t \to tV$, $(\exp_p)_*$ is represented by curve $\exp(tV) = \gamma_V(t)$ with $\overset{\bullet}{\gamma}_V(0) = V$. Thus $(\exp_p)_*V = V$. If $\phi\colon M \to \overline{M}$ is an isometry, then $\phi\gamma_V = \gamma_{\phi_*V}$ (2.58) so that $\phi\exp_p(V) = \phi\gamma_V(1) = \gamma_{\phi_*V}(1) = \exp_{\phi(p)}(\phi_*V)$.

The *geodesic vector field* G on TM is the defined by

$$G(V)f = \frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0}(f\,\overset{\bullet}{\gamma}_V(t)), \qquad f \in C^\infty(TM)$$

where $\overset{\bullet}{\gamma}_V(t)$ is curve on TM obtained by taking the velocity of the geodesic γ_V. Note that $\gamma_V(t_0+t) = \gamma_{\overset{\bullet}{\gamma}_V(t_0)}(t)$ so that

$$G(\overset{\bullet}{\gamma}_V\,(t_0))f = \frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0} f(\overset{\bullet}{\gamma}_V\,(t+t_0)) = \frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=t_0} f(\overset{\bullet}{\gamma}_V\,(t))$$

where t_0 is an arbitrary point in the open interval of definition for γ_V. This means that integral curves for the vector field G are velocity fields along geodesics. By a general theorem, there exists a smooth map $\theta\colon \mathcal{O} \to TM$, defined on an open subspace $\mathcal{O} \subset \mathbf{R} \times TM$ containing $0 \times TM$, such that $\theta(t, V)$ is the maximal integral curve for G through $V \in TM$ at time $t = 0$. Hence

$$\mathcal{E} = \{V \in TM \mid (1, V) \in \mathcal{O}\} = i_1^{-1}\mathcal{O}, \qquad i_1(V) = (1, V),$$

is open and $\exp(V) = \pi\theta(1, V)$ is smooth as a composition of smooth maps ($\pi\colon TM \to M$ is the projection). □

2.66. Example. Let $N = (0, 0, 1)$ be the North Pole of $S^2 \subset \mathbf{R}^3$ and let V be a unit vector in the tangent space $T_N S^2 = N^\perp$. The exponential map $\exp_N\colon T_N S^2 = N^\perp \to S^2$ takes the unit speed radial line tV to the unit speed geodesic whose trace is the intersection of S^2 with the plane through $0 \in \mathbf{R}^3$ containing N and V.

Normal coordinates is special coordinate system *determined by the metric.*

Let $U_p \subset T_pM$ be an open subset, star-shaped around 0, of the tangent space such that $\exp_p\colon U_p \to U$ is a diffeomorphism between U_p and an open subset U of M. Choose an orthonormal basis E_i for T_pM (with inner product g_p) and an orthonormal basis e_i for $\mathbf{R}^n$ (with standard inner product $\overline{g}$). Let $E\colon (\mathbf{R}^n, \overline{g}) \to (T_pM, g_p)$ be the isometry given by $Ee_i = E_i$. Then

$$x = E^{-1} \circ \exp_p^{-1}\colon M \supset U \to x(U) \subset \mathbf{R}^n \qquad \begin{tikzcd} M & T_pM \arrow[l, "\exp_p"'] & \mathbf{R}^n \arrow[l, "E"'] \arrow[from=1-1, to=1-3, bend right, "x"'] \end{tikzcd} \tag{2.67}$$

are *normal coordinates* around p. (We will often forget to mention E so that V can stand for a tangent vector $V \in T_pM$ as well as a vector $V \in \mathbf{R}^n$.) The smooth function

$$r\colon U - p \to \mathbf{R}, \qquad r(q) = |x(q)| = \sqrt{\sum x^i(q)^2} = |V|_g \quad (\exp_p(V) = q) \tag{2.68}$$

is the *radial distance function* and the *unit radial vector field* on $U - p$, denoted

$$\frac{\partial}{\partial r}(q), \qquad q \in U - p, \tag{2.69}$$

is the vector field formed by the velocity vectors of the unit speed radial geodesic; to any $q \in U - p$ it associates the velocity of the unit speed radial geodesic through q.

If $B_R(0) \subset U_p$, then

$$\begin{aligned} \overline{B}_R(p) &= \exp_p(\overline{B}_R(0)) = \{q \in x(U) \mid r(q) \le R\} \\ B_R(p) &= \exp_p(B_R(0)) = \{q \in x(U) \mid r(q) < R\} \\ S_R(p) &= \exp_p(\partial B_R(0)) = \{q \in x(U) \mid r(q) = R\} \end{aligned}$$

is a *(closed) geodesic ball*, respectively, a *geodesic sphere* around p. (Make a drawing of the situation!)

2.70. Proposition (Properties of normal coordinates). *Let $x\colon U \to x(U)$ be normal coordinates (2.67) around p.*

(1) $x(\gamma_{EV}(t)) = tV$ *for all* $V \in T_pM$ *and for all small* t. *(In normal coordinates, the geodesics through p are straight lines through* 0.*)*
(2) $x(p) = 0$, $\frac{\partial}{\partial x^i}(p) = E_i$, $g_{ij}(p) = \delta_{ij}$, $\Gamma^k_{ij}(p) = 0$, $\partial_k g_{ij}(p) = 0$.
(3) $\frac{\partial}{\partial r} = \frac{x^i(q)}{r(q)}\frac{\partial}{\partial x^i}$ *for all* $q \in U - p$.

Proof. (1) $x^{-1}(tV) = \exp_p(tEV) = \gamma_{EV}(t)$.
(2) $x^{-1}(0) = \exp_p(0) = p$. Let $\gamma(t) = x^{-1}(te_i)$ be the ith coordinate axis. Then $\overset{\bullet}{\gamma}\,(t) = \frac{\partial}{\partial x^i}(\gamma(t))$ in general (2.33). In this case, $\gamma(t) = x^{-1}(te_i) = \exp_p(tE_i) = \gamma_{E_i}(t)$ is the geodesic through $\gamma(0) = p$ with velocity $\overset{\bullet}{\gamma}\,(0) = E_i$. Thus $E_i = \frac{\partial}{\partial x^i}(p)$. The components of the Riemannian metric

g at p are $g_{ij}(p) = g(\frac{\partial}{\partial x^i}(p), \frac{\partial}{\partial x^j}(p)) = g(E_i, E_j) = \delta_{ij}$. The radial curve $\gamma(t) = x^{-1}(t(e_i + e_j))$ is a geodesic so its coordinates γ^k satisfy the ODEs (2.38) which in this particular case means that $\Gamma^k_{ij}(\gamma(t)) = 0$ for all t. For $t = 0$, we get $\Gamma^k_{ij}(p) = 0$. In other words, $\nabla_{\partial_i}\partial_j(p) = 0$ for all i, j. Then also

$$\partial_k g_{ij}(p) = \partial_k \langle \partial_i, \partial_j \rangle (p) = \langle \nabla_{\partial_k}\partial_i(p), \partial_j(p)\rangle + \langle \partial_i(p), \nabla_{\partial_k}\partial_j(p)\rangle = 0 + 0 = 0$$

since the Riemannian connection is compatible with the Riemannian metric (2.46).
(3) Let $q \in U - p$. Suppose that $x(q) = V \in \mathbf{R}^n$. Then $x^i(q) = V^i$ and $r(q) = |x(q)| = |V|$ (2.68). The unit speed radial geodesic from p to q is $\gamma(t) = \exp_p(tV/|V|)$, $0 \le t \le |V|$. Its velocity vector is $\overset{\bullet}{\gamma}(t) = \frac{V^i}{|V|}\frac{\partial}{\partial x^i}(\gamma(t))$ (2.33); at q, in particular, its velocity is $\frac{V^i}{|V|}\frac{\partial}{\partial x^i}(q) = \frac{x^i(q)}{r(q)}\frac{\partial}{\partial x^i}(q)$. □

Let $\Gamma\colon (-\varepsilon, \varepsilon) \times [a, b] \to M$ be a variation of the curve $\gamma(t) = \Gamma(0, t)$. We write (s, t) for points in $(-\varepsilon, \varepsilon) \times [a, b]$ (so that the interval $[a, b]$ is placed on the vertical t-axis!). Let $\Gamma_s(t) = \Gamma(s, t) = \Gamma^t(s)$ so that Γ_s is a a curve in the t-direction (a *main* curve) and Γ^t is a curve in the s-direction (a *transverse* curve). Let

$$\partial_t\Gamma(s,t) = \frac{\mathrm{d}}{\mathrm{dt}}\Gamma_s(t) = \overset{\bullet}{\Gamma}_s(t) = \Gamma_*(\frac{\partial}{\partial t}), \qquad \partial_s\Gamma(s,t) = \frac{\mathrm{d}}{\mathrm{ds}}\Gamma^t(s) = \overset{\bullet}{\Gamma}{}^t(s) = \Gamma_*(\frac{\partial}{\partial s}),$$

be the velocities of the main, respectively, the transverse curves. We may view the main velocity field $\partial_t\Gamma$ as a vector field along a transverse curve Γ^t and consider its covariant derivative $D_s\partial_t\Gamma$ along Γ^t. Similarly, we may view the transverse velocity field $\partial_s\Gamma$ as a vector field along a main curve Γ_s and consider its covariant derivative $D_t\partial_s\Gamma$ along Γ_s.

2.71. Lemma (Symmetry Lemma). *$D_s\partial_t\Gamma = D_t\partial_s\Gamma$ or $D_s\overset{\bullet}{\Gamma}_s = D_t\overset{\bullet}{\Gamma}{}^t$.*

Proof. This is a local question. Choose a coordinate system x around $\Gamma(s_0, t_0)$. In local coordinates, $x\Gamma(s,t) = (\Gamma^1(s,t), \dots, \Gamma^n(s,t))$ and (2.33) $\partial_t = \overset{\bullet}{\Gamma}_s = \frac{\partial\Gamma^i}{\partial t}\partial_i$, $\partial_s = \overset{\bullet}{\Gamma}{}^t = \frac{\partial\Gamma^i}{\partial s}\partial_i$. In local coordinates there is a formula for the covariant derivative along a curve (2.36). Using that formula, we get that

$$D_s\partial_t\Gamma = \left(\frac{\partial^2\Gamma^k}{\partial s\partial t} + \Gamma^k_{ij}\frac{\partial\Gamma^i}{\partial s}\frac{\partial\Gamma^j}{\partial t}\right)\partial_k$$

and there is a similar formula for $D_t\partial\Gamma_t$ except that the s and t swap places. The point is now that the Riemannian connection is symmetric so that the Christoffel symbols are symmetric in i and j (2.45). □

The *variational field*

$$V(t) = \overset{\bullet}{\Gamma}{}^t(t, 0) = \Gamma_{*(0,t)}(\frac{\partial}{\partial s}) \tag{2.72}$$

is the restriction to $\gamma(t)$ of the transverse vector field $\overset{\bullet}{\Gamma}{}^t$.

2.73. Lemma (First Variation of smooth curves). $\frac{d}{ds}L(\Gamma_s)(0) = \left\langle V, \overset{\bullet}{\gamma}\right\rangle\Big|_a^b - \int_a^b \left\langle V, D_t\overset{\bullet}{\gamma}\right\rangle dt$ *when $\gamma(t)$ is a unit speed curve.*

Proof. We differentiate the function $s \to L(\Gamma_s)$ and then evaluate the result at $s = 0$. Using that the connection is compatible with the metric (2.52) and The Symmetry lemma we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{ds}}L(\Gamma_s) = \frac{\mathrm{d}}{\mathrm{ds}}\int_a^b |\overset{\bullet}{\Gamma}_s(t)|dt = \int_a^b \frac{\partial}{\partial s}\left\langle\overset{\bullet}{\Gamma}_s, \overset{\bullet}{\Gamma}_s\right\rangle^{1/2} dt &\overset{2.52}{=} \int_a^b \frac{1}{|\overset{\bullet}{\Gamma}_s|}\left\langle D_s\overset{\bullet}{\Gamma}_s, \overset{\bullet}{\Gamma}_s\right\rangle dt \\ &\overset{2.71}{=} \int_a^b \frac{1}{|\overset{\bullet}{\Gamma}_s|}\left\langle D_t\overset{\bullet}{\Gamma}{}^t, \overset{\bullet}{\Gamma}_s\right\rangle dt\end{aligned}$$

When $s=0$, $\dot{\Gamma}_s=\dot{\Gamma}_0=\dot{\gamma}$, $|\dot{\gamma}|=1$, and $\dot{\Gamma}_s\,(0,t)=V(t)$ so that

$$\frac{\mathrm{d}}{\mathrm{ds}}L(\Gamma_s)(0)=\int_a^b\left\langle D_tV,\dot{\gamma}\right\rangle dt\overset{2.52}{=}\int_a^b\left(\frac{\mathrm{d}}{\mathrm{dt}}\left\langle V,\dot{\gamma}\right\rangle-\left\langle V,D_t\,\dot{\gamma}\right\rangle\right)dt$$
$$=\left\langle V(b),\dot{\gamma}\,(b)\right\rangle-\left\langle V(a),\dot{\gamma}\,(a)\right\rangle-\int_a^b\left\langle V,D_t\,\dot{\gamma}\right\rangle dt$$

□

2.74. Lemma (Gauss lemma). *In a normal neighborhood of a point p we have:*

(1) *The geodesic spheres are orthogonal to the geodesic rays.*
(2) $\langle\frac{\partial}{\partial r},Y\rangle=Y(r)=dr(Y)$ *for any tangent vector $Y\in T_qM$, $q\neq p$ (In other words, $\operatorname{grad}r=\frac{\partial}{\partial r}$).*

Proof. Let $q=\exp_p(V)$ where $V\neq 0$. Then $x(q)=V$ and $r(q)=|V|=R$. The claim is that

$$W\perp V\Longrightarrow(\exp_p)_*W\perp\dot{\gamma}_V\,(1)$$

for any $W\in T_VT_pM=T_pM$ (where $V^\perp=T_VS_R(0)$). Let $\sigma(s)$ be a curve in $S_R(0)$ that represents W, $\sigma(0)=V$, $\dot{\sigma}\,(0)=W$. Consider the variation $\Gamma(s,t)=\exp_p(t\sigma(s))$ of $\gamma_V(t)=\exp_p(tV)$. Put $S=\partial_s\Gamma$ and $T=\partial_t\Gamma$. Note that the curves in the t-direction, $t\to\exp_p(t\sigma(s))$, are geodesics of velocity T and speed $|T|=|\sigma(s)|=R$. Hence $D_tT=0$ and $|T|=R$ is constant. It follows that

$$\frac{\partial}{\partial t}\langle S,T\rangle\overset{(2.52)}{=}\langle D_tS,T\rangle+\langle S,D_tT\rangle=\langle D_tS,T\rangle\overset{(2.71)}{=}\langle D_sT,T\rangle\overset{(2.52)}{=}\frac{1}{2}\frac{\partial}{\partial s}|T|^2=0$$

where we use that the connection is compatible with the metric and symmetric. Thus

$$\langle S,T\rangle(0,0)=\langle S,T\rangle(0,1)$$

since $\langle S,T\rangle(s,t)$ is independent of t. We know compute

$$S(0,0)=\dot{\Gamma}^0\,(0)=0,\quad S(0,1)=\dot{\Gamma}^1\,(0)=(\exp_p)_*W,\quad T(0,1)=\dot{\gamma}_V\,(1)$$

as $\Gamma^0(s)=p$, $\Gamma^1(s)=\exp_p(\sigma(s))$, and $\Gamma_0(t)=\exp_p(tV)=\gamma_V(t)$. It follows that

$$0=\langle S,T\rangle(0,0)=\langle S,T\rangle(0,1)=\langle(\exp_p)_*W,\dot{\gamma}_V\,(1)\rangle$$

which is the first item of the lemma.

We now know that there is an orthogonal decomposition $T_qM=\mathbf{R}\frac{\partial}{\partial r}(q)\oplus T_qS_R(p)$ for we have already seen that $\frac{\partial}{\partial r}(q)$ is the unit vector proportional to $\dot{\gamma}_V\,(1)$. Any $Y\in T_qM$ therefore admits an orthogonal decomposition of the form $Y=\alpha\frac{\partial}{\partial r}(q)+X$ where $\alpha\in\mathbf{R}$ and $X\in T_qS_R(p)$ is tangent to the geodesic sphere. Hence

$$\left\langle\frac{\partial}{\partial r}(q),Y\right\rangle=\alpha|\frac{\partial}{\partial r}(q)|^2=\alpha$$

because $X(r)=0$ as r is constant on $S_R(p)$. On the other hand,

$$Y_*(r)=(\alpha\frac{\partial}{\partial r}(q)+X)(r)=\alpha\frac{\partial}{\partial r}(q)(r)$$

where

$$\frac{\partial}{\partial r}(q)(r)=\left(\frac{x^i}{r}\partial_i\right)\left(\sum(x^i)^2\right)^{1/2}=\sum\frac{x^i}{r}\frac{2x^i}{2r}=\frac{r^2}{r^2}=1$$

and we have proved also the second item of the lemma. □

03.05

9. The Riemann distance function

Let M be a connected Riemannian manifold.

2.75. DEFINITION. *A* regular curve *on M is a smooth map $\gamma\colon [a,b] \to M$ such that $\overset{\bullet}{\gamma}(t) \neq 0$ for all $t \in [a,b]$. The real number*

$$L\gamma = \int_a^b |\overset{\bullet}{\gamma}(t)|dt$$

is the length *of the regular curve γ.*

A piecewise regular curve on M is a continuous map $\gamma\colon [a,b] \to M$ such that $\gamma|[a_{i-1},a_i]$ is regular for some subdivision $a = a_0 < a_1 < \cdots < a_n = b$ of $[a,b]$. The real number

$$L\gamma = \sum L(\gamma|[a_{i-1},a_i])$$

is the length *of the piecewise regular curve γ.*

A piecewise regular curve γ on M has a velocity vector $\overset{\bullet}{\gamma}(t)$ at all points t which is not one of the break points a_i. At a break point a_i, we let

$$\Delta_i \overset{\bullet}{\gamma} = \overset{\bullet}{\gamma}(a_i^+) - \overset{\bullet}{\gamma}(a_i^-)$$

denote the jump between the velocity from the left, $\overset{\bullet}{\gamma}(a_i^-) \in T_{\gamma(a_i)}$, and from the right, $\overset{\bullet}{\gamma}(a_i^+)$.

2.76. PROPOSITION. (1) *The length of a (piecewise) regular curve is invariant under reparametrization.*
(2) *Any regular curve has a unit speed parameterization.*

PROOF. Let $\gamma\colon [a,b] \to M$ be a regular curve.
(1) Let $t\colon [c,d] \to [a,b]$ be a bijective smooth map with $t'(s) \neq 0$ for all $s \in [c,d]$. Then

$$(\gamma \circ t)^\bullet(s) = t'(s)\,\overset{\bullet}{\gamma}(t(s))$$

so that

$$\int_c^d |(\gamma \circ t)^\bullet(s)|ds = \pm \int_c^d |\overset{\bullet}{\gamma}(t(s))|t'(s)ds = \int_a^b |\overset{\bullet}{\gamma}(t)|dt$$

where the $+$ applies if $t' > 0$ and the $-$ applies if $t' < 0$.
(2) Let $s\colon [a,b] \to [0, L(\gamma)]$ be the smooth map $s(t) = \int_a^t |\overset{\bullet}{\gamma}(t)|dt$. Then $s'(t) = |\overset{\bullet}{\gamma}(t)|$ by the Fundamental theorem of Calculus. Let t be the inverse function. Then $(\gamma \circ t)^\bullet(s) = t'(s)\,\overset{\bullet}{\gamma}(t(s)) = \frac{1}{s'(t)}\overset{\bullet}{\gamma}(t)$ is a unit speed curve. □

2.77. LEMMA. *Any two points in M can be connected by a piecewise regular curve.*

PROOF. Since connected and locally path-connected spaces are path-connected, M is path-connected [**5**, §17]. Given any two points, p and q in M, there exists a continuous curve $\gamma\colon [0,1] \to M$ connecting them. By the Lesbesgue number lemma [**5**, §19], there is a subdivision $0 = a_0 < a_1 < \cdots < a_n = 1$ of $[0,1]$ such that $\gamma([a_{i-1},a_i])$ is contained in a coordinate neighborhood $x\colon U \to \mathbf{R}^n$ such that $x(U)$ is a ball. Replace $\gamma([a_{i-1},a_i])$ by a smooth curve within this coordinate neighborhood between the two end-points. □

2.78. DEFINITION. *The function $d\colon M \times M \to [0,\infty)$ given by*

$$d(p,q) = \inf\{L(\gamma) \mid \gamma \text{ is a piecewise regular curve from } p \text{ to } q\}$$

is the Riemann distance *function.*

We all know from general topology that all topological manifolds are metrizable. On a Riemannian manifold we can construct an explicit metric.

2.79. LEMMA. *d is a metric on the topological space M.*

Let $\Gamma\colon [-\epsilon,\epsilon] \times [a,b] \to M$ be a fixed endpoint variation of the unit speed piecewise regular curve $\gamma(t) = \Gamma(0,t)$.

For each s, the length of the piecewise regular curve Γ_s is $L(\Gamma_s)$. What is the rate of change of the length of the main curves near γ (hoping that this length function is smooth)?

THEOREM 2.80 (First variation formula for piecewise smooth curves). *Let γ be a unit speed piecewise regular curve and Γ any piecewise smooth variation of γ. Then*

$$\frac{d}{ds}\Big|_{s=0} L(\Gamma_s) = -\int_a^b \left\langle V, D_t \overset{\bullet}{\gamma} \right\rangle dt - \sum_{i=1}^{n-1} \left\langle V(a_i), \Delta_i \overset{\bullet}{\gamma} \right\rangle$$

where V is the variational field along γ.

PROOF. Since $L(\Gamma_s) = \sum L(\Gamma_s|[a_{i-1}, a_i])$ is a sum, just add the contributions from each subinterval $[a_{i-1}, a_i]$ where we are in the smooth situation (2.73). Remember that the endpoints are fixed under the variation so that the variational field is 0 at the endpoints. □

The formula shows that the length decreases when we vary γ in the direction of the jumps at the points a_i or vary γ is the direction of the acceleration $D_t \overset{\bullet}{\gamma}$ between the points a_i.

2.81. LEMMA. *Any vector field V (which vanishes at the endpoints) along a piecewise smooth curve is the variational field of some (fixed endpoint) variation of the curve.*

PROOF. Thanks to compactness, we can find $\epsilon > 0$ so that $\pm\epsilon V(t) \in \mathcal{E} \subset TM$ for all $t \in [a, b]$. Let $\Gamma(s, t) \exp(sV(t))$ for $(s, t) \in (-\epsilon, \epsilon) \times [a, b]$. Then Γ is a piecewise smooth variation of γ whose transverse curves are geodesics with velocity $\frac{\mathrm{d}}{\mathrm{ds}}\Big|_{s=0} \Gamma(s, t) = \frac{\mathrm{d}}{\mathrm{ds}}\Big|_{s=0} \exp(sV(t)) \overset{2.65}{=} V(t)$. □

2.82. COROLLARY. (1) *Piecewise regular minimizing curves of constant speed are geodesics.*
(2) *Geodesics are locally minimizing curves.*

PROOF OF FIRST PART OF 2.82. Let γ be a minimizing curve. For any vector field V along γ, the expression on the right hand side of the equation in 2.80 is 0 as $L(\gamma_s)$ has a minimum at $s = 0$. Use this to show first that $D_t \overset{\bullet}{\gamma} (t_0) = 0$ at any point which is not a break point. Thus γ is a piecewise geodesic. Next show that there are no break points so that γ is in fact a geodesic. □

What we showed was in fact that geodesics are critical points of the functional L.

Is there a minimizing curve between any two points of M? Are minimizing curves unique? No, there are uncountably many minimizing curves between the North Pole and the South Pole. Or look at the situation where you want to go to the other shore of a lake, there are usually two possibilities. Only if two points are sufficiently close, then there is in fact a *unique* minimizing curve between them.

THEOREM 2.83. *Let p be a point of M and $\overline{B}_R(p)$ a closed geodesic ball around p.*

(1) *For any point $q \in \overline{B}_R(p)$ in the geodesic ball there is a unique minimizing curve from p to q, namely the radial geodesic. Then $d(p, q) = r(q)$ where r is the radial distance function (2.68).*
(2) *For any point $q \notin \overline{B}_R(p)$ outside the geodesic ball there is a point $x \in S_R(p)$ such that $d(p, q) = R + d(x, q)$. Then $d(p, q) > R$.*

PROOF. (1) Suppose that $q \in \overline{B}_R(p)$ lies in the geodesic ball around p and that $r(q) = r$ where r is the radial distance function (2.68) for that ball. The radial unit speed geodesic from p to q is $\gamma(t) = \exp_p(tV)$, $t \in [0, r]$, where $V \in T_pM$ is the unit vector with $\exp_p(rV) = q$. This curve has length r so that $d(p, q) \leq r$.

Now let $\sigma \colon [a, b] \to M$ be any piecewise regular unit speed curve from p to q. Let a_0 be the last point where $r(\sigma(t)) = 0$ and b_0 the first point after a_0 such that $r(\sigma(t)) = r$. Then $\sigma|[a_0, b_0]$ runs inside the closed geodesic ball of radius r. On the interval $(a_0, b_0]$ we decompose the velocity

ke a drawing!

$$\overset{\bullet}{\sigma} (t) = \alpha(t) \frac{\partial}{\partial r}(\alpha(t)) + X(t) \tag{2.84}$$

into its radial component along the unit radial vector field (2.69) and a component $X(t) \perp \frac{\partial}{\partial r}$. Then

$$\alpha(t) = \left\langle \overset{\bullet}{\sigma} (t), \frac{\partial}{\partial r} \right\rangle \overset{2.74}{=} \overset{\bullet}{\sigma} (t)(r) = \frac{\mathrm{d}}{\mathrm{dt}}(r\sigma), \qquad 1 = |\overset{\bullet}{\sigma} (t)|^2 = |\alpha(t)|^2 + |X(t)|^2 \geq |\alpha(t)|^2$$

for $t \in (a_0, b_0]$ and therefore

$$b - a = L(\sigma) \geq L(\sigma|[a_0, b_0]) = b_0 - a_0 \geq \lim_{\delta\to 0} \int_{a_0+\delta}^{b_0} \alpha(t)dt = \lim_{\delta\to 0} \int_{a_0+\delta}^{b_0} \frac{\mathrm{d}}{\mathrm{dt}}(r\sigma)dt$$
$$= \lim_{\delta\to 0}(r(q) - r(\sigma(a_0 + \delta))) = r(q) = r$$

Since σ was an arbitrary curve, we have shown that the radial geodesic γ is minimizing and that $d(p, q) = R$.

We will now show that γ is the *only* minimizing curve (up to reparametrization). Suppose that $\sigma\colon [0, r] \to M$ is some minimizing unit speed curve from p to q. Then the inequalities in the above computation are in fact equalities so that $\alpha(t) = 1$ for all t. Then $X(t) = 0$ and (2.84) says that $\sigma(t)$ is an integral curve through p for the radial unit vector field. So is γ. Uniqueness of integral curves implies that $\sigma(t) = \gamma(t)$ for all $t \in [0, r]$.

(2) Suppose that $q \notin \overline{B}_R(p)$. By compactness, there is a point $x \in S_R(p)$ such that $d(x, q)$ is minimal. Then $d(p, q) \leq d(p, x) + d(x, q) = R + d(x, q)$. Suppose that $d(p, q) < d(p, x) + d(x, q) = R + d(x, q)$. Then there exists a piecewise smooth curve σ connecting p and q of length $L(\sigma) < R + d(x, q)$. Let σ_1 be the first part of σ that runs entirely inside the closed geodesic ball from p to a point on $S_R(p)$, and let σ_2 be the last part of σ that runs entirely outside the closed geodesic ball from a point of $S_R(p)$ to q. Then

$$R + d(x, q) > L(\sigma) \geq L(\sigma_1) + L(\sigma_2) \geq R + L(\sigma_2)$$

because $L(\sigma_1) \geq R$ by the first part of this theorem. Now $L(\sigma_2) < d(x, q)$ so that the start-point of σ_2 is a point on $S_R(p)$ that is closer to q than x. Contradiction! □

05.04.05

A smooth curve $\gamma\colon I \to M$ is *locally minimizing* if any $t_0 \in I$ has a neighborhood such that $\gamma|[t_1, t_2]$ is minimizing for all $t_1 < t_2$ in this neighborhood.

Proof of second part of 2.82. Let γ be a geodesic and $\gamma(t_0)$ a point on γ. Choose a uniformly geodesic neighborhood W around $\gamma(t_0)$. The preimage $\gamma^{-1}(W)$ is a union of open intervals. Let $I_0 \subset I$ be the interval containing t_0. If $t_1, t_2 \in I_0$ then $\gamma|[t_1, t_2]$ is geodesic in $W \subset B_\delta(\gamma(t_1))$ through $\gamma(t_1)$ so it is a radial geodesic in $B_\delta(\gamma(t_1))$, hence (2.83) minimizing. □

Also when we go beyond where $\exp_p$ is injective we can sometimes find minimizing curves, they may no longer be unique, though. (Look at curves on S^2 from N to S.)

2.85. Lemma. *Suppose that there is a point $p \in M$ such that the exponential map at p is defined on the whole tangent space T_pM. Then there is a minimizing curve, of the form $t \to \exp_p(tV)$, $0 \leq t \leq d(p, q)$ for some unit vector $V \in T_pM$, from p to any other point in M.*

Proof. Let q be some point different from p and let $T = d(p, q) > 0$ be the distanec between p and q.

Choose a closed geodesic ball, $\overline{B}_R(p)$, around p. We may assume that q is outside this ball, ie $T > R$, for otherwise we already know that there exists a minimizing curve from p to q (2.83). Let x be a point on $S_R(p)$ that realizes the distance between $S_R(p)$ and q and let γ be the unit speed radial geodesic from p through x. By assumption, γ is defined for all $t \geq 0$. The miracle is that γ goes through q: $\gamma(T) = q$.

To see this, consider the set

$$\mathcal{S} = \{b \in [0, T] \mid d(p, q) = d(p, \gamma(b)) + d(\gamma(b), q)\}$$

By using the continuity of the distance function d one can show that $\mathcal{S}$ is closed (take a sequence of points in $\mathcal{S}$). From 2.83 we know that $[0, R] \subset \mathcal{S}$. Let $A = \sup \mathcal{S}$ and put $y = \gamma(A)$. Then $T = d(p, q) = d(p, y) + d(y, q) = A + d(y, q)$ as $A \in \mathcal{S}$. Suppose that $A < T$. Choose a closed geodesic ball $\overline{B}_\delta(y)$ around y where $0 < \delta < T - A$. Let z be the point on the geodesic sphere $S_\delta(y)$ such that $d(y, q) = \delta + d(z, q)$ (2.83) and let τ be the unique radial unit speed geodesic from y to z. The piecewise smooth curve $\gamma|[0, A] \cup \tau$ from p to z has length $A + \delta$ and as

$$d(p, z) \geq d(p, q) - d(z, q) = T - (d(y, q) - \delta) = T - (T - A - \delta) = A + \delta$$

it is minimizing, hence a geodesic (2.82), in particular a smooth curve with no breaks. Thus $\gamma|[0,A]$ and τ must fit together to form the geodesic $\gamma|[0,A+\delta]$ by uniqueness of geodesics. Now $z=\gamma(A+\delta)$ and $d(p,z)=A+\delta$ so that

$$d(p,z)+d(z,q)=(A+\delta)+(T-A-\delta)=T=d(p,q)$$

which contradicts that A is the supremum of $\mathcal{S}$.

Now we know that $T\in\mathcal{S}$ so that

$$T=d(p,q)=d(p,\gamma(T))+d(\gamma(T),q)=L(\gamma|[0,T])+d(\gamma(T),q)=T+d(\gamma(T),q)$$

so that $d(\gamma(T),q)=0$ and $\gamma(T)=q$. □

Now comes a definition that will only be used for a very short time!

2.86. Definition. *A Riemannian manifold is* geodesically complete *if all maximal geodesics are defined for all of* $\mathbf{R}$.

04.05

$\mathbf{R}^2$ is geodesically complete, $B_1^2(0)$ is not.

Recall that a metric space is complete if all Cauchy sequences converge. Compact metric spaces are complete.

Theorem 2.87 (Hopf–Rinow). *M is complete as a metric space $\iff$ M is geodesically complete.*

Proof. Suppose that M is complete as a metric space. The claim is that all geodesics are defined for all time. Suppose that there is some unit speed maximal geodesic $\gamma\colon(a,b)\to M$ that cannot be extended beyond b. Let $t_i\in(a,b)$ be an increasing sequence of points converging to b. Then $\gamma(t_i)$ is Cauchy for $d(\gamma(t_i),\gamma(t_j))\le L(\gamma|[t_i,t_j])=|t_i-t_j|$. Let $q=\lim q(t_i)$. Choose a uniformly geodesic neighborhood W around q and a $\delta>0$ such that $W\subset B_\delta(p)$ for all $p\in W$. This means that any unit speed geodesic through a point of W exists at least in a time span of δ. Choose t_j so that $t_j>b-\delta$ and $q(t_j)\in W$. Then we can extend the geodesic γ near t_j for at least time δ beyond t_j. Contradiction!

Next sssume that there is a point $p\in M$ such that $\exp_p$ is defined for all T_pM. Let (q_i) be a Cauchy sequence. We can assume that $p\neq q_i$ for all i (if not, throw away some of the q_i). Choose $V_i\in T_pM$ such that $t\to\exp_p(tV_i)$, $0\le t\le 1$, is a minimizing radial geodesic from p to q_i (2.85). Then $d(p,q_i)=\int_0^1|V_i|dt=|V_i|$. Since Cauchy sequences are bounded, the sequence (V_i) is bounded in T_pM. Any bounded sequence in the inner product space T_pM contains a convergent subsequence by compactness. Suppose that $V_{i_k}\to V\in T_pM$. Then $q_{i_k}=\exp_p(V_{i_k})\to\exp_p(V)\in M$ by continuity of $\exp_p$. Any Cauchy sequence containing a convergent subsequence is itself convergent. Thus $\lim q_i=\exp_p(V)$. □

In the future we will not bother to say 'geodesically complete' but just say 'complete'.

We actually proved that the conditions

(1) M is metric complete
(2) All maximal geodesics in M are defined on $\mathbf{R}$
(3) All maximal geodesics through one point of M are defined on $\mathbf{R}$

are equivalent and that any of these conditions imply

(4) There is a minimizing curve between any two points of M.

The Heine–Borel theorem holds in any complete Riemannian manifold.

CHAPTER 3

Curvature

1. The Riemann curvature tensor

The Riemann curvature tensor is the obstruction to flatness.

3.1. Definition. *The* Riemann curvature tensor *is the (5.6)* $(3,1)$*-tensor* $R(X,Y,Z,\omega) = \omega(R(X,Y)Z)$ *corresponding to the* $C^\infty(M)$*-multilinear map*

$$T(M)\times T(M)\times T(M) \xrightarrow{R} T(M), \quad (X,Y,Z)\to R(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

(in a somewhat unorthodox notation) called the Riemann curvature endomorphism.

3.2. Proposition. *The map from 3.1 is* $C^\infty(M)$*-multilinear.*

Proof. The function is clearly $\mathbf{R}$-multilinear.

Note that $[fX,Y] = f[X,Y] - Y(f)X$ by computing $[fX,Y](g)$. The computation

$$\begin{aligned}\nabla_{fX}\nabla_Y Z - \nabla_Y\nabla_{fX}Z - \nabla_{[fX,Y]}Z &= f\nabla_X\nabla_Y Z - \nabla_Y(f\nabla_X Z) - f\nabla_{[X,Y]}Z + Y(f)\nabla_X Z\\ = f\nabla_X\nabla_Y Z - Y(f)\nabla_X Z - f\nabla_Y\nabla_X Z - f\nabla_{[X,Y]}Z + Y(f)\nabla_X Z &= f(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z)\end{aligned}$$

shows that the function is $C^\infty(M)$-linear in the X-variable.

The function is anti-symmetric in the X and Y-variables, so it is also $C^\infty(M)$-linear in the Y-variable.

A direct computation as above shows that is also $C^\infty(M)$-linear in the Z-variable. □

Equivalently, the Riemann curvature tensor is the $(4,0)$-tensor field $\mathrm{Rm} = R^\flat$ given by (5.8)

$$\mathrm{Rm}(X,Y,Z,W) = R(W,Y,Z,W^\flat) = W^\flat(R(X,Y)Z) = \langle R(X,Y)Z, W\rangle$$

In local coordinates (x^i), the components of the curvature tensors

$$R = R_{ijk}{}^{\ell} dx^i\otimes dx^j\otimes dx^k\otimes\partial_\ell, \qquad \mathrm{Rm} = R_{ijk\ell}dx^i\otimes dx^j\otimes dx^k\otimes dx^\ell$$

are given by

$$R(\partial_i,\partial_j)\partial_k = R_{ijk}{}^{\ell}\partial_\ell$$

so that

$$R_{ijk\ell} = \langle R(\partial_i,\partial_j)\partial_k,\partial_\ell\rangle = \left\langle R_{ijk}{}^{m}\partial_m,\partial_\ell\right\rangle = g_{\ell m}R_{ijk}{}^{m}, \qquad R_{ijk}{}^{m} = g^{m\ell}R_{ijk\ell} \tag{3.3}$$

3.4. Example. In Euclidean geometry $(\mathbf{R}^n,\overline{g})$, $\overline{\nabla}_X(Y^k\partial_k) = X(Y^k)\partial_k$, and (2.22) shows that $R(\partial_i,\partial_j)\partial_k = 0$ as the basis vector fields commute, $[\partial_i,\partial_j]=0$. Thus $R=0$ on $\mathbf{R}^n$. On S^2 with spherical coordinates the curvature tensor is nonzero in that $R(\partial_\theta,\partial_\phi)\partial_\phi = -\partial_\theta$.

3.5. Lemma (Symmetries in the Riemann curvature tensor). *Let R and Rm be the curvature tensor of a Riemannian manifold.*

1. *Rm is anti-symmetric in the first two variables: $Rm(X,Y,Z,W) = -Rm(Y,X,Z,W)$*
2. *Rm is anti-symmetric in the last two variables: $Rm(X,Y,Z,W) = -Rm(X,Y,W,Z)$*
3. *Rm is symmetric between the first two variables and the last two variables: $Rm(X,Y,Z,W) = -Rm(Z,W,X,Y)$*
4. *Rm satisfies a cyclic permutation property of the first three variables:*

$$Rm(X,Y,Z,W) + Rm(Z,X,Y,W) + Rm(Y,Z,X,W) = 0$$

known as the First Bianchi identity.

(5) *∇Rm satisfies a cyclic permutation property of the last three variables:*

$$\nabla Rm(X,Y,Z,V,W) + \nabla Rm(X,Y,W,Z,V) + \nabla Rm(X,Y,V,W,Z) = 0$$

known as the Second or Differential Bianchi identity.

In a local coordinate frame these identities are equivalent to

(1) $R_{ijk\ell} = -R_{jik\ell}$
(2) $R_{ijk\ell} = -R_{ij\ell k}$
(3) $R_{ijk\ell} = -R_{k\ell ij}$
(4) $R_{ijk\ell} + R_{kij\ell} + R_{jki\ell} = 0$
(5) $R_{ijk\ell;m} + R_{ijmk;\ell} + R_{ij\ell m;k} = 0$

Proof. Each item is a more or less clever calculation.
(1) Clear.
(2) It is enough to show that $\mathrm{Rm}(X,Y,Z,Z)=0$ as we see by expanding $\mathrm{Rm}(X,Y,Z+W,Z+W)$ using multilinearity. Now

$$\begin{aligned}
&XY|Z|^2 - YX|Z|^2 = [X,Y]|Z|^2 \\
&\overset{2.46}{\iff} 2(\langle \nabla_X\nabla_Y Z, Z\rangle + \langle \nabla_Y Z, \nabla_X Z\rangle) - 2(\langle \nabla_Y\nabla_X Z, Z\rangle + \langle \nabla_X Z, \nabla_Y Z\rangle) = 2\left\langle \nabla_{[X,Y]}Z, Z\right\rangle \\
&\iff \langle \nabla_X\nabla_Y Z, Z\rangle - \langle \nabla_Y\nabla_X Z, Z\rangle = \left\langle \nabla_{[X,Y]}Z, Z\right\rangle \iff \mathrm{Rm}(X,Y,Z,Z) = 0
\end{aligned}$$

□

3.6. Proposition. *The Riemann curvature tensor Rm is invariant under local isometries.*

Proof. If $\phi\colon M \to \widetilde{M}$ is a (local) isometry then $\widetilde{\mathrm{Rm}}(\phi_*X, \phi_*Y, \phi_*Z, \phi_*W) = \mathrm{Rm}(X,Y,Z,W)$ by (2.57.(3)). □

3.7. Example. The curvature tensor in Euclidean geometry is $R=0$. The curvature tensor of S^2 is not zero as, for instance, $R(\frac{\partial}{\partial\phi}, \frac{\partial}{\partial\theta})\frac{\partial}{\partial\phi} = -\frac{\partial}{\partial\theta}$ so that $\mathrm{Rm}(\frac{\partial}{\partial\phi}, \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}) = g_{11} = -R^2\sin^2\phi$ using spherical coordinates as in [**4**, Exercise 5.7]. Thus S^2 is not locally isometric to $\mathbf{R}^2$.

A Riemannian manifold is *flat* if any point has a neighborhood isometric to an open subspace of $\mathbf{R}^n$.

Theorem 3.8. *M is flat $\iff$ $R=0$.*

Proof. We have already established one direction. What remains is to show that if $R=0$ is a neighborhood of a point p then there are coordinates (y^i) near p so that $g_{ij} = \delta_{ij}$ in these coordinates. Since this is a local question we may as well assume that $M = \mathbf{R}^n$ and $p=0$.

Put $E_i = \partial_i(0)$ so that $(E_1,\ldots,E_n)$ is the standard orthonormal basis for $T_0\mathbf{R}^n$.

First, we extend E_j to a vector field on $\mathbf{R}^n$. Let E_j be the unique parallel vector field along the x^1 axis $t \to (t,0,\ldots,0)$ with $E_j(0) = E_j$. Next, for each fixed x_0^1, let E_j be the unique parallel vector field along the line $t \to (x_0^1, t, 0, \ldots, 0)$ satisfying the initial condition that at $t=0$ it is $E_j(x_0^1, t, 0, \ldots, 0)$. The vector field E_j is now defined in the x^1x^2-plane. Continue this way. The result is a smooth vector field on $\mathbf{R}^n$.

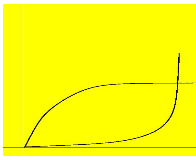
$\underline{R = 0 \Longrightarrow E_j \text{ is parallel.}}$ By construction, E_j is parallel along the x^1-axis. Thus $\nabla_{\partial_1}E_j = 0$ at any point on the x^1-axis. By construction, E_j is parallel along the lines $(x_0^1, x^2, 0\ldots,0,\ldots,0)$. Thus $\nabla_{\partial_2}E_j = 0$ at any point in the x^1x^2-plane. Also the vector field $\nabla_{\partial_1}E_j$ is parallel along the lines $(x_0^1, x^2, 0\ldots,0,\ldots,0)$, $\infty < x_2 < \infty$, for

$$\nabla_{\partial_2}\nabla_{\partial_1}E_j = \nabla_{\partial_1}\nabla_{\partial_2}E_j = \nabla_{\partial_1}0 = 0$$

since $R=0$ and $[\partial_1,\partial_2]=0$. Thus $\nabla_{\partial_1}E_j$ is the parallel vector field along this line with value $\nabla_{\partial_1}E_j(x^1,0,\ldots,0) = 0$ at $x^2=0$. That vector field is the zero vector field. We conclude that $\nabla_{\partial_1}E_j = 0$ at all points in the x^1x^2-plane. Continue this way and conclude that $\nabla_{\partial_1}E_j = 0, \ldots, \nabla_{\partial_n}E_j = 0$ at all points in $x^1x^2\cdots x^n$-space. Thus $\nabla_X E_j = 0$ for any vector field X by $C^\infty(\mathbf{R}^n)$-linearity.

Compatibility $\implies (E_1,\dots,E_n)$ is an orthonormal frame. Since the Riemannian connection is compatible with the inner product, parallel translation preserves the inner product (2.52): For a point in the x^1x^2-plane for instance

$$\big\langle E_i(x^1,x^2,0,\dots,0),E_j(x^1,x^2,0,\dots,0)\big\rangle = \big\langle E_i(x^1,,0,\dots,0),E_j(x^1,0,\dots,0)\big\rangle \\ = \langle E_i(0,\dots,0),E_j(0,\dots,0)\rangle = \langle E_i,E_j\rangle = \delta_{ij}$$



In general, we see in this way that $(E_1,\dots,E_n)$ is an orthonormal moving frame.
Symmetry $\implies$ the vector fields $E_1,\dots,E_n$ commute. We have $[E_i,E_j] = \nabla_{E_i}E_j - \nabla_{E_j}E_i = 0$ since the Riemannian connection is symmetric (2.44).

Finally, a theorem of elementary differential geometry says that the vector fields E_i are coordinate vector fields for some coordinate system (y^i) near 0. Thus $g_{ij}=\delta_{ij}$ for this coordinate system. □

14.04.05

2. Ricci curvature, scalar curvature, and Einstein metrics

3.9. Definition. *Ricci curvature is the trace on the first and last variable of the Riemann curvature endomorphism:* $Rc = \operatorname{tr}(R) = \operatorname{tr}_g(Rm) \in \mathcal{T}_0^2(M)$.

This simply means that Ricci curvature is the tensor given by

$$\operatorname{Rc}(X,Y) = \operatorname{tr}(U \to R(U,X)Y)$$

Ricci curvature is a $(2,0)$-tensor with components

$$\operatorname{Rc}_{ij} = R_{kij}{}^k = g^{k\ell}R_{kij\ell} = g^{\ell k}R_{kij\ell} = R^\ell{}_{ij\ell} = R^k{}_{ijk}$$

The symmetries of the Riemann curvature give

- $\operatorname{Rc}_{ij} = R_{kij}{}^k = R_j{}^k{}_{ki} = R_{jk}{}^k{}_i$
- $\operatorname{Rc}_{ij} = R_{kij}{}^k = -R_{ikj}{}^k = -R_{ki}{}^k{}_j$
- $\operatorname{Rc}_{ij} = R_j{}^k{}_{ki} = R^k{}_{jik} = \operatorname{Rc}_{ji}$

where the last line means that the Ricci tensor is symmetric.

3.10. Definition. *Scalar curvature is the trace with respect to g (5.26) of the Ricci curvature:* $S = \operatorname{tr}_g Rc$.

Scalar curvature is the smooth function on M given by

$$S = \operatorname{Rc}_j{}^j = R_{ij}{}^{ji}$$

3.11. Definition. *The* divergence *operators div are the maps*

$$\mathcal{T}_\ell^0(M) \xrightarrow{\nabla} \mathcal{T}_\ell^1(M) \xrightarrow{\operatorname{tr}} \mathcal{T}_{\ell-1}^0(M), \qquad \mathcal{T}_0^k(M) \xrightarrow{\nabla} \mathcal{T}_0^{k+1}(M) \xrightarrow{\operatorname{tr}_g} \mathcal{T}_0^{k-1}(M)$$

where in the last case the trace is taken with respect to the covariant differentiation index and some other lower index.

For $X \in \mathcal{T}_\ell^0(M)$, $\operatorname{div}(X)^{j_1\cdots j_{\ell-1}} = X^{j_1\cdots j_{\ell-1}i}{}_{;i}$ and for $X \in \mathcal{T}_0^k(M)$, $\operatorname{div}(X)_{i_1\cdots i_{k-1}} = X_{i_1\cdots i_{k-1}j;}{}^j$. For example

- the divergence of a vector field $X = X^j\partial_j \in \mathrm{T}_1^0(M)$ is the smooth function $\operatorname{div}(X) = X^i{}_{;i}$
- the divergence of the Ricci curvature tensor $\operatorname{Rc} \in \mathcal{T}_0^2(M)$ is the covector field $\operatorname{div}(\operatorname{Rc}) \in \mathcal{T}_0^1(M)$ with components
$$\operatorname{div}(\operatorname{Rc})_m = \operatorname{Rc}_{mj;}{}^j = R_{imj}{}^i{}_;{}^j$$
We compute this tensor below (3.12).
- the divergence of the metric g is $\operatorname{div}(g) = \operatorname{tr}_g(\nabla g) = \operatorname{tr}_g(0) = 0$ because $\nabla g = 0$ as the connection and the metric are compatible (2.52).
- the divergence of the product $Sg = S\otimes g$ of the scalar curvature and the metric is $\operatorname{div}(Sg) = \operatorname{tr}_g(\nabla(Sg)) = \operatorname{tr}_g(\nabla S\otimes g + S\otimes\nabla g) = \operatorname{tr}_g(\nabla S\otimes g) = S_{;j}g_i{}^j = S_{;j}\delta_i^j = S_{;i} = \nabla S$ because $(S\otimes g)_{mij} = S_{;m}g_{ij}$.

3.12. Lemma (Contracted Bianchi identity). $div(Rc) = \frac{1}{2}\nabla S$ *or* $S_{;m} = 2Rc_{mj;}{}^j = 2R_{imj}{}^i{}_;{}^j$.

PROOF. Start with the Differential Bianchi identity

$$R_{ijk\ell;m} + R_{ijmk;\ell} + R_{ij\ell m;k} = 0$$

and take the g-trace over i and ℓ,

$$R_{ijk\ ;m}{}^{i} + R_{ijmk;}{}^{i} + R_{ij}{}^{i}{}_{m;k} = 0$$

and take the g-trace over j and k,

$$R_{ij}{}^{ji}{}_{;m} + R_{ijm}{}^{j}{}_{;}{}^{i} + R_{ij}{}^{i}{}_{m;}{}^{j} = 0$$

The first term we recognize as $S_{;m}$. The second term is $-R_{jim}{}^{j}{}_{;}{}^{i} = -\mathrm{div}(\mathrm{Rc})_m$. The third term is $-R_{ijm}{}^{i}{}_{;}{}^{j} = -\mathrm{div}(\mathrm{Rc})_m$. □

04.05

3.13. DEFINITION. *The metric g is an Einstein metric if it is proportional to its Ricci curvature at any point: $\mathrm{Rc} = \lambda g$ for some smooth function λ on M.*

In fact, the function λ is implicitly given by the Einstein equation:

$$\mathrm{Rc} = \lambda g \Longrightarrow \mathrm{tr}_g(\mathrm{Rc}) = \mathrm{tr}_g(\lambda g) \iff S = \lambda n \iff \lambda = \frac{1}{n} S$$

because $\mathrm{tr}_g(g) = n = \dim M$ (5.26).

3.14. PROPOSITION. *Any connected Riemannian manifold of dimension > 2 with an Einstein metric has constant scalar curvature.*

PROOF. $\mathrm{Rc} = \frac{1}{n} S \otimes g \Longrightarrow \mathrm{div}(\mathrm{Rc}) = \frac{1}{n}\mathrm{div}(S \otimes g) \iff \frac{1}{2}\nabla S = \frac{1}{n}\nabla S \overset{n>2}{\Longrightarrow} \nabla S = 0$. so that S is constant on each component of M. □

3.15. EXAMPLE (Curvature of surfaces). Let M be a Riemannian manifold of dimension 2, a Riemann surface. Let $K = \frac{1}{2} S$ denote the function that is half of the scalar cuvature (we shall later call it the *Gaussian curvature* of the surface). Let (E_1, E_2) be an orthonormal basis for the tangent space T_pM at some point p of M.
Riemann curvature: The Riemann curvature tensor has $2^4 = 16$ components $R_{ijk\ell} = \mathrm{Rm}(E_i, E_j, E_k, E_\ell)$, $1 \le i, j \le 2$. However, for (anti-)symmetry reasons $R_{iik\ell} = 0 = R_{ijkk}$, so that

$$R_{1221} = R_{2112} = -R_{2121} = -R_{1212}$$

are the only nonzero components.
Ricci curvature: The components of $\mathrm{Rc} = \mathrm{tr}_g(R)$ are $\mathrm{Rc}_{ij} = R_{1ij}{}^{1} + R_{2ij}{}^{2} = R_{1ij1} + R_{2ij2}$. Here we use that $R_{1ij}{}^{1} = g^{1k} R_{1ijk} = \delta_1^k R_{1ijk} = R_{1ij1}$ since the basis is orthonormal so that the matrix for g and its inverse are identity matrices. Hence

$$\begin{aligned} \mathrm{Rc}_{11} &= R_{2112} = R_{1221} & \qquad \mathrm{Rc}_{12} &= 0 \\ \mathrm{Rc}_{21} &= 0 & \mathrm{Rc}_{22} &= R_{1221} \end{aligned}$$

are the components of Ricci curvature.
Scalar curvature: $S = \mathrm{tr}_g(\mathrm{Rc}) = \mathrm{Rc}_1{}^1 + \mathrm{Rc}_2{}^2 = \mathrm{Rc}_{11} + \mathrm{Rc}_{22} = 2R_{1221}$. so that $R_{1221} = \frac{1}{2} S = K$.

We conclude that $K = R_{1221} = \mathrm{Rc}_{11} = \mathrm{Rc}_{22}$ so that scalar curvature determines Ricci and Riemann curvature for surfaces.

Until now we have been working with an orthonormal basis. Let us now consider an arbitrary basis (X, Y) for T_pM. Then

$$E_1 = \frac{X}{|X|}, \qquad E_2 = \frac{Y - \left\langle Y, \frac{X}{|X|} \right\rangle \frac{X}{|X|}}{|Y - \left\langle Y, \frac{X}{|X|} \right\rangle \frac{X}{|X|}|}$$

is the orthonormal basis obtained by applying the Gram–Schmidt process to (X, Y). From the above computations

$$\begin{aligned} \mathrm{Rc}(X, Y) = \mathrm{Rc}(X^1 E_1 + X^2 E_2, Y^1 E_1 + Y^2 E_2) = X^1 Y^1 \mathrm{Rc}_{11} + X^2 Y^2 \mathrm{Rc}_{22} \\ = K X^1 Y^1 + K X^2 Y^2 = K \langle X, Y \rangle \end{aligned}$$

$$\mathrm{Rm}(X, Y, Z, W) = K(\langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle)$$

For the last equation, note that both sides are multilinear and equal in case $X, Y, Z, W \in \{E_1, E_2\}$; try for instance $(X, Y, Z, W) = (E_1, E_2, E_2, E_1)$. In the special case where $Z = Y$ and $W = X$ this equation gives that

$$K = \frac{\mathrm{Rm}(X, Y, Y, X)}{|X|^2|Y|^2 - \langle X, Y\rangle^2} \tag{3.16}$$

The denominator here is the area of the parallelogram spanned by vectors X and Y.

See 'A compendium of Surfaces' in [**8**] for much more information about surfaces.

3. Riemannian submanifolds

Let $(\widetilde{M}, g)$ be a Riemannian manifold and $M \subset \widetilde{M}$ an embedded submanifold equipped with the induced metric, also called g. The 2nd fundamental form of the Riemannian submanifold M is the difference between the Riemannian connections $\widetilde{\nabla}$ and ∇. (The 1st fundamental form is the metric g.)

The ambient tangent bundle $T\widetilde{M}|M \to M$ splits

$$T\widetilde{M}|M = TM \oplus NM$$

into the orthogonal direct sum of the tangent bundle of M with the normal bundle $NM \to M$.

Any section of $T\widetilde{M}|M$ splits orthogonally into a direct sum of its tangential and normal part. If $X, Y \in \mathcal{T}(M)$ are smooth vector fields on M, then $\widetilde{\nabla}_X Y$ is a well-defined (2.56) section of $T\widetilde{M}|M$. We already know (2.55) that $\nabla_X Y$ is the tangential component of $\widetilde{\nabla}_X Y$. If we write $II(X, Y)$ for the normal component then the orthogonal splitting of $\widetilde{\nabla}_X Y$ has the form

$$\widetilde{\nabla}_X Y = (\widetilde{\nabla}_X Y)^T + (\widetilde{\nabla}_X Y)^\perp = \underbrace{\nabla_X Y}_{T_pM} + \underbrace{II(X, Y)}_{N_pM} \qquad \text{(Gauss formula)} \tag{3.17}$$

where the normal component $II(X, Y) \in NM$ is called the *second fundamental form.* Equivalently,

$$II(X, Y) = \widetilde{\nabla}_X Y - \nabla_X Y$$

is the difference between the extrinsic connection $\widetilde{\nabla}$ and the intrinsic connection ∇.

3.18. Lemma. *Let $M \subset \widetilde{M}$ be a Riemannian submanifold and $II\colon \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{N}(M)$ its second fundamental form.*

(1) *II $C^\infty(M)$-bilinear and symmetric.*
(2) *If $X, Y \in \mathcal{T}(M)$ are vector fields and $N \in \mathcal{N}(M)$ a normal field on M then*

$$\left\langle \widetilde{\nabla}_X N, Y \right\rangle = -\langle N, II(X, Y)\rangle \qquad \textit{(Weingarten equation)} \tag{3.19}$$

(3) *If $X, Y, Z, W \in \mathcal{T}(M)$ are vector fields on M then*

$$\widetilde{Rm}(X, Y, Z, W) = Rm(X, Y, Z, W) - \langle II(X, W), II(Y, Z)\rangle + \langle II(X, Z), II(Y, W)\rangle \tag{3.20}$$

(Gauss equation)

(4) *Let $\gamma\colon I \to M$ be a curve in M and V a vector field in M along γ. Then*

$$\widetilde{D}_t V = D_t V + II(\dot{\gamma}, V) \qquad \textit{(Gauss formula along a curve)} \tag{3.21}$$

Proof. (2) $\left\langle \widetilde{\nabla}_X Y, N \right\rangle + \left\langle Y, \widetilde{\nabla}_X N \right\rangle = \widetilde{\nabla}_X \langle N, Y\rangle = \widetilde{\nabla}_X(0) = 0$ where $\left\langle \widetilde{\nabla}_X Y, N \right\rangle = \langle II(X, Y), N\rangle$.
(3) Riemann curvature computed in ambient space is

$$\widetilde{\mathrm{Rm}}(X, Y, Z, W) = \left\langle \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]} Z, W \right\rangle$$

Observe that $\left\langle \widetilde{\nabla}_{[X,Y]} Z, W \right\rangle = \langle \nabla_{[X,Y]} Z, W\rangle$ because the normal part of the first vector does not contribute to the inner product with a vector tangent to M. Observe also that

$$\left\langle \widetilde{\nabla}_X \widetilde{\nabla}_Y Z, W \right\rangle = \left\langle \widetilde{\nabla}_X (\nabla_Y Z + II(Y, Z)), W \right\rangle = \left\langle \widetilde{\nabla}_X \nabla_Y Z, W \right\rangle + \left\langle \widetilde{\nabla}_X II(Y, Z), W \right\rangle$$
$$= \langle \nabla_X \nabla_Y, W\rangle - \langle II(Y, Z), II(X, W)\rangle$$

It now follows that

$$\widetilde{\mathrm{Rm}}(X,Y,Z,W) = \mathrm{Rm}(X,Y,Z,W) - \langle II(Y,Z), II(X,W)\rangle + \langle II(X,Z), II(Y,W)\rangle$$

which is Gauss's equation. □

3.21. Remark (Gauss formula for the velocity field along a curve). If we apply the Gauss formula along a curve to the special case where $V = \dot{\gamma}$ is the velocity field then we get that

$$\widetilde{D}_t \dot{\gamma} = D_t \dot{\gamma} + II(\dot{\gamma}, \dot{\gamma})$$

and we see that

- If γ is a geodesic in M, then $\widetilde{D}_t \dot{\gamma} = II(\dot{\gamma}, \dot{\gamma})$. Thus $II(V,V)$, $V \in T_pM$, is the acceleration at p in $\widetilde{M}$ of the geodesic γ_V in M.
- If the curve γ in M is a geodesic in $\widetilde{M}$, then γ is also a geodesic in M and $II(\dot{\gamma}, \dot{\gamma}) = 0$. Thus $II(V,V) = 0$ if the geodesic γ_V in $\widetilde{M}$, $V \in T_pM \subset T_p\widetilde{M}$, happens to stay inside M.

Since the second fundamental form is a symmetric bilinear form is it completely determined by its quadratic function $V \to II(V,V)$.

3.22. Gaussian and mean curvature of codimension one Euclidean embeddings. We consider the simplest case of a Riemannian submanifold, namely that of (an orientable) hypersurface in Euclidean space, $M^n \subset \mathbf{R}^{n+1}$. We shall associate curvature to the embedding.

Choose a normal field N that is nonzero at every point of M. (This is possible if M is orientable; in any case it is possible to choose such a normal field locally.) Using N, we may write the second fundamental form $II \colon \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{N}(M)$ as

$$II(X,Y) = h(X,Y)N \text{ so that } \langle \overline{\nabla}_X Y, N\rangle = \langle II(X,Y), N\rangle = |N|^2 h(X,Y)$$

where $h \in \mathcal{T}^2_0(M)$ is symmetric $(2,0)$-tensor on M, the *scalar second fundamental form.* In this case of a codimension one embedding into Euclidean space $\mathbf{R}^{n+1}$ the Gauss and Weingarten formulas specialize to

$$\overline{\nabla}_X Y = \overbrace{\nabla_X Y}^{T_pM} + \overbrace{h(X,Y)N}^{N_pM} \quad \text{(Gauss formula)} \tag{3.23}$$

$$|N|^2 h(X,Y) = \langle \overline{\nabla}_X Y, N\rangle = -\langle \overline{\nabla}_X N, Y\rangle \quad \text{(Weingarten equation for } N\text{)} \tag{3.24}$$

$$\mathrm{Rm}(X,Y,Z,W) = |N|^2 (h(X,W)h(Y,Z) - h(X,Z)h(Y,W)) \quad \text{(Gauss equation)} \tag{3.25}$$

For the Gauss equation (3.25) note that $\widetilde{\mathrm{Rm}} = 0$ in Euclidean space and that the inner product $\langle II(X,W), II(Y,Z)\rangle = \langle h(X,W)N, h(Y,Z)N\rangle = |N|^2 h(X,W)h(Y,Z)$.

The *shape operator* $s \in \mathcal{T}^1_1(M)$ is defined by

$$\forall X,Y \in \mathcal{T}(M) \colon \langle sX, Y\rangle = |N| h(X,Y) = \frac{1}{|N|}\langle \overline{\nabla}_X Y, N\rangle = -\frac{1}{|N|}\langle \overline{\nabla}_X N, Y\rangle \tag{3.26}$$

meaning that $s^\flat = |N|h$ or $s = |N|h^\sharp$ (5.12) is obtained from h by raising an index. The shape operator

$$sX = -\frac{1}{|N|}\overline{\nabla}_X N$$

informs about the shape of M since it measures the variation of the normal field N as it moves on M. The shape operator is self-adjoint,

$$\langle sX, Y\rangle = |N|h(X,Y) = |N|h(Y,X) = \langle sY, X\rangle = \langle X, sY\rangle$$

because h is symmetric. Therefore T_pM has an orthonormal basis $E_1, \dots, E_n$ of eigenvectors, $sE_i = \kappa_i E_i$, for s.

3.27. Definition. *The* principal directions *and the* principal curvatures *of the embedding $M \subset \widetilde{M}$ are the orthonormal eigenvectors and the eigenvalues of the shape operator. The* Gauss curvature *and the* mean curvature *of the embedding $M \subset \widetilde{M}$ is the determinant, $K = \det s = \prod \kappa_i$, and $1/n$th of the trace, $H = \frac{1}{n}\operatorname{tr} s = \frac{1}{n}\sum \kappa_i$, of s.*

Principal curvatures are invariants of the *embedding* and not invariants of the manifold; see [**4**, p 5–6] for examples. It therefore came as a total surprise that the product of the two principal curvatures of an embedded surface does not depend on the embedding but only on the surface itself.

3.28. Remark (The shape operator in a local frame). Suppose that (E_i) is a local frame. The components of $s = |N|h^\sharp$ are (5.27)

$$s_i{}^j = |N|h_i{}^j = |N|g^{jk}h_{ik}$$

and we obtain the formulas

$$H = \frac{1}{n}\operatorname{tr} s = \frac{1}{n}|N|g^{ij}h_{ij}, \qquad K = \det s = |N|^n\frac{\det(h_{ij})}{\det(g_{ij})}$$

for the mean and Gaussian curvature. If N is a unit normal field, the Gaussin curvature is the determinant of the 2nd fundamental form relative to the determinant of the 1st fundamental form. Inserting $h_{ij} = h(E_i, E_j) = \frac{1}{|N|^2}\left\langle \overline{\nabla}_{E_i}E_j, N\right\rangle$ from the Weingarten equation (3.24) we can also write

$$H = \frac{1}{n|N|}g^{ij}\left\langle \overline{\nabla}_{E_i}E_j, N\right\rangle, \qquad K = \frac{1}{|N|^n}\frac{\det(\left\langle \overline{\nabla}_{E_i}E_j, N\right\rangle)}{\det(g_{ij})}$$

for the mean and Gaussian curvature.

Theorem 3.29 (Theorema Egregium, Gauss 1828). *The Gaussian curvature of a Riemannian embedding $M^2 \subset \mathbf{R}^3$ of a Riemannian surface in $\mathbf{R}^3$ does not depend on the embedding but only on the Riemann surface itself. (In fact, the Gaussian curvature of the embedding equals half the scalar curvature of the surface.)*

Proof. Choose an orthonormal local frame (E_1, E_2) and a unit normal field N for M (to make life a little easier). Then the Gaussian curvature of the Riemannian *sub*manifold M

$$K \overset{3.28}{=} \det(h_{ij}) = h(E_1,E_1)h(E_2,E_2) - h(E_1,E_2)^2 \overset{(3.25)}{=} \operatorname{Rm}(E_1,E_2,E_2,E_1) \overset{3.15}{=} \frac{1}{2}S$$

equals half of the scalar curvature of the Riemannian manifold M. □

In Gauss' original formulation the theorem goes something like

> If an area in E^3 can be developed (i.e. mapped isometrically) into another area of E^3, the values of the Gaussian curvatures are identical in corresponding points

Gauss received a prize from the University of Copenhagen for this theorem.

We can therefore speak of the 'Gaussian curvature' of an (orientable) Riemann surface (as we did in (3.15)).

3.30. Example (The shape of a parameterized surface in $\mathbf{R}^3$). Let $X(u, v)$ be a parameterization of a surface in $M \subset \mathbf{R}^3$. The vectors $X_u = X_*(\partial_u) = \frac{\partial X_i}{\partial u}\partial_i$ and $X_v = X_*(\partial_v) = \frac{\partial X_i}{\partial v}\partial_i$ (1.2) form a basis for the tangent space of the surface. In this basis, the metric, the 1st fundamental form, is

$$g = \begin{pmatrix} \langle X_u, X_u\rangle & \langle X_u, X_v\rangle \\ \langle X_v, X_u\rangle & \langle X_v, X_v\rangle \end{pmatrix}, \qquad g^{-1} = \frac{1}{\det g}\begin{pmatrix} \langle X_v, X_v\rangle & -\langle X_u, X_v\rangle \\ -\langle X_v, X_u\rangle & \langle X_u, X_u\rangle \end{pmatrix},$$

The cross product $N = X_u \times X_v$ is an everywhere nonzero normal field. The vector $X_{uu} = \overline{\nabla}_{X_u}X_u$,

$$X_{uu} = \overline{\nabla}_{X_u}X_u = \overline{\nabla}_{X_u}(\tfrac{\partial X^i}{\partial u}\partial_i) \overset{(2.22)}{=} X_u(\tfrac{\partial X^i}{\partial u})\partial_i \overset{(2.34)}{=} \frac{\partial^2 X^i}{\partial^2 u}\partial_i$$

is simply obtained by differentiating each of the three coordinate functions in X twice wrt u. The 2nd fundamental form is

$$h = \frac{1}{2|N|}\begin{pmatrix} \langle X_{uu}, N\rangle & \langle X_{uv}, N\rangle \\ \langle X_{vu}, N\rangle & \langle X_{vv}, N\rangle \end{pmatrix}$$

The formulas from 3.28 take the form

$$H = \frac{1}{2|N|}\frac{\langle X_v, X_v\rangle \langle X_{uu}, N\rangle - 2\langle X_u, X_v\rangle \langle X_{uv}, N\rangle + \langle X_u, X_u\rangle \langle X_{vv}, N\rangle}{\langle X_u, X_u\rangle \langle X_v, X_v\rangle - \langle X_u, X_v\rangle^2}$$

$$K = \frac{1}{|N|^2}\frac{\langle X_{uu}, N\rangle \langle X_{vv}, N\rangle - \langle X_{uv}, N\rangle^2}{\langle X_u, X_u\rangle \langle X_v, X_v\rangle - \langle X_u, X_v\rangle^2}$$

In particular, if M is the graph of the function f, then $X(u,v) = (u, v, f(u,v))$, $X_u = (1, 0, f_u)$, $X_v = (0, 1, f_v)$, $X_{uu} = (0, 0, f_{uu})$, $X_{uv} = (0, 0, f_{uv})$, $X_{vv} = (0, 0, f_{vv})$. As $N = X_u \times X_v = (-f_u, -f_v, 1)$ the formula gives the equation

$$H = \frac{f_{uu}(1+f_v^2) + f_{vv}(1+f_v^2) - 2f_{uv}f_uf_v}{2(1+f_u^2+f_v^2)^{3/2}}, \qquad K = \frac{f_{uu}f_{vv} - f_{uv}^2}{(1+f_u^2+f_v^2)^2}$$

for the mean and Gaussian curvature of a graph of a function of two variables. We see that a graph is isometric to a plane if and only if $f_{uu}f_{vv} = f_{uv}f_{uv}$. (The next example is a generalization.)

3.31. Example (The shape of a graph). Let $M \subset \mathbf{R}^n$ be the graph of the smooth function $f\colon \mathbf{R}^n \to \mathbf{R}$. We shall write f_i for the ith partial derivative $\partial f/\partial u_i$ and f_{ij} for $\partial^2 f/\partial u_i \partial u_j$. The embedding $X(u) = (u, f(u))$, $u \in \mathbf{R}^n$, is a parameterization of the graph. As in (3.30) the vectors

$$X^i = \partial_i + f_i\partial_n = (0, \dots, 0, 1, 0, \dots, 0, f_i), \qquad 1 \le i \le n,$$

form a basis for the tangent space T_pM of the graph. Thus $g_{ij} = \langle X^i, X^j\rangle = \delta_{ij} + f_if_j$ and

$$N = -\sum f_i\partial_i + \partial_n = (-f_1, -f_2, \dots, -f_n, 1)$$

is a nonzero normal field. Now, $\overline{\nabla}_{X^i}X^j = f_{ij}\partial_n$ as in (3.30) so that $\langle \overline{\nabla}_{X^i}X^j, N\rangle = f_{ij}$. From (3.28) we get the expressions

$$H = \frac{f_{ij}g^{ij}}{2\sqrt{1+\sum_i f_i^2}}, \qquad K = \frac{1}{|N|^n}\frac{\det(f_{ij})}{\det(g_{ij})} = \frac{\det(f_{ij})}{(1+\sum_i f_i^2)^{n/2}\det(\delta_{ij}+f_if_j)}$$

for the mean and the Gaussian curvature of a graph.

4. Sectional curvature

Let (M, g) be a Riemannian manifold. We shall give a geometric interpretation of the Riemann curvature Rm tensor of M.

Let p be a point of M. For each 2-dimensional subspace Π of the tangent space T_pM, let $S_\Pi \subset M$ be the surface in M that is the image under $\exp_p$ of Π, or rather the image of the part of Π where $\exp_p$ is a diffeomorphism, $S_\Pi = \exp_p(\Pi \cap \varepsilon_p)$. We give S_Π the induced metric so that $S_\Pi \subset M$ is a Riemannian embedding. Note that the tangent space of S_Π is $T_pS_\Pi = T_p\exp_p(\Pi \cap \varepsilon_p) = (\exp_p)_*T_0\Pi = T_0\Pi = \Pi \subset T_pM$ (2.34, 2.65).

3.32. Definition. *Sectional curvature at $p \in M$ is the function that to any tangent plane $\Pi \subset T_pM$ to the manifold associates the Gaussian curvature $K(\Pi) = K(S_\Pi)_p$ at p of the Riemannian surface S_π.*

3.33. Proposition. *The sectional curvature of the tangent plane $\Pi \subset T_pM$ is*

$$K(\Pi) = \frac{Rm(X, Y, Y, X)}{|X|^2|Y|^2 - \langle X, Y\rangle^2}$$

where X, Y is any basis for Π.

Proof. For any $V \in T_pS_\Pi = \Pi$, the geodesic γ_V in M runs inside S_Π. Gauss formula for the velocity of a curve (3.21) implies that $II(V, V) = 0$. Since the second fundamental form vanishes at p, Gauss' equation (3.20) says that $\mathrm{Rm}(X, Y, Z, W)$ is the Riemann curvature tensor for the surface S_Π at p for all $X, Y, Z, W \in T_pS_\Pi$. Now (3.16) tells us about the Gaussian curvature of S_Π at p. □

In particular, if the oriented Riemannian manifold $M^n \subset \mathbf{R}^{n+1}$ sits as an embedded codimension one manifold in $\mathbf{R}^{n+1}$ with Euclidean or Minkowski metric then (3.33, 3.25) the sectional curvature is

$$(3.34)\quad K(X,Y) = |N|^2 \frac{h(X,X)h(Y,Y) - h(X,Y)^2}{g(X,X)g(Y,Y) - g(X,Y)^2} = \frac{1}{|N|^2} \frac{\langle \overline{\nabla}_X X, N\rangle \langle \overline{\nabla}_Y Y, N\rangle - \langle \overline{\nabla}_X Y, N\rangle^2}{\langle X,X\rangle \langle Y,Y\rangle - \langle X,Y\rangle^2}$$

where N is a nowhere zero normal field.

3.35. Proposition. *The sectional curvatures determine the Riemann curvature tensor of M.*

Exactly how you get Riemann curvature out of sectional curvature is a bit complicated [**1**, (1.10)]. It is easier to explain how you get Ricci and scalar curvature.

Let $E_1 \in T_pM$ be a unit vector. Expand (E_1) to an orthonormal basis $E_1, E_2, \ldots, E_n$ for T_pM. Then

$$\mathrm{Rc}(E_1,E_1) = \mathrm{Rc}_{11} = \sum R_{k11}{}^k = \sum_k R_{k11k} = \sum_k \mathrm{Rm}(E_1,E_k,E_k,E_1) = \sum_{k>1} K(E_1,E_k)$$

$$S = \mathrm{Rc}_k^k = \sum_k \mathrm{Rc}_{kk} = \sum_{j,k} R_{jkkj} = \sum_{j,k} \mathrm{Rm}(E_j,E_k,E_k,E_j) = \sum_{j\neq k} K(E_j,E_k)$$

where we write $K(X,Y)$ for the Gaussian curvature $K(\mathrm{span}\{X,Y\})$ of the plane spanned by the independent vectors X and Y.

We shall next see that the three model geometries, Euclidean, spherical, and hyperbolic, are Riemannian manifolds of constant sectional curvature, space forms.

3.36. Euclidean geometry. In Euclidean geometry (2.1, 2.59) the curvature tensor $\mathrm{Rm} = 0$ (3.4) so that the sectional curvature $K(\Pi) = 0$ for any plane $\Pi \subset T_p\mathbf{R}^n$ at any point $p \in \mathbf{R}^n$.

3.37. Spherical geometry. In spherical geometry (2.2, 2.60), the tangent space $T_pS^n_R = p^\perp$ (1.5) so that the normal space $N_pS^n_R = \mathbf{R}p$ is the line through p. Thus $N_p = \frac{1}{R}p$ is the outward pointing unit normal vector field. For any point $p \in S^n_R$ and any unit tangent vector $V \in T_pS^n_R$, Gauss formula for the velocity field of a curve (3.21) tells us that

$$II(V,V) = -\frac{1}{R}N \text{ and } h(V,V) = -\frac{1}{R}$$

as this is the acceleration of the geodesic through p with unit speed V, the great circle through p tangent to $V \in p^\perp$. Consider for instance the geodesic γ through $(0,\ldots,0,R)$ with unit speed $\dot{\gamma}(0) = (1,0,\ldots,0)$ as in (2.61). Its acceleration is

$$\ddot{\gamma}(0) = -\frac{1}{R^2}\gamma(0) = -\frac{1}{R}(\frac{1}{R}\gamma(0)) = -\frac{1}{R}N(\gamma(0))$$

Since $h(V,V) = -\frac{1}{R}$ for all unit tangent vectors V, $h(U,V) = 0$ for any orthonormal pair U,V of tangent vectors. Formula (3.34) tells us that the sectional curvature of S^n_R

$$K(\Pi) = \frac{1}{R^2}$$

is constant. 27.04.05

3.38. Hyperbolic geometry. In hyperbolic geometry (2.14, 2.62), the tangent space $T_pH^n_R = p^\perp$ (2.15) so that the normal space $N_pH^n_R = \mathbf{R}p$ is the line through p. Thus $N_p = \frac{1}{R}p$ is a normal vector field of constant square length $|N|^2 = -1$. For any point $p \in H^n_R$ and any unit tangent vector $X \in T_pH^n_R$, Gauss formula for the velocity field of a curve (3.21) tells us that

$$II(V,V) = \frac{1}{R}N \text{ and } h(V,V) = \frac{1}{R}$$

as this is the acceleration of the geodesic through p with unit speed V, the great hyperbola through p tangent to $V \in p^\perp$. Consider for instance the geodesic γ through $(0,\ldots,0,R)$ with unit speed $\dot{\gamma}(0) = (1,0,\ldots,0)$ as in (2.63). Its acceleration is

$$\ddot{\gamma}(0) = \frac{1}{R^2}\gamma(0) = \frac{1}{R}(\frac{1}{R}\gamma(0)) = \frac{1}{R}N(\gamma(0))$$

Since $h(V,V) = \frac{1}{R}$ for all unit tangent vectors V, $h(U,V) = 0$ for any orthonormal pair U, V of tangent vectors. Formula (3.34) tells us that the sectional curvature of H^n_R

$$K(\Pi) = -\frac{1}{R^2}$$

is constant.

For space forms there are explicit formulas for the curvature tensors.

3.39. Proposition. *In a Riemannian manifold with constant sectional curvature C,*

$$R(X,Y)Z = C\left[\langle Y,Z\rangle X - \langle X,Z\rangle Y\right]$$
$$Rm(X,Y,Z,W) = C\left[\langle X,W\rangle \langle Y,Z\rangle - \langle X,Z\rangle \langle Y,W\rangle\right]$$
$$Rc(X,Y) = (n-1)C\,\langle X,Y\rangle$$
$$S = n(n-1)C$$

Proof. The first and the second equation are equivalent. Both sides of the second equation are tensors with the symmetry properties of the curvature tensor. Thus they are equal if they are equal for $(X,Y,Z,W) = (X,Y,Y,X)$ with orthonormal X,Y (3.35). In that case the LHS is $K(X,Y) = C$ and the RHS is C.

Both sides of the third equation are symmetric tensors. Thus they are equal if they are equal for $(X,Y) = (X,X)$ with X a unit vector. In that case the LHS is $(n-1)C$ by the formula below 3.35 and the RHS is also $(n-1)C$.

The formula below 3.35 says that constant sectional curvature C implies constant scalar curvature of value $n(n-1)C$. □

3.40. Example (The shape of the sphere). Since we know the 2nd fundamental form of the sphere (3.37) we also know the shape operator. In fact, the shape operator (3.26) for the codimension one Euclidean embedding $S^n_R \subset \mathbf{R}^{n+1}$ is the symmetric isomorphism

$$s\colon T_pS^n_R = p^\perp \to T_pS^n_R = p^\perp, \qquad s(V) = -\frac{1}{R}V$$

for this is clearly self-adjoint and its bilinear form, or quadratic function, is given by $\langle sV,V\rangle = -\frac{1}{R} = h(V,V)$ when $|V| = 1$. Thus the principal curvatures all equal $-\frac{1}{R}$ and the mean and Gaussian curvatures of the embedding $S^n_R \subset \mathbf{R}^{n+1}$ are

$$H = \frac{1}{n}\operatorname{tr}(s) = -\frac{1}{R}, \qquad K = \det(s) = \frac{(-1)^n}{R^n}$$

The sign depends on the choice of unit normal field.

5. Jacobi fields

Let $\gamma\colon [a,b] \to M$ be a geodesic from $\gamma(a) = p$ to $\gamma(b) = q$.

3.41. Definition. *A smooth vector field J along γ is a* Jacobi field *if*

$$D_tD_tJ + R(J,\dot{\gamma})\,\dot{\gamma} = 0 \qquad \textit{(Jacobi equation)}$$

3.42. Proposition (Existence and uniqueness of Jacobi fields). *There is an isomorphism of vector spaces*

$$\{\textit{Jacobi fields along } \gamma\} \to T_pM \times T_pM\colon J \to (J(a), D_tJ(a))$$

Proof. The claim is that for any given $X,Y \in T_pM$ there exists a unique Jacobi field J along γ such that $J(p) = X$ and $D_tJ(p) = Y$. Let $(E_i(t))$ be an orthonormal frame of parallel vector fields along γ. Write $J = J^iE_i$. The vector field J is Jacobi if and only if

$$\forall i\colon\ \langle D_tD_tJ, E_i\rangle = -\left\langle R(J,\dot{\gamma})\,\dot{\gamma}, E_i\right\rangle$$

The LHS is

$$\langle D_tD_tJ, E_i\rangle = \frac{d^2}{dt^2}\langle J, E_i\rangle = \frac{d^2J^i}{dt^2}E_i$$

and the RHS is

$$-\left\langle R(J,\dot{\gamma})\,\dot{\gamma}, E_i\right\rangle = -\left\langle R(E_j,\dot{\gamma})\,\dot{\gamma}, E_i\right\rangle J^j = -\mathrm{Rm}(E_j,\dot{\gamma},\dot{\gamma},E_i)J^j$$

In matrix notation the requirement is

$$\frac{d^2}{dt^2}\begin{pmatrix} J^1 \\ \vdots \\ J^n \end{pmatrix} = -\Big((\mathrm{Rm}(E_j, \overset{\bullet}{\gamma}, \overset{\bullet}{\gamma}, E_i)\Big)\begin{pmatrix} J^1 \\ \vdots \\ J^n \end{pmatrix}$$

This 2nd order linear system of ODEs has a unique solution with the given initial conditions. □

3.43. Lemma. *Suppose that V is smooth vector field along a smooth variation of γ through geodesics $\Gamma\colon (-\varepsilon,\varepsilon)\times[a,b]\to M$. Then*

$$D_sD_tV - D_tD_sV = R(\partial_s\Gamma, \partial_t\Gamma)V$$

Proof. Choose local coordinates $x\colon M\to \mathbf{R}^n$. Let $x\Gamma(s,t) = (\Gamma^1(s,t),\dots,\Gamma^n(s,t))$ and $V = V^i(s,t)\partial_i$ be the local expressions for Γ and V. By properties of the covariant derivative along a curve (2.36)

$$D_tV = D_t(V^i\partial_i) = \frac{\partial V^i}{\partial t}\partial_i + V^iD_t\partial_i$$

and

$$\begin{aligned} D_sD_tV = D_s(\frac{\partial V^i}{\partial t}\partial_i + V^iD_t\partial_i) &= D_s(\frac{\partial V^i}{\partial t}\partial_i) + D_s(V^iD_t\partial_i) \\ &= \frac{\partial^2 V^i}{\partial s\partial t}\partial_i + \frac{\partial V^i}{\partial t}D_s\partial_i + \frac{\partial V^i}{\partial s}D_t\partial_i + V^iD_sD_t\partial_i \end{aligned}$$

When we compute the difference $D_sD_tV - D_tD_sV$ many terms cancel and we are left with

$$D_sD_tV - D_tD_sV = V^i(D_sD_t - D_tD_s)\partial_i$$

Since the vector field ∂_i and its covariant derivative are extendible,

$$V^i(D_sD_t - D_tD_s)\partial_i = V^i(\nabla_{\partial_s\Gamma}\nabla_{\partial_t\Gamma}\partial_i - \nabla_{\partial_t\Gamma}\nabla_{\partial_t\Gamma}\partial_i) = V^i(\nabla_{\partial_s\Gamma}\nabla_{\partial_t\Gamma} - \nabla_{\partial_t\Gamma}\nabla_{\partial_t\Gamma})\partial_i = R(\partial_s\Gamma,\partial_t\Gamma)V$$

where the last equality uses that $[\partial_s\Gamma, \partial_t\Gamma] = [\Gamma_*(\partial_s), \Gamma_*(\partial_t)] = \Gamma_*[\partial_s,\partial_t] = \Gamma_*(0) = 0$. □

3.44. Proposition.

$$\{\textit{Jacobi fields along } \gamma\} = \{\textit{Variational fields of smooth variations of } \gamma \textit{ through geodesics}\}$$

Proof. We shall only prove sup. Let $\Gamma\colon(-\varepsilon,\varepsilon)\times[a,b]\to M$ be a smooth variation of γ through geodesics. Lemma 3.43 applied to the vector field $\partial_t\Gamma$ along Γ says that

$$D_sD_t\partial_t\Gamma - D_tD_s\partial_t\Gamma = R(\partial_s\Gamma,\partial_t\Gamma)\partial_t\Gamma$$

In the first term, $D_t\partial_t\Gamma = 0$ as each main curve is a geodesic. In the second term, $D_s\partial_t\Gamma = D_t\partial_s\Gamma$ by the Symmetry Lemma 2.71. Thus the equation says that

$$D_tD_t\partial_s\Gamma + R(\partial_s\Gamma,\partial_t\Gamma)\partial_t\Gamma = 0$$

At $s=0$, $\partial_s\Gamma = V$ is the variational field and $\partial_t\Gamma = \overset{\bullet}{\gamma}$ is the velocity field of γ so that we have the Jacobi equation $D_tD_tV + R(V,\overset{\bullet}{\gamma})\overset{\bullet}{\gamma} = 0$.

The opposite inclusion is also true [**1**], but we shall need it here. □

For each $Y\in T_pM$, let J_Y be the Jacobi field along γ with $J_Y(a) = 0$ and $D_tJ_Y(a) = 0$. Here is an explicit description of J_Y.

3.45. Proposition (Jacobi fields vanishing at p). *Let $\gamma(t) = \exp_p(tT)$, $T\in T_pM$, $|T| = 1$, be the unit speed parametrization of γ (defined for t such that $tT\in\varepsilon_p\subset T_pM$). For any $Y\in T_pM$,*

$$J_Y(t) = t(\exp_p)_{*tT}(Y)$$

Proof. The variation $\Gamma(s,t) = \exp_p(t(T+sY))$ is a smooth variation of γ through the radial geodesics $t\to\exp_p(t(T+sY))$. Its variational field J is therefore a Jacobi field. The tangent vector $J(t)$ at $\gamma(t)$ is represented by the curve

$$s\to\exp_p(tT+stY)$$

and hence $J(t) = (\exp_p)_{*tT}(tY) = t(\exp_p)_{*tT}(Y) = tU(t)$ and $D_tJ = D_t(tU) = U + tD_tU$ where $U(t) = (\exp_p)_{*tT}(Y)$. In particular, $J(0) = 0$ and $D_tJ(0) = U(0)Y = (\exp_p)_*(Y) = Y$. □

A Jacobi field J along γ is *normal* if $\left\langle J(t), \dot{\gamma}(t) \right\rangle = 0$ for all t.

3.46. Lemma. *A Jacobi field J along γ is normal iff $J(0)$ and $D_tJ(0)$ are orhtogonal to $\dot{\gamma}(0)$ iff J vanishes at two points.*

Proof. The point is that $\left\langle J, \dot{\gamma} \right\rangle$ is an affine function, $t \to at + b$, of t because

$$\frac{d^2}{dt^2}\left\langle J, \dot{\gamma} \right\rangle = \frac{d}{dt}\left\langle D_tJ, \dot{\gamma} \right\rangle = \left\langle D_tD_tJ, \dot{\gamma} \right\rangle = -\left\langle R(J, \dot{\gamma})\,\dot{\gamma}, \dot{\gamma} \right\rangle = \mathrm{Rm}(J, \dot{\gamma}, \dot{\gamma}, \dot{\gamma}) = 0$$

by anti-symmetry of the Riemann curvature tensor. If such a function vanishes at two points or itself and its derivative vanishes at one point, then it is identically zero. $\square$

The vector space of normal Jacobi fields is isomorphic to $\dot{\gamma}(0)^\perp \times \dot{\gamma}(0)^\perp$ of dimension $2n-2$. The vector space of normal Jacobi fields J along γ with $J(p) = 0$ is isomorphic to $\dot{\gamma}(0)^\perp$ of dimension $n-1$.

The Jacobi fields that are *not* normal are linear combinations of the two Jacobi fields $J_0 = \dot{\gamma}$ and $J_1 = tJ_0$ with $J_0(0) = \dot{\gamma}(0), D_tJ_0(0) = 0$ and $J_1(0) = 0, D_tJ_1(0) = \dot{\gamma}(0)$ and $D_tD_tJ_0 = 0 = D_tD_tJ_1$.

3.47. Conjugate points. We say that two points on a geodesic segment γ are *conjugate* if there exists a nonzero Jacobi (necessarily normal) field along γ that vanishes at p and q. The dimension of the vector space of all such Jacobi fields is the *multiplicity* of conjugacy.

3.48. Corollary. *Consider the smooth function $\exp_p \colon \varepsilon_p \to M$. Let $q = \exp_p(rT) \neq p$ be a point on the geodesic $\gamma(t) = \exp_p(tT)$. Then*

$$\exp_p \textit{ is not a local diffeomorphism at } rT \iff p \textit{ and } q \textit{ are conjugate points}$$

Proof. The Jacobi fields that vanish at p are $J_Y = t(\exp_p)_{*tT}(Y)$ (3.45). The value of such a Jacobi field at $q = \exp_p(rT)$ is $J_Y(r) = t(\exp_p)_{*tT}(Y)$. Thus there is nonzero Jacobi field that vanishes at p and q iff there is a nonzero vector in the kernel of $(\exp_p)_{*rT}$. $\square$

3.49. Jacobi fields in constant sectional curvature manifolds. Jacobi fields have particularly simple descriptions in Riemannian manifolds with constant sectional curvature.

3.50. Proposition (Normal Jacobi fields in constant sectional curvature manifolds). *Suppose that M is a Riemannian manifold of constant sectional curvature C. Let γ be a unit speed geodesic in M with $\gamma(0) = p$ and $\dot{\gamma}(0) = T$ where T is a unit vector in T_pM. For any $Y \in T^\perp$, the normal Jacobi field J_Y is*

$$J_Y(t) = u_C(t)Y(t)$$

where $Y(t)$ is the parallel vector field along γ with $Y(0) = Y$ and

$$u_C(t) = \begin{cases} R\sin(t/R) & C = 1/R^2 \\ t & C = 0 \\ R\sinh(t/R) & C = -1/R^2 \end{cases}$$

is the solution to $\ddot{u}_C + Cu_C = 0$, $u_C(0) = 0$, $\dot{u}_C(0) = 1$.

Proof. Put $J(t) = u_C(t)Y(t)$. Then $D_tJ = u_C'Y$ and $D_tD_tJ = u_C''Y = -CJ$ as $Y(t)$ is parallel. Thus $J(0) = 0$ and $D_tJ(0) = Y$. By construction, J is orthogonal to γ (2.52). It is a Jacobi field because (3.39)

$$R(J, \dot{\gamma})\,\dot{\gamma} = C\left[\langle \dot{\gamma}, \dot{\gamma} \rangle J - \langle J, \dot{\gamma} \rangle\, \dot{\gamma}\right] = CJ = -D_tD_tJ$$

as γ has unit speed and J is normal to γ. We conclude that $J = J_Y$. $\square$

Look at concrete Jacobi fields along geodesics in $\mathbf{R}^n$ and S^n_R obtained by variations through geodesics.

3.51. PROPOSITION (Space form metrics in normal coordinates). *Let M be a Riemannian manifold of constant sectional curvature C and p a point of M. Consider the smooth map $\exp_p\colon \varepsilon_p \to M$. Let $T \in T_pM$ be a unit vector and $r > 0$ a positive scalar such that $rT \in \varepsilon_p$. Put $q = \exp_p(rT)$. The induced isomorphism $(\exp_p)_{*rT}\colon T_pM \to T_qM$ satisifies*

$$|(\exp_p)_{*rT}Y|_q^2 = |Y^\perp|_p^2 + \frac{u_C(r)^2}{r^2}|Y^T|_p^2, \qquad Y \in T_{rT}T_pM = T_pM$$

where $Y = Y^\perp + Y^T$ are the components of Y orthogonal and tangent to the sphere through rT.

PROOF. Let $\gamma(t) = \exp_p(tT)$ be the unit speed geodesic from p to q.

The Gauss lemma (2.74) says that

$$|(\exp_p)_{*rT}Y|_q^2 = |(\exp_p)_{*rT}(Y^\perp + Y^T)|_q^2 = |(\exp_p)_{*rT}(Y^\perp)|_q^2 + |(\exp_p)_{*rT}(Y^T)|_q^2$$

as the components $(\exp_p)_{*rT}(Y^\perp)$ and $(\exp_p)_{*rT}(Y^T)$ are orthogonal.

The radial component $Y^\perp$ equals λT for some $\lambda > 0$. Thus $(\exp_p)_{*rT}Y^\perp$ is represented by the radial curve

$$s \to \exp_p(rT + s\lambda T) = \gamma(r + s\lambda)$$

and therefore $(\exp_p)_{*rT}(Y^\perp) = \lambda \overset{\bullet}{\gamma}(r)$ has length $|\lambda| = |\lambda T|_p = |Y^\perp|_p$. Thus $(\exp_p)_{*rT}$ preserves length in the radial direction.

The normal Jacobi field $J = J_{\frac{1}{r}Y^T}$ (3.45) has value 0 at p and value $J(r) = r(\exp_p)_{*rT}(\frac{1}{r}Y^T) = (\exp_p)_{*rT}(Y^T)$ at q. Now,

$$J(t) = \frac{1}{r}u_C(t)Y^T(t)$$

according to the description of normal Jacobi fields in manifolds of constant sectional curvature (3.50) so that we may compute the length of J at any point $\gamma(t)$ on γ as

$$|J(t)|_{\gamma(t)}^2 = \frac{u_C(t)^2}{r^2}|Y^T|_p^2$$

for the length of any parallel vector field is constant (2.52). In particular, the length of J at q is

$$|(\exp)_{*rT}(Y^T)|_q^2 = |J(r)|_q^2 = \frac{u_C(r)^2}{r^2}|Y^T|_p^2$$

as asserted. □

The above proposition says that constant sectional curvature metrics expressed in normal coordinates (2.67)

$$M \xleftarrow{\exp_p} T_pM \xleftarrow{\cong} \mathbf{R}^n$$

only depend on the function u_C so only depend on C.

3.52. COROLLARY (Local uniqueness of manifolds of constant sectional curvature). *Let M_0 and M_1 be two Riemannian manifolds of constant sectional curvature C. Any two points, $p_0 \in M_0$ and $p_1 \in M_1$, have isometric neighborhoods.*

PROOF. An isometry can be constructed as the composite in

$$\begin{tikzcd}
(T_{p_0}M_0, g_0) \arrow[d, "\exp_0"'] & (\mathbf{R}^n, \overline{g}^n) \arrow[l, "\cong"'] \arrow[r, "\cong"] & (T_{p_1}M_1, g_1) \arrow[d, "\exp_1"] \\
M_0 \arrow[rr, dashed] & & M_1
\end{tikzcd}$$

where the upper horizontal maps are linear isometries and the other maps are understood to be defined only locally and to be diffeomorphisms. Since a space form metric in a normal neighborhood around a point is determined by the the inner product at the tangent space of the point, this is an isometry. □

3.53. Second variation formula. For any variation Γ of the geodesic $\gamma(t) = \Gamma(0,t)$ we know that

$$\gamma \text{ is minimizing} \Longrightarrow \frac{d}{ds}L(\Gamma_s)(0) = 0 \text{ and } \frac{d^2}{ds^2}L(\Gamma_s)(0) \geq 0$$

But what is the second derivative?

3.54. Lemma. *Let $\gamma\colon [a,b] \to M$ be a unit speed geodesic and $\Gamma\colon (-\epsilon,\epsilon)\times[a,b] \to M$ a smooth variation of γ. Then*

$$\frac{d^2}{ds^2}L(\Gamma_s)(0) = \int_a^b \left(|D_tV^\perp|^2 - Rm(V^\perp, \dot\gamma, \dot\gamma, V^\perp)\right) dt + \left\langle D_s\partial_s\Gamma, \dot\gamma\right\rangle\Big|_a^b$$

where $V^\perp$ is the normal part of V.

Proof. In the proof of 2.73 we saw that the first derivative of the curve length function is

$$\frac{d}{ds}L(\Gamma_s) = \int_a^b \frac{\langle D_t\partial_s\Gamma, \partial_t\Gamma\rangle}{|\partial_t\Gamma|}dt = \int_a^b \langle D_t\partial_s\Gamma, \partial_t\Gamma\rangle \langle\partial_t\Gamma, \partial_t\Gamma\rangle^{-1/2}\, dt$$

Differentiating once more we get that the second derivative is

$$\begin{aligned}\frac{d^2}{ds^2}L(\Gamma_s) &= \int_a^b \frac{d}{ds}\left(\langle D_t\partial_s\Gamma, \partial_t\Gamma\rangle \langle\partial_t\Gamma, \partial_t\Gamma\rangle^{-1/2}\right) dt \\ &= \int_a^b \left(\frac{\langle D_sD_t\partial_s\Gamma, \partial_t\Gamma\rangle}{|\partial_t\Gamma|} + \frac{\langle D_t\partial_s\Gamma, D_s\partial_t\Gamma\rangle}{|\partial_t\Gamma|} - \frac{\langle D_t\partial_s\Gamma, \partial_t\Gamma\rangle\langle D_s\partial_t\Gamma, \partial_t\Gamma\rangle}{|\partial_t\Gamma|^3}\right) dt\end{aligned}$$

where

$$D_sD_t\partial_s\Gamma = D_tD_s\partial_s\Gamma + R(\partial_s\Gamma, \partial_t\Gamma)\partial_s\Gamma \qquad (3.43)$$
$$D_s\partial_t\Gamma = D_t\partial_s\Gamma \qquad (2.71)$$

At $s=0$, $|\partial_t\Gamma| = |\dot\gamma| = 1$, and $\partial_s\Gamma = V$ so that

$$\frac{d^2}{ds^2}L(\Gamma_s)(0) = \int_a^b \left(\left\langle D_tD_s\partial_s\Gamma, \dot\gamma\right\rangle - \mathrm{Rm}(V, \dot\gamma, \dot\gamma, V) + |D_tV|^2 - \left\langle D_tV, \dot\gamma\right\rangle^2\right) dt$$

In the first term we still use $\partial_s\Gamma$ and not V. Since γ is a geodesic and $D_t\dot\gamma = 0$, the first term is

$$\int_a^b \left\langle D_tD_s\partial_s\Gamma, \dot\gamma\right\rangle dt = \int_a^b \frac{d}{dt}\left\langle D_s\partial_s\Gamma, \dot\gamma\right\rangle dt = \left\langle D_s\partial_s\Gamma, \dot\gamma\right\rangle\Big|_a^b$$

Now look at the last two terms. Split $V = V^T + V^\perp$ into the tangential and the normal part. The tangential component, $V^T = \left\langle V, \dot\gamma\right\rangle\dot\gamma$, has covariant derivative

$$D_t(V^T) = D_t(\left\langle V, \dot\gamma\right\rangle\dot\gamma) = \left\langle D_tV, \dot\gamma\right\rangle\dot\gamma = (D_tV)^T$$

and therefore also $D_t(V^\perp) = (D_tV)^\perp$. It follows that

$$|D_tV|^2 = |(D_tV)^T|^2 + |(D_tV)^\perp|^2 = \left\langle D_tV, \dot\gamma\right\rangle^2 + |D_tV^\perp|^2$$

or $|D_tV|^2 - \left\langle D_tV, \dot\gamma\right\rangle^2 = |D_tV^\perp|^2$. Finally we note that the tangential part of V does not contribute to $\mathrm{Rm}(V, \dot\gamma, \dot\gamma, V)$ because $\mathrm{Rm}(\dot\gamma, \dot\gamma, \dot\gamma, V) = 0 = \mathrm{Rm}(V, \dot\gamma, \dot\gamma, \dot\gamma)$. $\square$

Theorem 3.55 (Second Variational Formula). *Let $\gamma\colon [a,b] \to M$ be a unit speed geodesic and $\Gamma\colon (-\epsilon,\epsilon)\times[a,b] \to M$ a piecewise smooth proper variation of γ. Then*

$$\frac{d^2}{ds^2}L(\Gamma_s)(0) = \int_a^b \left(|D_tV^\perp|^2 - Rm(V^\perp, \dot\gamma, \dot\gamma, V^\perp)\right) dt$$

where $V^\perp$ is the normal part of V.

PROOF. We appply the formula from 3.54 to each of the rectangels $(-\epsilon,\epsilon)\times[a_{i-1},a_i]$ where Γ is smooth. This gives that $\frac{d^2}{ds^2}L(\Gamma_s)(0)$ equals

$$\int_a^b \Big(|D_tV^\perp|^2 - \mathrm{Rm}(V^\perp,\dot\gamma,\dot\gamma,V^\perp)\Big)\,dt + \sum_{i=1}^n \Big\langle D_s\partial_s\Gamma(a_i^-),\dot\gamma\,(a_i)\Big\rangle - \Big\langle D_s\partial_s\Gamma(a_{i-1}^+),\dot\gamma\,(a_i)\Big\rangle$$

Note that $D_s\partial_s\Gamma(a_i^-)=D_s\partial_s\Gamma(a_i^+)$ because this vector field only depends on Γ along the line $t=a_i$. Thus the contributions from the points a_i telescope to $\Big\langle D_s\partial_s\Gamma(b),\dot\gamma\,(b)\Big\rangle - \Big\langle D_s\partial_s\Gamma(a),\dot\gamma\,(a)\Big\rangle$ which is 0 since the variation keeps the end-points fixed. □

The expression in the Second Variational Formula is the quadratic function $I(V^\perp,V^\perp)$ of the bilienar form (the *index form*)

$$I(V,W)=\int_a^b \Big(\langle D_tV,D_tW\rangle - \mathrm{Rm}(V,\dot\gamma,\dot\gamma,W)\Big)\,dt \tag{3.56}$$

defined for all piecewise smooth normal proper vector fields V and W along γ. If V and W happen to be smooth then the integrand is

$$\frac{d}{dt}\langle D_tV,W\rangle - \langle D_tD_tV,W\rangle - \mathrm{Rm}(V,\dot\gamma,\dot\gamma,W) = -\Big\langle D_tD_tV + R(V,\dot\gamma)\,\dot\gamma,W\Big\rangle + \frac{d}{dt}\langle D_tV,W\rangle$$

so that the index form may be written

$$I(V,W) = -\int_a^b \Big\langle D_tD_tV + R(V,\dot\gamma)\,\dot\gamma,W\Big\rangle\,dt + \langle D_tV,W\rangle\Big|_a^b$$

In general, there may be points $a=a_0<a_1<\cdots a_k=b$ where V and W are not smooth and then

$$I(V,W) = -\int_a^b \Big\langle D_tD_tV + R(V,\dot\gamma)\,\dot\gamma,W\Big\rangle\,dt - \sum_{i=1}^{k-1}\langle D_tV(a_i^+)-D_tV(a_i^-),W(a_i)\rangle \tag{3.57}$$

which shows the connection to the Jacobi equation.

3.58. COROLLARY. *If γ is a unit speed minimizing curve then γ is a geodesic and $I(V,V)\ge 0$ for any picewise smooth proper normal vector field V along γ.*

PROOF. Any such V is the variational field for some proper variation of γ (2.81). Since γ is minimizing, $\frac{d}{ds}L(\Gamma_s)(0)=0$ so that γ is a unit speed geodesic (2.82) and $\frac{d^2}{ds^2}L(\Gamma_s)(0)=I(V,V)\ge 0$ by the Second Variation Formula (3.55). □

THEOREM 3.59. *Let γ be a geodesic segment from p to q. If γ contains an interior point conjugate to p, then γ is not minimizing.*

PROOF. It suffices to find a proper normal vector field V along γ with index form $I(V,V)<0$.
By assumption there exists a nontrivial Jacobi field J vanishing at p and at some interior point $\gamma(c)$. The Jacobi field J is normal since it vanishes at two points (3.46), and $D_tJ(c)\neq 0$ as J is not the zero field (3.42). Extend J to a piecewise smooth normal vector field by $J=0$ from $\gamma(c)$ to q.
Now choose a smooth normal proper vector field W along γ such that $W(c)=-D_tJ(c^-)$ (by using a smooth bump function). Since J is a piecewise smooth Jacobi field $I(J,J)=0$ and $I(J,W)=-|D_tJ(c^-)|^2$ (3.57). For any $\epsilon\in\mathbf{R}$ we have

$$I(J+\epsilon W,J+\epsilon W) = -2\epsilon|D_tJ(c^-)|^2+\epsilon^2 I(W,W)$$

which is negative when ϵ is close to 0. □

6. Comparison theorems

THEOREM 3.60 (Jacobi field Comparison). *Suppose that the sectional curvature is bounded from above by some constant C, $K \leq C$, in the manifold M. For $T, Y \in T_pM$, $T \perp Y$, let γ be the unit speed geodesic through $p \in M$ in direction $T \in T_pM$ and J_Y the normal Jacobi field along γ with $J_Y(p) = 0$ and $D_tJ_Y(p) = Y$. Then the inequality*

$$|J_Y(t)| \geq u_C(t)|Y|$$

holds for $t \in [0, \pi R]$ if $C = 1/R^2 > 0$ and for all $t \geq 0$ if $C \leq 0$. (The Jacobi field is longer than it would have been had the sectional curvature been constant.)

PROOF. We may as well assume that Y is a unit vector. The claim is then that $|J(t)| \geq |u_C(t)|$ for certain values of t where J is short for J_Y. The function u_C is the solution to $\ddot{u} + Cu = 0$ so we want to apply the Sturm comparison theorem with $u = u_C$ and $v = |J|$. We need to verify that $v(0) = 0$, that v is differentiable at 0 with $\dot{v}(0) = 1$, and that $\frac{d^2}{dt^2}|J(t)| + C|J(t)| \geq 0$ in an interval $[0, T]$.
$\underline{v(0) = 0}$: This is clear because $J(0) = 0$.
$\underline{\dot{v}(0) = 0}$: It is not a priori clear that the function $v = |J, J| = \langle J, J\rangle^{1/2}$ is differentiable at 0. But $\underline{J(t) = tU(t)}$ for a certain vector field U (3.45) so that the difference quotient

$$\frac{|J(t)| - |J(0)|}{t - 0} = |U(t)|, \qquad t > 0,$$

converges to $|U(0)| = |D_tJ(0)| = |Y| = 1$ for $t \to 0^+$. Thus v is differentiable at 0 and $\dot{v}(0) = 1$.

Since $v(0) = 0$ and $\dot{v}(0) = 1 > 0$, $v(t) = |J(t)|$ is postive on some interval $(0, T)$. In this interval $v(t) = |J(t)| = \langle J, J\rangle^{1/2}$ is smooth and we can compute its derivatives.
$\underline{\ddot{v} + Cv \geq 0}$: We compute (using the Schwartz inequality and the Jacobi equation)

$$\begin{aligned}
\frac{d^2}{dt^2}|J| = \frac{d}{dt}\left(\langle D_tJ, J\rangle \frac{1}{|J|}\right) &= \frac{\langle D_t^2J, J\rangle + \langle D_tJ, D_tJ\rangle}{|J|} - \frac{\langle D_tJ, J\rangle^2}{|J|^3} \\
&= \frac{-\mathrm{Rm}(J, \dot{\gamma}, \dot{\gamma}, J)}{|J|} + \frac{|D_tJ|^2}{|J|} - \frac{\langle D_tJ, J\rangle^2}{|J|^3} \\
&\geq \frac{-\mathrm{Rm}(J, \dot{\gamma}, \dot{\gamma}, J)}{|J|} + \frac{|D_tJ|^2}{|J|} - \frac{|D_tJ|^2|J|^2}{|J|^3} \\
&= \frac{-\mathrm{Rm}(J, \dot{\gamma}, \dot{\gamma}, J)}{|J|} = \frac{-\mathrm{Rm}(J, \dot{\gamma}, \dot{\gamma}, J)}{|J|^2}|J| = -K(J, \dot{\gamma})|J| \\
&\geq -C|J|
\end{aligned}$$

and obtain the required estimate.

We can therefore conclude from the Sturm comparison theorem that $|J(t)| \geq |u_C(t)|$ on $[0, T]$. If $C \leq 0$, the function u_C has no zeros on $(0, \infty)$, so $|J(t)|$ can not become 0 either on this interval and we can take $T = \infty$. If $C = 1/R^2 > 0$, then $|J(t)|$ can not attain a zero before πR and we can take $T = \pi R$. □

3.61. COROLLARY (Conjugate Point Comparison Theorem). *Suppose that the sectional curvature is bounded from above by some constant C, $K \leq C$, in the manifold M. If $C = 1/R^2 > 0$, then there are no points conjugate to p in the geodesic ball $B_{\pi R}(p)$ of radius πR. If $C \leq 0$, then there are no points at all conjugate to p.*

THEOREM 3.62 (Metric Comparison Theorem). *Suppose that $K \leq C$ in the manifold M. Then we have that*

$$|(\exp_p)_{*rT}Y|_q^2 \geq |Y^\perp|_p^2 + \frac{u_C(r)^2}{r^2}|Y^T|_p^2, \qquad Y \in T_{rT}T_pM = T_pM$$

in the situation of 3.51. If $C \leq 0$, then

$$|(\exp_p)_{*rT}Y|_q^2 \geq |Y|_p^2$$

meaning that $\exp_p$ does not decrease length.

PROOF. The proof is the same as in 3.51 execpt that at some point we must replace an equality by the inequality from 3.60. □

THEOREM 3.63 (Cartan–Hadamard). *Let M be a connected, complete Riemannian manifold of nonpositive sectional curvature. Then* $\exp_p\colon T_pM \to M$ *is the universal covering space of M (for any point $p \in M$). In particular, M is a $K(\pi, 1)$, and if M is simply connected, then* $\exp_p\colon T_pM \to M$ *is a diffeomorphism.*

3.64. LEMMA. *Let $\phi\colon \widetilde{M} \to M$ be local isometry between two connected Riemannian manifolds. If $\widetilde{M}$ is complete, then ϕ is a covering map.*

PROOF OF THEOREM 3.63. The exponential map $\exp_p$ is defined on all of T_pM because M is complete (2.87); it is a local diffeomorphism since p has no conjugate points (3.48, 3.61). If we let $\widetilde{g} = \exp_p^*(g)$ be the pull-back metric, then $\exp_p$ is a local isometry $(T_pM, \widetilde{g}) \to (M, g)$. The Riemannian manifold $(T_pM, \widetilde{g})$ is complete because the straight line $t \to tY$, $Y \in T_pM$, defined for all t, is a geodesic as its image under the local isometry $\exp_p$ is a geodesic. Lemma 3.64 now implies that $\exp_p$ is a covering map. □

CHAPTER 4

Space-times

Hall's book on relativity [**2**] is in the library. Hawking and Ellis [**3**] or O'Neill [**7**] are also a possibilities.

Let now M be a 4-dimensional smooth manifold with a Lorentz metric. Einstein's gravitational tensor is

$$G = \mathrm{Rc} - \frac{1}{2}Sg = \mathrm{Rc} - \frac{1}{2}\operatorname{tr}_g(\mathrm{Rc})g$$

The g-trace of G is

$$\operatorname{tr}_g(G) = \operatorname{tr}_g(\mathrm{Rc}) - \frac{1}{2}4S = S - 2S = -S$$

since $\operatorname{tr}_g(g) = 4$ in this case (5.26). Therefore

$$\mathrm{Rc} = G + \frac{1}{2}Sg = G - \frac{1}{2}(-S)g = G - \frac{1}{2}\operatorname{tr}_g(G)g$$

and we see that G and Rc determine each other, they contain the same information. The gravitational tensor has zero divergence (3.11),

$$\operatorname{div}(G) = \operatorname{div}(\mathrm{Rc} - \frac{1}{2}Sg) = \frac{1}{2}\nabla S - \frac{1}{2}\nabla S = 0$$

as $\operatorname{div}(\mathrm{Rc}) = \frac{1}{2}\nabla S$ by the Contracted Bianchi identity (3.12).

Einstein says that gravitation is not a force but a geometrical property of spacetime. A particle in free fall, under the influence of gravity alone, will follow a geodesic in spacetime. When you throw a stone it will trace out a geodesic in spacetime. Einstein's equation

$$T = \frac{1}{8\pi}G$$

says that the stress-energy tensor T of matter is proportional to the gravitational tensor. This is not a mathematical statement but (an assertion about) a law of nature just like Newton's law of inertia. Einstein's equation tells how matter generates Ricci curvature of space-time. The equation $\operatorname{div}(T) = 0$ tells how Ricci curvature moves matter. There is no definition of matter – it is something that has stress–energy ... Check out

Baez and Bunn: The meaning of Einstein's equation
Sean M. Carroll: Lecture Notes on General Relativity
Clifford M. Will: The Confrontation between General Relativity and Experiment.

Apparently T is determined by the distribution of mass/energy and the equation then expresses the impact of matter on the Ricci curvature of spacetime [**7**].

CHAPTER 5

Multilinear Algebra

We review some vector space constructions from linear algebra.

1. Tensors

An endomorphism of a vector space V is a special case of a tensor on V. Thus tensors are generalized endomorphisms or, in coordinates, generalized matrices. The formal definition goes as follows.

Let V be a finite dimensonal real vector space and $V^* = \mathrm{Hom}(V, \mathbf{R})$ the dual space of linear forms on V.

5.1. Lemma. *There are natural isomorphisms between the following three vector spaces*

- *Bilinear maps* $V^* \times V \xrightarrow{A} \mathbf{R}$
- *Linear maps* $V^* \xrightarrow{B} \mathrm{Hom}(V, \mathbf{R})$
- *Linear maps* $V \xrightarrow{C} \mathrm{Hom}(V^*, \mathbf{R})$

given by $A(\omega, v) = B(\omega)(v) = C(v)(\omega)$. *In particular, the evaluation morphism* $ev\colon V^* \times V \to \mathbf{R}$, *given by* $ev(\omega, v) = \omega(v)$, *corresponds to the identity* $V^* = \mathrm{Hom}(V, \mathbf{R})$ *and to the isomorphism* $V \to \mathrm{Hom}(V^*, \mathbf{R})$ *given by* $v \to (\omega \to \omega(v))$.

5.2. Definition. *The* $\binom{k}{\ell}$-tensor space *of* V *is the vector space* $T^k_\ell(V)$ *of all* $\mathbf{R}$*-multilinear homomorphisms*

$$\underbrace{V^* \times \cdots \times V^*}_{\ell} \times \underbrace{V \times \cdots \times V}_{k} \to \mathbf{R}$$

An element of $T^k_\ell(V)$ *is called a* $\binom{k}{\ell}$*-tensor.*

The tensor product *of* $A \in T^{k_1}_{\ell_1}(V)$ *and* $B \in T^{k_2}_{\ell_2}(V)$, *is the multilinear map* $A \otimes B \in T^{k_1+k_2}_{\ell_1+\ell_2}(V)$ *given as the composite*

$$\underbrace{V^* \times \cdots \times V^*}_{\ell_1+\ell_2} \times \underbrace{V \times \cdots \times V}_{k_1+k_2} = \underbrace{V^* \times \cdots \times V^*}_{\ell_1} \times \underbrace{V \times \cdots \times V}_{k_1} \times \underbrace{V^* \times \cdots \times V^*}_{\ell_2} \times \underbrace{V \times \cdots \times V}_{k_2}$$
$$\xrightarrow{A \times B} \mathbf{R} \times \mathbf{R} \xrightarrow{\cdot} \mathbf{R}$$

where $\cdot$ *is multiplication in the ring* $\mathbf{R}$.

A $\binom{k}{\ell}$-tensor $A \in T^k_\ell(V)$ eats ℓ covectors $\omega^1, \dots, \omega^\ell \in V^*$ and k vectors $v_1, \dots, v_k \in V$ and spits out a real number $A(\omega^1, \dots, \omega^\ell, v_1, \dots, v_k) \in \mathbf{R}$.

Some special cases of tensor spaces are

- $T^0_0(V) = \mathbf{R}$
- $T^1_0(V) = \mathrm{Hom}(V, \mathbf{R}) = V^*$
- $T^0_1(V) = \mathrm{Hom}(V^*, \mathbf{R}) \overset{5.1}{\cong} V$
- $T^1_1(V) = \{V^* \times V \xrightarrow{A} \mathbf{R}\} = \{V \to \mathrm{Hom}(V^*, \mathbf{R})\} = \{V \xrightarrow{s} V\} = \mathrm{Hom}(V, V)$, $A(\omega, v) = \omega(sv)$.
- $T^k_\ell(V) \cong T^\ell_k(V^*)$
- T^k_0 is a contravariant and T^0_ℓ a covariant endofunctor of the category of finite dimensional real vector spaces.

There is a bilinear map

$$T^0_1(V) \times T^1_0(V) = V \times V^* \xrightarrow{\otimes} T^1_1(V) = \{V^* \times V \to \mathbf{R}\}$$

given by $(u \otimes \phi)(\psi, v) = \psi(u)\phi(v)$. Note that if E_i is a basis for V and ϕ^j the dual basis for V^*, then $E_i \otimes \phi^j$ is a basis for $T^1_1(V) = \mathrm{Hom}(V, V)$.

Guided by the principle that tensors are generalized endomorphisms we define, more generally, the *tensor product* as the bilinear map

$$\begin{gathered} T^{k_1}_{\ell_1}(V) \times T^{k_2}_{\ell_2}(V) \xrightarrow{\otimes} T^{k_1+k_2}_{\ell_1+\ell_2}(V) \\ (A, B) \to A \otimes B \end{gathered}$$

given by

$$\begin{aligned} &(A \otimes B)(\omega^1, \dots, \omega^{\ell_1}, \omega^{\ell_1+1}, \dots, \omega^{\ell_1+\ell_2}, v_1, \dots, v_{k_1}, v_{k_1+1}, \dots, v_{k_1+k_2}) \\ &= A(\omega^1, \dots, \omega^{\ell_1}, v_1, \dots, v_{k_1}) B(\omega^{\ell_1+1}, \dots, \omega^{\ell_1+\ell_2}, v_{k_1+1}, \dots, v_{k_1+k_2}) \end{aligned}$$

5.3. Lemma. $\dim T^k_\ell(V) = (\dim V)^{k+\ell}$

Proof. Let $E_1, \dots, E_n$ be a basis for V and $\phi^1, \dots, \phi^n$ the dual basis for V^* given by $\phi^i(E_j) = \delta^j_i$, $n = \dim V$. Then the set

$$\{E_{j_1} \otimes \cdots \otimes E_{j_\ell} \otimes \phi^{i_1} \otimes \cdots \otimes \phi^{i_k} \mid 1 \leq i_1, \dots, i_k, j_1, \dots, j_\ell \leq n\} \tag{5.4}$$

is a basis for $T^k_\ell(V)$: These multilinear maps are clearly linearly independent and since

$$A = \sum A^{j_1 \cdots j_\ell}_{i_1 \cdots i_k} E_{j_1} \otimes \cdots \otimes E_{j_\ell} \otimes \phi^{i_1} \otimes \cdots \otimes \phi^{i_k}, \quad A^{j_1 \cdots j_\ell}_{i_1 \cdots i_k} = A(\phi^{j_1}, \dots, \phi^{j_\ell}, E_{i_1}, \dots, E_{i_k}) \tag{5.5}$$

for any multilinear map $A \in T^k_\ell(V)$, they generate $T^k_\ell(A)$ □

The convention is that vectors have a lower index so that *components* of vectors have an upper index: $V = V^j E_j$. Similarly, covectors have an upper index and their components have a lower index: $\omega = \omega_i \phi^i$.

Now comes a more natural way of considering tensors (as generalized endomorphisms). We have already seen the special case of $T^1_1(V) = \mathrm{Hom}(V, V)$. In general, we may view $T^k_{\ell+1}(V)$ as the vector space of all multilinear maps $(V^*)^\ell \times V^k \to V$.

5.6. Lemma. *There is an isomorphismbetween the vector space $T^k_{\ell+1}(V)$ of all multilinear homomorphisms*

$$\underbrace{V^* \times \cdots \times V^*}_{\ell+1} \times \underbrace{V \times \cdots \times V}_{k} \xrightarrow{A} \mathbf{R}$$

and the vector space of all multilinear homomorphisms

$$\underbrace{V^* \times \cdots \times V^*}_{\ell} \times \underbrace{V \times \cdots \times V}_{k} \xrightarrow{B} V$$

given by

$$A(\omega, \omega^1, \dots, \omega^\ell, v_1, \dots, v_k) = \omega B(\omega^1, \dots, \omega^\ell, v_1, \dots, v_k)$$

Proof. As in 5.1 there is an isomorphism bewteen

- the vector space of multilinear maps $V^* \times (V^*)^\ell \times V^k \xrightarrow{A} \mathbf{R}$ and
- the vector space of multilinear maps $(V^*)^\ell \times V^k \xrightarrow{B} \mathrm{Hom}(V^*, \mathbf{R}) \cong V$.

given by $A(\omega, \psi, v) = \omega B(\psi, v)$. □

The trace homomorphism $\mathrm{tr} \colon T^1_1(V) = \mathrm{Hom}(V, V) \to \mathbf{R} = T^0_0(V)$ is the unique homomorphism that makes the diagram

$$\begin{tikzcd} V \times V^* \arrow[rr, "\otimes"] \arrow[dr, "\mathrm{ev}"'] & & T^1_1(V) \arrow[dl, "\mathrm{tr}"] \\ & \mathbf{R} & \end{tikzcd}$$

commutative; it is given by $\mathrm{tr}(v \otimes \phi) = \phi(v)$. For eample, $\mathrm{tr}(\mathrm{id}) = \dim V$. More generally, we define the $\mathrm{tr}\colon T^{k+1}_{\ell+1} \to T^k_\ell(V)$ to be the unique homomorphism that makes the diagram

$$\begin{tikzcd} V \times T^k_\ell(V) \times V^* \arrow[rr, "\otimes"] \arrow[dr, "\mathrm{ev}"'] & & T^{k+1}_{\ell+1}(V) \arrow[dl, "\mathrm{tr}"] \\ & \mathbf{R} & \end{tikzcd}$$

commutative; it is given by $\mathrm{tr}(v \otimes A \otimes \phi) = \phi(v)A$ for all tensors $A \in T^k_\ell(V)$.

5.7. Lemma (Contraction of tensors). *For each k and ℓ there exits a unique $\mathbf{R}$-linear map*

$$\mathrm{tr}\colon T^{k+1}_{\ell+1}(V) \to T^k_\ell(V)$$

such that $\mathrm{tr}(v \otimes A \otimes \phi) = \phi(v)A$ *for all* $v \in V = T^0_1(V)$, $A \in T^k_\ell(V)$ *and* $\phi \in V^* = T^1_0(V)$.

Proof. The only possibility is the linear map whose values on the basis (5.4) is

$$\mathrm{tr}(E_j \otimes E_{j_1} \otimes \cdots \otimes E_{j_\ell} \otimes \phi^{i_1} \otimes \cdots \otimes \phi^{i_k} \otimes \phi^i) = \delta^j_i E_{j_1} \otimes \cdots \otimes E_{j_\ell} \otimes \phi^{i_1} \otimes \cdots \otimes \phi^{i_k}$$

and this map actually has the property because

$$\mathrm{tr}(v \otimes A \otimes \omega) = \mathrm{tr}(v^i E_i \otimes A \otimes \omega_j \phi^j) = \omega_j v^i \delta^j_i A = \omega(v)A$$

for all $v \in V$ and $\omega \in V^*$. □

The tensor algebra of V is the graded algebra $T^*(V) = \bigoplus_{k=0}^\infty T^k(V)$ with tensor product as product. T^* is a contravariant functor.

2. Tensors of inner product spaces

If V has an inner product then there is an isomorphism

$$T^1_1(V) = \mathrm{Hom}(V, V) \xrightarrow{\cong} \{V \times V \to \mathbf{R}\} = T^2_0(V)$$

under which the endomorphism $s\colon V \to V$ and the bilinear map $h\colon V \times V \to \mathbf{R}$ correspond to each other if $\langle su, v\rangle = h(u, v)$. Since tensors are generalized endomorphisms, there should be a similar correspondence for all tensor spaces.

Suppose that V has an inner product $g = \langle\ ,\ \rangle \in T^2(V)$. The inner product induces isomorphisms

$$\left\langle v, \omega^\sharp \right\rangle = \omega(v) \qquad V \underset{\sharp}{\overset{\flat}{\rightleftarrows}} V^* \qquad v^\flat(u) = \langle u, v\rangle$$

or, more generally,

$$\underbrace{V^* \times \cdots \times V^*}_{\ell} \times V \times \underbrace{V \times V \cdots \times V}_{k} \underset{\sharp}{\overset{\flat}{\rightleftarrows}} \underbrace{V^* \times \cdots \times V^*}_{\ell} \times V^* \times \underbrace{V \times V \cdots \times V}_{k}$$

inverse to each other. Composition with these isomorphism induces isomorphism of tensor spaces

$$T^{k+1}_\ell(V) \underset{\flat}{\overset{\sharp}{\rightleftarrows}} T^k_{\ell+1}(V)$$

given by

$$\begin{aligned} A^\flat(\phi^1, \ldots, \phi^\ell, v_1, v_2, \ldots, v_{k+1}) &= A(\phi^1, \ldots, \phi^\ell, v_1^\flat, v_2, \ldots, v_{k+1}), & A &\in T^k_{\ell+1}(V) \\ B^\sharp(\phi^1, \ldots, \phi^\ell, \phi^{\ell+1}, v_1, \ldots, v_k) &= B(\phi^1, \ldots, \phi^\ell, (\phi^{\ell+1})^\sharp, v_1, \ldots, v_k), & B &\in T^{k+1}_\ell(V) \end{aligned}$$

and again inverse to each other.

If we consider the tensor space $T^k_{\ell+1}(V)$ as the vector space of all multilinear maps $(V^*)^\ell \times V^k \to V$ as in (5.6) (and now act on the last variable) then

$$\begin{aligned} \text{(5.8)}\quad A^\flat(\phi^1, \ldots, \phi^\ell, v_1, \ldots, v_k, v_{k+1}) &= A(\phi^1, \ldots, \phi^\ell, v_1, \ldots, v_k, v_{k+1}^\flat) \\ &= v_{k+1}^\flat A(\phi^1, \ldots, \phi^\ell, v_1, \ldots, v_k) = \left\langle A(\phi^1, \ldots, \phi^\ell, v_1, \ldots, v_k), v_{k+1} \right\rangle \end{aligned}$$

and, equivalently,

$$B(\phi^1,\dots,\phi^\ell,v_1,\dots,v_k,v_{k+1}) = (B^\flat)^\sharp(\phi^1,\dots,\phi^\ell,v_1,\dots,v_k,v_{k+1}) \\ \langle B^\sharp(\phi^1,\dots,\phi^\ell,v_1,\dots,v_k),v_{k+1}\rangle \tag{5.9}$$

for $B \in T^{k+1}_\ell(V)$. In other words, A and B correspond to each other under the isomorphisms

$$\{(V^*)^\ell \times V^{k+1} \to \mathbf{R}\} = T^{k+1}_\ell(V) \underset{\flat}{\overset{\sharp}{\rightleftarrows}} T^k_{\ell+1}(V) = \{(V^*)^\ell \times V^{k+1} \to V\}$$

if and only if

$$A(\phi^1,\dots,\phi^\ell,v_1,\dots,v_{k+1}) = \langle B(\phi^1,\dots,\phi^\ell,v_1,\dots,v_k),v_{k+1}\rangle \tag{5.10}$$

5.11. Definition. *The* trace with respect to g *is the linear map*

$$\mathrm{tr}_g \colon T^{k+1}_\ell(V) \xrightarrow{\sharp} T^k_{\ell+1}(V) \xrightarrow{\mathrm{tr}} T^{k-1}_\ell(V)$$

where we dualize the first V-variable and take the trace with respect to the new V^* and the next V-variable.

5.12. Example. The $(2,0)$-tensor $h \in T^2_0(V)$ and the $(1,1)$-tensor $s \in \mathrm{Hom}(V,V)$ correspond to each other, $s = h^\sharp$ or $h = s^\flat$, if and only if $h(v_1,v_2) = \langle s(v_1),v_2\rangle$. The trace with respect to g of h is $\mathrm{tr}_g h = \mathrm{tr}\, s$.

3. Coordinate expressions

Let V be a real vector space with basis $E_1,\dots,E_n$ and dual basis $\phi^1,\dots,\phi^n$ so that the set (5.4) is a basis for the tensor space $T^k_\ell(V)$. For any tensor $A \in T^k_\ell(V)$, the coordinates of A (5.5) with respect to this basis,

$$A^{j_1\cdots j_\ell}_{i_1\cdots i_k} = A(\phi^{j_1},\dots,\phi^{j_\ell},E_{i_1},\dots,E_{i_k}),$$

are called the *components* of A.

5.13. Kronecker's δ. The components of the $\begin{pmatrix}1\\1\end{pmatrix}$-tensor $\delta \in T^1_1(V)$ given by $\delta(\omega,v) = \omega(v)$ are

$$\delta^j_i = \delta(\phi^j,E_i) = \phi^j(E_i) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{5.14}$$

5.15. Endomorphisms. Let $A \colon V \to V$ be an endomorphism of V. The components of A, viewed (5.6) as the $\begin{pmatrix}1\\1\end{pmatrix}$-tensor $A \in T^1_1(V)$ given by $A(\omega,v) = \omega(Av)$, are $A^j_i = A(\phi^j,E_i) = \phi^j AE_i$, the entries in the matrix for A wrt basis E_i.

5.16. Tensor product. If $A \in T^{k_1}_{\ell_1}(V)$ and $B \in T^{k_2}_{\ell_2}(V)$ then

$$(A\otimes B)^{j_1\cdots j_{\ell_1} j_{\ell_1+1}\cdots j_{\ell_1+\ell_2}}_{i_1\cdots i_{k_1} i_{k_1+1}\cdots i_{k_1+k_2}} = A^{j_1\cdots j_{\ell_1}}_{i_1\cdots i_{k_1}} B^{j_{\ell_1+1}\cdots j_{\ell_1+\ell_2}}_{i_{k_1+1}\cdots i_{k_1+k_2}} \tag{5.17}$$

5.18. Trace. If the tensor $A \in T^{\ell+1}_k(V)$ has components $A^{j_1\cdots j_{\ell+1}}_{i_1\cdots i_{k+1}}$ meaning that

$$A = \sum A^{j_1\cdots j_{\ell+1}}_{i_1\cdots i_{k+1}} E_{j_1}\otimes\cdots\otimes E_{j_{\ell+1}}\otimes\phi^{i_1}\otimes\cdots\otimes\phi^{i_{k+1}}$$

then

$$\begin{aligned} \mathrm{tr}(A) &= \sum A^{j_1\cdots j_{\ell+1}}_{i_1\cdots i_{k+1}} \mathrm{tr}(E_{j_1}\otimes E_{j_2}\otimes\cdots\otimes E_{j_{\ell+1}}\otimes\phi^{i_1}\otimes\cdots\otimes\otimes\phi^{i_k}\otimes\phi^{i_{k+1}}) \\ &= \sum A^{j_1\cdots j_{\ell+1}}_{i_1\cdots i_{k+1}}\delta^{j_1}_{i_{k+1}} E_{j_2}\otimes\cdots\otimes E_{j_{\ell+1}}\otimes\phi^{i_1}\otimes\cdots\otimes\phi^{i_k} \\ &= \sum A^{mj_2\cdots j_{\ell+1}}_{i_1\cdots i_k m} E_{j_2}\otimes\cdots\otimes E_{j_{\ell+1}}\otimes\phi^{i_1}\otimes\cdots\otimes\phi^{i_k} \end{aligned}$$

which means that the components of $\mathrm{tr}(A) \in T^k_\ell(V)$ are

$$\mathrm{tr}(A)^{j_2\cdots j_{\ell+1}}_{i_1\cdots i_k} = A^{mj_2\cdots j_{\ell+1}}_{i_1\cdots i_k m} \tag{5.20}$$

a sum of the components of A.

5.21. Inner product. The inner product on V is a symmetric $(2,0)$-tensor $g \in T^2_0(V)$ whose components

$$g_{ij} = g(E_i, E_j) = \langle E_i, E_j \rangle \tag{5.22}$$

form a symmetric matrix (g_{ij}).

5.23. Raising and lowering of the index. Since

$$(v^j E_j)^\flat = v^j E_j^\flat = v^j E_j^\flat(E_i)\phi^i = v^j \langle E_i, E_j \rangle \phi^i = g_{ij} v^j \phi^i$$

we may write

$$(v^j E_j)^\flat = v_i \phi^i \text{ where } v_i = g_{ij} v^j, \qquad (\omega_i \phi^i)^\sharp = \omega^j E_j \text{ where } \omega^j = g^{ij}\omega_i$$

because the linear map $V \xrightarrow{\flat} V^*$ has matrix (g_{ij}) so that the inverse map $V \xleftarrow{\sharp} V^*$ has $(g_{ij})^{-1} = (g^{ij})$ as its matrix. We get v_i from v^j by *lowering the index* and ω^j from ω_i by *raising the index.*

More generally, if $A \in T^k_{\ell+1}(V)$, then the components of $A^\flat \in T^{k+1}_\ell(V)$ are

$$(A^\flat)^{j_1 \cdots j_\ell}_{i i_1 \cdots i_k} = g_{ij} A^{j_1 \cdots j_\ell j}_{i_1 \cdots i_k}$$

because

$$\begin{aligned} A^\flat(\phi^{j_1}, \dots, \phi^{j_\ell}, E_i, E_{i_1}, \dots, E_{i_k}) &= A(\phi^{j_1}, \dots, \phi^{j_\ell}, E_i^\flat, E_{i_1}, \dots, E_{i_k}) \\ &= g_{ij} A(\phi^{j_1}, \dots, \phi^{j_\ell}, \phi^j, E_{i_1}, \dots, E_{i_k}) = g_{ij} A^{j_1 \cdots j_\ell j}_{i_1 \cdots i_k} \end{aligned}$$

and if $B \in T^{k+1}_\ell(V)$, then the components of $B^\sharp \in T^k_{\ell+1}(V)$ are

$$(B^\sharp)^{j_1 \cdots j_\ell j}_{i_1 \cdots i_k} = g^{ij} B^{j_1 \cdots j_\ell}_{i i_1 \cdots i_k} \tag{5.25}$$

because

$$\begin{aligned} B^\sharp(\phi^{j_1}, \dots, \phi^{j_\ell}, \phi^j, E_{i_1}, \dots, E_{i_k}) &= B(\phi^{j_1}, \dots, \phi^{j_\ell}, (\phi^j)^\sharp, E_{i_1}, \dots, E_{i_k}) \\ &= g^{ij} B(\phi^{j_1}, \dots, \phi^{j_\ell}, E_i, E_{i_1}, \dots, E_{i_k}) = g^{ij} B^{j_1 \cdots j_\ell}_{i i_1 \cdots i_k} \end{aligned}$$

Here are some special cases:

- ${g_i}^j = g^{kj} g_{ik} = \delta^j_i$.
- If the basis E_i is orthonormal so that (g_{ij}) and its inverse are identity matrices, $(A^\flat)^{j_1 \cdots j_\ell}_{i i_1 \cdots i_k} = g_{ij} A^{j_1 \cdots j_\ell j}_{i_1 \cdots i_k} = \delta^i_j A^{j_1 \cdots j_\ell j}_{i_1 \cdots i_k} = A^{j_1 \cdots j_\ell i}_{i_1 \cdots i_k}$ and $(B^\sharp)^{j_1 \cdots j_\ell j}_{i_1 \cdots i_k} = B^{j_1 \cdots j_\ell}_{j i_1 \cdots i_k}$.

5.26. Trace with respect to g. If $B \in T^{k+1}_\ell(V)$ has components $B^{j_1 \cdots j_\ell}_{i_1 \cdots i_{k+1}}$ then the components of $\operatorname{tr}_g(B) \in T^{k-1}_\ell(V)$ are

$$\operatorname{tr}_g(B)^{j_1 \cdots j_\ell}_{i_3 \cdots i_{k+1}} \overset{(5.20)}{=} (B^\sharp)^{j_1 \cdots j_\ell j}_{j i_3 \cdots i_{k+1}} \overset{(5.25)}{=} g^{ij} B^{j_1 \cdots j_\ell}_{i j i_3 \cdots i_{k+1}}$$

if we remember to take the trace at the right places.

5.27. Example. Let $h \in T^2_0(V)$ is a $(2,0)$-tensor and $s \in T^1_1(V)$ the corresponding endomorphism of V as in 5.12. Then

$$h_{ij} = h(E_i, E_j) = \langle sE_i, E_j \rangle = \left\langle {s_i}^k E_k, E_j \right\rangle = {s_i}^k g_{kj}$$

or, equivalently, ${s_i}^j = g^{jk} h_{ik}$. Lowering an index in s gives h and rasing an index in h gives s. The problem we are solving here (hidden behind a lot of formalistic notation) is how to construct $s\colon V \to V$ from knowledge of $\langle sX, Y \rangle = h(X, Y$. Then $\operatorname{tr}_g h = \operatorname{tr} s = {s_i}^i = g^{ij} h_{ij}$. In particular if $h = g$ is the metric, then $\operatorname{tr}_g(g) = {g_i}^i = n$.

CHAPTER 6

Non-euclidean geometry

Historical account of non-euclidean geometry.

6.1. Axiom (Parallel axiom of euclidean geometry). For any straight line L and any point P outside L there is a unique line through P that does not meet L.

6.2. Axiom (Parallel axiom of spherical geometry). For any straight line L and any point P outside L there is no line through P that does not meet L.

6.3. Axiom (Parallel axiom of hyperbolic geometry). For any straight line L and any point P outside L there are at least two lines through P that do not meet L.

1. The hyperbolic plane

The hyperbolic plane of radius R is the Riemannian manifold

$$H^2_R = \{(\xi,\tau) \in \mathbf{R}^2 \times \mathbf{R}_+ \mid |\xi|^2 - \tau^2 = -R^2\} \subset \mathbf{R}^2 \times \mathbf{R} = \mathbf{R}^3$$

with metric induced from the Minkowski metric on $\mathbf{R}^3$. Equivalently, the hyperbolic plane of radius R is the disc of radius R

$$B^2_R(0) = \{u \in \mathbf{R}^2 \mid \ |u| < R\}$$

equipped with the metric induced from H^2_R under hyperbolic stereographic projection

$$\pi\colon H^2_R \to B^2_R(0), \qquad \pi^{-1}(u) = \left(\frac{2R^2 u}{R^2 - |u|^2}, R\frac{R^2+|u|^2}{R^2-|u|^2}\right)$$

which is a diffeomorphism.

Stereograhic projection induces a correspondence between geodesics in the two models since isometries preserve geodesics. Since the geodesics in the hyperboloid model H^2_R are the curves

$$\{(\xi,\tau) \in H^2_R \mid (\xi,\tau)\cdot(\alpha,\beta) = 0\}, \qquad \mathbf{R}^3 \ni (\alpha,\beta) \neq 0,$$

the geodesics in the disc model are the curves

$$\{u \in B^2_R(0) \mid \pi^{-1}(u)\cdot(\alpha,\beta) = 0\} = \{u \in B^2_R(0) \mid 2Ru\cdot\alpha + (R^2+|u|^2)\beta = 0\}$$

parametrized as constant speed curves. If $\beta = 0$, this is a straight line through 0. If $\beta \neq 0$, we replace (α,β) by a parallel vector of the form $(-\alpha, 1)$ and get

$$\{u \in B^2_R(0) \mid |u - R\alpha|^2 + |R\alpha|^2 = R^2\} = \{u \in B^2_R(0) \mid |u-R\alpha|^2 = R^2(1-|R\alpha|^2)\}$$

If $|\alpha| \geq 1$, this is the empty set. When $|\alpha| < 1$, this is part of a circle with center $R\alpha$ and radius $R\sqrt{1-|\alpha|^2}$. If we let u_0 denote either of the two points where the circle meets the boundary of the disc, then we see that

$$|u_0 - R\alpha|^2 + |R\alpha|^2 = |u_0|^2$$

which means that $u_0 - R\alpha \perp u_0$ so that the center, $R\alpha$, of the circle lies on the tangent to the boundary circle at u_0. We conclude that the geodesics in the ball model are straight lines through 0 and circular arcs meeting the boudary circle in right angles.

The geodescis crossing the boundary of the hyperbolic plane of radius 1 at the point $(1,0)$ are

- circular arcs with radius $V > 0$ and center $(1, \pm V)$ on T
- the straight line through $(1,0)$ perpendicular to T

where T is the tangent at $(1,0)$ to the boundary circle.

6.4. Definition. *The Voronoï distance between the point P and the circle c with centre C and radius R is $V(c,P) = \sqrt{|CP|^2 - R^2}$.*

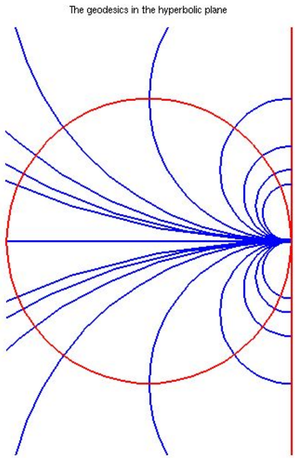


FIGURE 1. Geodesics in the hyperbolic plane

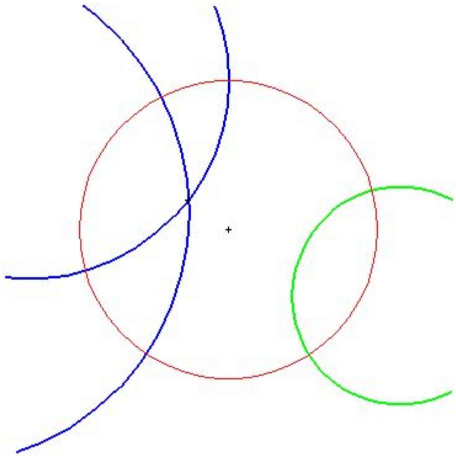


FIGURE 2. Two lines through P that do not meet the green line!

If P is outside the circle, $V(c, P)$ is the length of the tangent segments through P; $V(c, P) = 0$ when P is on the circle; and $V(c, P)$ is purely imaginary when P is inside the circle.

The centers of the geodesic circular arcs through a point P_0 in the hyperbolic plane $B^2_R(0)$ of radius R are the points P for which

$$|P_0P| = V(\partial B^2_R(0), P)$$

meaning that the euclidean distance to P_0 equals the Voronoï distance to the boundary circle $\partial B^2_R(0)$. This locus is the perpendicular bisector of P_0P_0' where P_0' is the point where the radius through P_0 crosses the boundary circle.

If P_1 and P_2 are two given points in $B^2_R(0)$, not on the same diameter, then the center of the geodesic through P_1 and P_2 is the point Q for which

$$|P_1Q| = V(\partial B^2_R(0), Q) = |P_2Q|$$

meaning that Q is the intersection of the two bisecting lines associated to P_1 and P_2 as just described.

It is known that the hyperbolic plane embeds isometrically into euclidean $\mathbf{R}^5$ but not into $\mathbf{R}^3$. It is unknown if there exists an isometric embedding of the hyperbolic plane into $\mathbf{R}^4$.

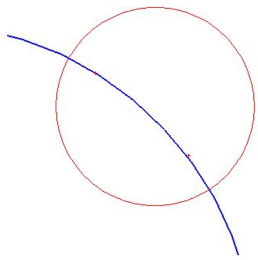


FIGURE 3. The unique geodesic through two given points

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