CHEVALLEY p-LOCAL FINITE GROUPS

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Abstract. We describe the spaces of homotopy fixed points of unstable Adams operations acting on p-compact groups and also of unstable Adams operations twisted with a finite order automorphism of the p-compact group. We obtain new exotic p-local finite groups.

Contents

1. Introduction

The main purpose of this paper is the description of the structure of the spaces of homotopy fixed points of unstable Adams operations ψ^q acting on p-compact groups and also of unstable Adams operations twisted by automorphisms of p-compact groups $\tau\psi^q$.

In the classical case, for a prime number p, a prime power q, prime to p, a compact connected Lie group G, and a finite order automorphism τ of G, Friedlander showed that there is a homotopy pullback diagram

$$\begin{tikzcd} B\,{}^{\tau}G(q)^\wedge_p \arrow[r, "f"] \arrow[d, "f"'] & BG^\wedge_p \arrow[d, "\Delta"] \\ BG^\wedge_p \arrow[r, "{(1,\tau\psi^q)}"] & BG^\wedge_p \times BG^\wedge_p \end{tikzcd}$$

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where ${}^{\tau}G(q)$ is the twisted Chevalley group over $\mathbb{F}_q$ of type G, Δ is the diagonal map, and ψ^q an unstable Adams operation of exponent q [32, 33]. Here and throughout, p-completion is understood in the sense of Bousfield–Kan [10].

The concept of a p-compact group was introduced by Dwyer and Wilkerson in [25] as a p-local homotopy theoretic analogue of compact Lie group. A p-compact group is a triple (X, BX, e), where $H^*(X; \mathbb{F}_p)$ is finite, BX is a pointed p-complete space, and $e : X \to \Omega BX$ is a homotopy equivalence. We will usually refer to a p-compact group simply as X. BX is then understood as its classifying space, a concrete loop space structure imposed in the underlying space X. If G is a compact connected Lie group, then the p-completion of its classifying space $BG^\wedge_p$ is a p-compact group. A p-compact group that cannot be obtained in this way is called *exotic*. We postpone till Section 2 a more detailed description of the theory of p-compact groups.

Unstable Adams operations ψ^q, for any p-adic unit q, can be defined for any connected p-compact group X (see Section 2). Following the above pattern, if $\tau\psi^q$ a twisted Adams operation, then the space $B\,{}^{\tau}X(q)$ is defined by the homotopy pullback square

$$\begin{tikzcd} B\,{}^{\tau}X(q) \arrow[r, "f"] \arrow[d, "f"'] & BX \arrow[d, "\Delta"] \\ BX \arrow[r, "(1,\tau\psi^q)"] & BX \times BX\,. \end{tikzcd} \tag{1}$$

Thus if X is obtained as the p-completion of a compact Lie group G, and τ is a finite order automorphism of G, $B\,{}^{\tau}X(q)$ is homotopy equivalent to the p-completed classifying space of the twisted Chevalley group ${}^{\tau}G(q)$.

The concept of p-local finite group has been recently introduced in [13] as algebraic objects that are modeled on the p-local structure of finite groups and as such they have classifying spaces which are p-complete spaces. In turn, the classifying space of a p-local finite group determines its algebraic structure. Every finite group G determines a p-local finite group at a prime p with classifying space $BG^\wedge_p$. Like in the case of p-compact groups, p-local finite groups that do not arise in this way for any finite group G are called *exotic*. We refer to Section 3 for the precise definition and main properties of p-local finite groups. Our main result shows that $B\,{}^{\tau}X(q)$ is the classifying space of a p-local finite group. We will also determine the cases in which they are exotic p-local finite groups.

Theorem A. *Let p be an odd prime. If X is a 1-connected p-compact group, q is a prime power, prime to p, and τ is an automorphism of X of finite order prime to p, then the space of homotopy fixed points of BX by the action of $\tau\psi^q$, denoted $B\,{}^{\tau}X(q)$, is the classifying space of a p-local finite group.*

By analogy with the classical case, we will call the p-local finite group $X(q)$ (${}^{\tau}X(q)$) with classifying space $BX(q)$ ($B\,{}^{\tau}X(q)$) obtained in Theorem A a (twisted) *Chevalley p-local finite group of type X*.

Our arguments concentrate on the exotic p-compact groups at odd primes, and break into two separate steps. One deals with actions of finite groups of order not divisible by p on p-compact groups and the results obtained have an independent interest. The other step deals with the action of unstable Adams operations ψ^q where $q \equiv 1 \bmod p$ and it is the one leading to the new exotic examples of p-local finite groups.

Group actions will be understood in the weak sense of *proxy* actions; that is, we will say that an action of a group G on a space M is a fibration $M \xrightarrow{i} M_{hG} \xrightarrow{\mathrm{pr}} BG$ [25]. The total space M_{hG} is referred to as the homotopy quotient space. The space of homotopy fixed points is the space

$$M^{hG} = \{BG \xrightarrow{s} M_{hG} \mid \mathrm{pr} \circ s = \mathrm{id}_{BG}\}$$

of sections. Two actions will be considered equivalent if they are defined by fibre homotopy equivalent fibrations. If M is a G-space in the usual sense then M_{hG} is the Borel construction and M^{hG} is homeomorphic to the space $\mathrm{Map}_G(EG, M)$ of equivariant maps where EG is a contractible free G-space. When we specialize to p-compact groups X, an outer action of G is a homomorphism $\rho\colon G \to \mathrm{Out}(X)$, where $\mathrm{Out}(X)$ is the group of outer automorphisms of the p-compact group X, in other words, unpointed homotopy classes of self-equivalences of BX. By obstruction theory, it turns out that if G has finite order prime to p, then an outer action on a connected p-compact group X determines a unique action, up to equivalence, and the space of homotopy fixed points is again a connected p-compact group.

The space $B\,{}^{\tau}X(q)$ defined by pullback square (1) can also be viewed as a homotopy fixed point space $BX^{h\langle\tau\psi^q\rangle}$ for the action of the infinite cyclic group generated by $\tau\psi^q \in \mathrm{Out}(X)$. More details are given in as explained in Section 6.

Theorem B. *Let X be a connected p-compact group. If G is a finite group of order prime to p and $\rho\colon G \to \mathrm{Out}(X)$ an outer action, then*

(1) *ρ lifts to a unique action of G on X, up to equivalence.*
(2) *X^{hG} is a connected p-compact group with $H^*(BX^{hG};\mathbb{Q}_p) \cong S[QH^*(BX;\mathbb{Q}_p)_G]$, the symmetric algebra generated on the coinvariants $QH^*(BX;\mathbb{Q}_p)_G$.*
(3) (Harper splitting) *$X^{hG} \to X$ is a p-compact group monomorphism, there is a homotopy equivalence*
$$X \simeq X^{hG} \times X/X^{hG}$$
and X/X^{hG} is an H-space.
(4) *Assume that p is odd. If $H^*(BX;\mathbb{F}_p)$ is a polynomial ring, then $H^*(BX^{hG};\mathbb{F}_p)$ is also a polynomial ring.*

Here and throughout, $H^*(-;\mathbb{Q}_p)$ stands for $H^*(-;\mathbb{Z}_p)\otimes\mathbb{Q}$, and $QH^*(BX;\mathbb{Q}_p)$ denotes the module of the indecomposables in $H^*(BX;\mathbb{Q}_p)$.

Some interesting cases to which Theorem B applies are F_4 at the prime 3 and E_8 at the prime 5, where the p-compact groups $\mathbf{X}_{12}$, respectively, $\mathbf{X}_{31}$ split off (see Section 2 for notation). In the first case, Friedlander's exceptional isogeny φ of F_4 at the prime 3 gives rise to an automorphim of order 2 and the homotopy fixed point p-compact group ${F_4}^{hC_2}$ is the p-compact group $\mathbf{X}_{12} = DI_2$ whose cohomology realizes the rank 2 Dickson algebra $H^*(B\mathbf{X}_{12};\mathbb{F}_3) \cong \mathbb{F}_3[x_{12}, x_{16}]$ (subscripts of cohomology classes indicate degrees) over $\mathbb{F}_3$. This case was already considered in our previous work [15]. In the second case, a cyclic group of order 4 generated by the unstable Adams operation ψ^i, $i = \sqrt{-1}$, acts on E_8. The homotopy fixed point p-compact group $E_8^{hC_4}$ is the p-compact group $\mathbf{X}_{31}$ corresponding to the reflection group number 31 on the Clark–Ewing list, and its mod 5 cohomology ring is $H^*(B\mathbf{X}_{31};\mathbb{F}_5) = \mathbb{F}_5[x_{16}, x_{24}, x_{40}, x_{48}]$ (see A.12).

It turns out that $\mathbf{X}_{12}$ and $\mathbf{X}_{31}$ are the two exotic p-compact groups originally constructed by Zabrodsky [69], and later included in the Aguadé family [1]. Zabrodsky used the actions of these same automorphisms, φ and ψ^i, on the homotopy groups of BF_4 and BE_8, respectively, and realized the invariant subgroups as homotopy groups of new spaces, $B\mathbf{X}_{12}$ and $B\mathbf{X}_{31}$.

The corresponding splittings are $F_4 \simeq DI_2 \times F_4/DI_2$ at the prime 3, first discovered by Harper [36], and $E_8 \simeq X_{31} \times E_8/X_{31}$, that was obtained by Wilkerson [66]. Other examples appear in 5.4.

Our second step deals with the action of unstable Adams operations ψ^q of exponent $q \equiv 1 \bmod p$, $q \neq 1$, on connected p-compact groups X. These automorphism have infinite order and the effect now is opposite in some sense to the case of finite groups of order prime to p. The spaces of homotopy fixed points $BX(q)$ have the same p-rank as the original p-compact groups X, but the maximal tori $T^n \simeq ((S^1)^n)^\wedge_p$ are cut down to finite maximal tori $T^n_\ell \cong (\mathbb{Z}/p^\ell)^n$, $\ell = \nu_p(1-q)$, keeping the same Weyl group (see 7.5, 7.6).

We restrict our calculations in this part to p-compact groups for which the mod p cohomology ring $H^*(BX;\mathbb{F}_p)$ is a polynomial ring. For simplicity, we will refer to them as polynomial p-compact groups. At odd primes, these include all irreducible exotic examples and will therefore suffice to our purposes.

Theorem C. *Let q be a p-adic unit such that $q \equiv 1 \bmod p$, $q \neq 1$. If X is an irreducible 1-connected polynomial p-compact group, then $BX(q)$ is the classifying space of a p-local finite group.*

The proof is based on the classification theorem for p-compact groups at odd primes [7], see Section 2. The irreducible polynomial p-compact groups are

(1) $BSU(n)^\wedge_p$ (family 1 in the Clark–Ewing list),
(2) the generalized Grassmannians (family 2a in the Clark–Ewing list),
(3) the Clark–Ewing p-compact groups (p-compact groups with Weyl group of order prime to p), and
(4) the Aguadé family $\mathbf{X}_{12}$, $\mathbf{X}_{29}$, $\mathbf{X}_{31}$, $\mathbf{X}_{34}$ at primes $p = 3$, 5, 5, and 7, respectively, and of rank $p-1$. (The subscripts indicate the number of the Weyl group in the Clark–Ewing list.)

Theorem C is proved by considering separately these four cases in 11.1, 11.4 , 9.8, and 10.3, respectively.

In cases (1) and (3) we always obtain that $BX(q)$ is the p-completed classifying space of a finite group. The other two families contain the new exotic examples of p-local finite groups.

A complete description of the structure of the p-local finite groups $\mathbf{X}_i(q)$, $i = 12, 29, 31, 34$, is obtained in Section 10. Fix $q \equiv 1 \bmod p$ and let $\nu_3(1+2^{2n+1}) = \nu_3(1-q)$. For $\mathbf{X}_{12}(q)$, $p=3$, we obtain that $B\mathbf{X}_{12}(q) \simeq B({}^2F_4(2^{2n+1}))^\wedge_3$ (Example 10.7). For $\mathbf{X}_{31}(q)$, $p=5$, it turns out that if $\nu_5(1+2^{4m+2}) = \nu_5(1-q)$, then $B\mathbf{X}_{31}(q) \simeq BE_8(2^{2m+1})^\wedge_5$ (Example 10.8). In particular, we can obtain the p-compact groups $\mathbf{X}_{12}$ and $\mathbf{X}_{31}$ as telescopes of a sequence of p-completed classifying spaces of finite groups (see 10.9):

$$B\mathbf{X}_{12} \simeq \operatorname*{hocolim}_m B({}^2F_4(2^{3^m}))^\wedge_3 \,,$$
$$B\mathbf{X}_{31} \simeq \operatorname*{hocolim}_m BE_8(2^{5^m})^\wedge_5 \,.$$

The cases $B\mathbf{X}_{29}(q)$ and $B\mathbf{X}_{34}(q)$ at primes 5 and 7, respectively, are classifying spaces of exotic p-local finite groups (Example 10.6).

Family 2a in the Clark–Ewing list consists of the pseudoreflection subgroups of $GL_n(\mathbb{Z}_p)$ $G(m,r,n)$ with $r|m|(p-1)$ generated by the permutation matrices and the diagonal matrices $\mathrm{diag}(a_1, a_2, \dots, a_n)$ with ${a_i}^m = 1$ and $(a_1 a_2 \dots a_n)^{m/r} = 1$. We denote $\mathbf{X}(m,r,n)$ the p-compact group of rank n with Weyl group $G(m,r,n)$. We also prove that $B\mathbf{X}(m,r,n)(q)$ is the classifying space of an exotic p-local finite group provided $n \geq p$ and $r > 2$ (Proposition 11.5).

It is remarkable the list of new exotic p-local finite groups that arise in this way.

Theorem D. *For $q \equiv 1 \bmod p$, $q \neq 1$, the following are classifying spaces of exotic p-local finite groups:*

- $B\mathbf{X}_{29}(q)$ *and* $B\mathbf{X}_{34}(q)$ *at primes $p=5$ and $p=7$, respectively, and*
- $B\mathbf{X}(m,r,n)(q)$ *for $n \geq p$ and $r > 2$.*

Our next theorem provides the necessary arguments in order to deduce the general case of Theorem A from the two steps.

Theorem E. *Let p be an odd prime and X a connected p-compact group, τ an automorphism of X of order prime to p and ψ^q an unstable Adams operation of exponent a p-adic unit.*

(1) *If X is 1-connected and $q \equiv 1 \bmod p$, $q \neq 1$, then $B\,{}^{\tau}X(q) \simeq BX^{h\langle\tau\rangle}(q)$.*
(2) *If q' is another p-adic unit such that $q \equiv q' \bmod p$ and $\nu_p(1-q^r) = \nu_p(1-q'^r)$, where r is the order of $q \bmod p$, then $BX(q) \simeq BX(q')$.*

Since we can decompose a p-adic unit q as $q = \zeta q_0$ where ζ is a $(p-1)$st-root of unity and $q_0 \equiv 1 \bmod p$, part (1) of the above theorem will reduce the question of computing $BX(q)$ to the case where $q \equiv 1 \bmod p$ which turns out to be easier to handle in abstract calculations and concrete examples. The second part of the theorem tells us that $BX(q)$ does only depend on the order r of $q \bmod p$ and the p-adic valuation $\nu_p(1-q^r)$, so we can change the exact value of q at our convenience if we keep those parameters fixed.

Part (2) of Theorem E also explains the often observed fact that finite Chevalley groups $G(q)$ and $G(q')$ have same cohomology ring or identical p-local structure when q and q' are prime powers, with $q^r \equiv q'^r \equiv 1 \bmod p$ and $\nu_p(1-q^r) = \nu_p(1-q'^r)$, for some r, $1 \leq r \leq p-1$.

Proof of Theorem A. We consider $B\,{}^{\tau}X(q)$ as the homotopy fixed point space $BX^{h\langle\tau\psi^q\rangle}$ for the action on BX of the group generated by $\tau\psi^q$.

If we write $q = \zeta q_0$, where ζ is a $(p-1)$th root of unity and $q_0 \equiv 1 \bmod p$, $q_0 \neq 1$, so that $\tau\psi^q = \tau\psi^\zeta\psi^{q_0}$, then we have

$$B\,{}^{\tau}X(q) = BX^{h\langle\tau\psi^q\rangle} \simeq BX^{h\langle\tau\psi^\zeta\rangle}(q_0)\,,$$

according to Theorem E.

$X^{h\langle\tau\psi^\zeta\rangle}$ is a 1-connected p-compact group by Theorem B, hence it splits as a product of irreducible 1-connected p-compact groups [26, 54]

$$BX^{h\langle\tau\psi^\zeta\rangle} \simeq BX_1 \times \cdots \times BX_s\,,$$

and then, also, $BX^{h\langle\tau\psi^\zeta\rangle}(q_0) \simeq BX_1(q_0) \times \cdots \times BX_s(q_0)$. It remains to show that each $BX_i(q_0)$ is the classifying space of a p-local finite group.

If X_i is polynomial, Theorem C applies and $BX_i(q_0)$ is the classifying space of a p-local finite group.

If X_i is the p-completion of a compact Lie group G, then we can find a prime number q_0' with $q_0' \equiv q_0 \equiv 1 \mod p$ and $\nu_p(1-q_0) = \nu_p(1-q_0')$, and then $BX_i(q_0) \simeq BX_i(q_0')$ by Theorem E (cf. Remark 6.6), and this last is the p-completed classifying space of a finite Chevalley group of type G, by the classical result of Friedlander [33].

By the classification theorem of p-compact groups at odd primes [7] (see Section 2), every irreducible, simply-connected p-compact group is either polynomial or the p-completion of a compact Lie group, hence the proof is complete. □

Many authors have been interested in the cohomology rings of finite Chevalley groups at primes different from the defining characteristic. Quillen [59, Theorem 4], shows that for an odd prime p and a prime power q prime to p, if m is the order of q mod p and $\ell = \nu_p(1-q^m)$, then

$$H^*(BGL(n,q);\mathbb{F}_p) \cong P[x_1,\dots,x_{[\frac{n}{m}]}] \otimes E[y_1,\dots,y_{[\frac{n}{m}]}]$$

where $\deg(x_i) = 2mi$ and $\deg(y_i) = 2mi-1$.

Fiedorowicz and Priddy, [29, 30] computed the cohomology rings of Chevalley groups of classical type. Kleinerman [38] has computed the cohomology of Chevalley groups of exceptional Lie type at large primes. M. Mimura, M. Tezuka, and S. Tsukuda [43] have recently approached the cohomology rings of finite Chevalley groups at torsion primes, by newly constructing a spectral sequence of Eilenberg-Moore type.

The result that we include here is essentially due to L. Smith, at least part (1) already appears in [62]. We include it here for the convenience of the reader, as it is an important step in our arguments.

Theorem F. *Let X be a polynomial p-compact group with*

$$H^*(BX;\mathbb{F}_p) \cong P[x_1,\dots,x_n]$$

and q a p-adic unit with $q \equiv 1 \bmod p$, $q \neq 1$. Then:

(1) $H^*(BX(q);\mathbb{F}_p) \cong P[x_1,\dots,x_n] \otimes E[y_1,\dots,y_n]$ *with higher Bockstein relations* $\beta_{(\ell_i)}(y_i) = x_i$, $\ell_i = \nu_p(1-q^{d_i})$, $2d_i = \deg x_i$, $2d_i - 1 = \deg y_i$, *and*
(2) *the inclusion of the maximal finite torus* $i\colon BT^n_\ell \to BX(q)$, $\ell = \nu_p(1-q)$, *induces a monomorphism* $i^*\colon H^*(BX(q);\mathbb{F}_p) \to H^*(BT^n_\ell;\mathbb{F}_p)^{W_X}$.

The inclusion $i^*\colon H^*(BX(q);\mathbb{F}_p) \to H^*(BT^n_\ell;\mathbb{F}_p)^{W_X}$ is an isomorphism in many cases. This is checked by direct calculation of the relevant invariant rings. In cases in which X is a Clark–Ewing p-compact group or a generalized Grassmannian, i^* is an isomorphism (see Section 9). It is also an isomorphism in the case of the Aguadé p-compact groups $\mathbf{X}_i(q)$, $i = 29, 31, 34$, however, i^* is not an epimorphism in case of $\mathbf{X}_{12}(q)$, for which we obtain $H^*(B\mathbf{X}_{12}(q);\mathbb{F}_3) \cong P[x_{12},x_{16}] \otimes E[y_{11},y_{15}]$, while $H^*(BT^2_\ell;\mathbb{F}_3)^{W_{X_{12}}} \cong P[x_{12},x_{16}] \otimes E[y_{10},y_{11},y_{15}]/(y_{11}y_{15} - x_{16}y_{10},\ y_{10}y_{11},\ y_{10}y_{15})$ (see Example 9.7).

We have restricted our calculations at odd primes, although some of the results are also valid at the prime two. At present the classification of 2-compact groups has not been completed although a plausible conjecture is that the Dwyer-Wilkerson 2-compact group $DI(4)$ is the only irreducible exotic 2-compact group. The finite Chevalley versions of $DI(4)$, named $BSol(q)$, for odd prime powers q, have been first considered by Benson [8] and then by Levi and Oliver [39] who proved that they are classifying spaces of 2-local finite groups and their 2-local structure is in fact a system of fusion relations studied by Solomon [63] and defined over the Sylow 2-subgroup of $Spin(7,q)$.

The paper is organized as follows. In Sections 2 and 3 we review the definitions and main results from the theory of p-compact groups and p-local finite groups. In Section 4 we further develop some aspects of the theory of p-local finite groups concerning the homotopy characterization of classifying spaces of p-local finite groups. The main results in Sections 10 and 11 stating that $BX(q)$ is the classifying space of a p-local finite group if X is a p-compact group in the Aguadé family or a generalized Grassmannian are based in this homotopy characterization of classifying spaces.

Section 5 deals with what we have called *first step*. There is a discussion of different ways in which we can understand an action of a group on a p-compact group and it contains the proof of Theorem B. This Theorem states that a homotopy fixed point space X^{hG} is again a p-compact group if X was a connected p-compact group and G is a finite group of order prime to p. Identifying X^{hG} with a p-compact group in the classification list requires a close look to the restriction of the action to the maximal torus normalizer. This will be considered in Appendix A. In particular, Corollary A.6 contains a criterion for the recognition of the homotopy fixed point p-compact group by action of unstable Adams operations of finite order. This is applied to many examples through the Clark–Ewing list at the end of this appendix, A.7 through A.12.

Section 6 is devoted to the proof of Theorem E. It reduces the analysis of the structure of a general homotopy fixed point space $B^{\tau}X(q)$ to first analysing a homotopy fixed point p-compact group and then a homotopy fixed point space by the action of an unstable Adams operation $\psi^{q'}$ of exponent $q' \equiv 1 \bmod p$. This allows us to complete the argument for the proof of Theorem A from steps one and two.

The *second step* starts in Sections 7, 8, and 9, where we analyse the general subgroup structure of spaces $BX(q)$, where $q \equiv 1 \bmod p$, $q \neq 1$, and their cohomological properties. Theorem F is proved in Section 8. Some technical results concerning the Bousfield–Kan spectral sequence for the cohomology of a homotopy colimit are postponed to Appendix B.

Finally, sections 10 and 11, are devoted to the more specific properties of the p-compact groups in the Aguadé family and the generalized Grassmannians, respectively. With them, we complete the proof of theorems C and D.

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2. p-compact groups

A p-compact group is a triple (X, BX, e) where X is a space, BX is a p-complete connected pointed space, $H^*(X; \mathbb{F}_p)$ is finite, and $e\colon X \to \Omega BX$ is a homotopy equivalence from X to the space ΩBX of based loops in BX.

Throughout the paper, and when no confusion is possible, we will simply denote a p-compact group (X, BX, e) as X. We shall say that X is *connected* if $\pi_0(X)$ is a point and *simply connected* if also $\pi_1(X)$ is trivial. These spaces were introduced by Dwyer and Wilkerson in 1994 as p-local homotopy theoretic versions of compact Lie groups [25]. We present here a short summary of the theory of p-compact groups and refer to the surveys [47, 56, 22] for more information. Examples of p-compact groups include all simply connected p-complete spaces with polynomial $\mathbb{F}_p$-cohomology, and the p-completed classifying spaces of all compact Lie groups G such that $\pi_0(G)$ is a finite p-group. The p-compact group obtained in this way from a torus is called a p-compact torus. Thus a p-compact torus BT of rank n is simply a $K(\mathbb{Z}_p,2)^n$ and we have that $H_2(BT;\mathbb{Z}_p)=\mathbb{Z}_p^n$ is a finitely generated free $\mathbb{Z}_p$-module. A *maximal torus* of a p-compact group BX is a pointed map $BT\to BX$, satisfying an injectivity and a maximality condition, of a p-compact torus into BX. The *Weyl group* W of the maximal torus $BT\to BX$, which we may assume is a fibration, is the monoid of fibre homotopy classes $BT\to BT$ over BX. It turns out that all elements of W are invertible so that W is actually a group. Equivalently, the Weyl group is the group of components of the *Weyl space* which is the associative topological monoid of self-maps of BT over BX. The Borel construction, BN, for the action of the Weyl space on BT is called the *normalizer of the maximal torus*. The monomorphism $BT\to BX$ extends to a monomorphism $BN\to BX$ [25, 9.2,9.8].

Theorem 2.1 (Existence of maximal tori [25, 9.7])**.** *Any p-compact group X admits a maximal torus $BT_X\to BX$ and a Weyl group W_X. When X is connected, the Weyl group W_X acts faithfully on the finitely generated free $\mathbb{Z}_p$-module $L_X=H_2(BT_X;\mathbb{Z}_p)$, the pair (W_X,L_X) is a $\mathbb{Z}_p$-reflection group, and*

$$H^*(BX;\mathbb{Z}_p)\otimes\mathbb{Q}\to\big(H^*(BT_X;\mathbb{Z}_p)\otimes\mathbb{Q}\big)^{W_X}$$

is an isomorphism.

This theorem introduces a relationship between p-compact groups and $\mathbb{Z}_p$-reflection groups as defined below.

An automorphism of a finitely generated free $\mathbb{Z}_p$-module is a *reflection* if it acts as the identity on a hyperplane. A *$\mathbb{Z}_p$-reflection group* is a pair (W,L) where L is a finitely generated free $\mathbb{Z}_p$-module and W a subgroup of $\mathrm{Aut}_{\mathbb{Z}_p}(L)=GL(L)$ that is generated by the reflections that it contains. A *morphism* between two $\mathbb{Z}_p$-reflection groups, (W_1,L_1) and (W_2,L_2), is a pair (α,θ) consisting of a group homomorphism $\alpha\colon W_1\to W_2$ and an α-linear $\mathbb{Z}_p$-module homomorphism $\theta\colon L_1\to L_2$ [52, 4.1]. The $\mathbb{Z}_p$-reflection group (W,L) is *irreducible* if $L\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ is an irreducible $\mathbb{Q}_pW$-module. Using the Shephard–Todd classification of irreducible complex reflection groups [61], Clark and Ewing [18] produced the list of all finite irreducible $\mathbb{Z}_p$-reflection groups [52, 11.18]. At odd primes the list is as follows:

- Family 1: $(\Sigma_{n+1},S(\mathbb{Z}_p^{n+1}))$ where the symmetric group Σ_{n+1} permutes the $n+1$ factors of $\mathbb{Z}_p^{n+1}$ and $S(\mathbb{Z}_p^{n+1})=\{(x_1,\dots,x_{n+1})\in\mathbb{Z}_p^{n+1}\mid\sum x_i=0\}$.
- Family 2a: Let $r\geq 1$ and $m\geq 2$ natural numbers such that $r\mid m\mid p-1$. The cyclic group C_m of order m is contained in the group of units $\mathbb{Z}_p^\times$ for $\mathbb{Z}_p$. The $\mathbb{Z}_p$-reflection group $(G(m,r,n),\mathbb{Z}_p^n)$, $n\geq 2$, is the group generated by the subgroup Σ_n of all permutations of the n coordinates and the subgroup
$$A(m,r,n)=\{\mathrm{diag}(\theta_1,\dots,\theta_n)\in C_m^n\mid(\theta_1\cdots\theta_n)^{m/r}=1\}$$
consisting of diagonal matrices.
- Family 2b: $(D_{2m},\mathbb{Z}_p^2)$, $m>2$, when $m\equiv\pm1 \bmod p$ or $m=3,6$ if $p=3$ is the dihedral group of order $2m$, generated by matrices $\left(\begin{smallmatrix}0&-1\\1&\theta+\theta^{-1}\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, where θ is a primitive mth-root of unity. It is also usual to call them $G(m,m,2)$, following the notation of Shephard–Todd [61].
- Family 3: $(C_m,\mathbb{Z}_p)$ when $m\mid p-1$ and C_m is the order m cyclic subgroup of $\mathbb{Z}_p^\times$.
- Sporadic groups: 34 sporadic $\mathbb{Z}_p$-reflection groups G_i, $4\leq i\leq 37$.

See [6] for a more detailed description of this list of all irreducible $\mathbb{Z}_p$-reflection groups.

The automorphism group of the $\mathbb{Z}_p$-reflection group (W, L) is isomorphic to $N_{GL(L)}(W)$. There is an obvious homomorphism from this group to the group of trace preserving automorphisms of W. The kernel is the group $\mathrm{Aut}_{\mathbb{Z}_pW}(L)$ of automorphisms of the $\mathbb{Z}_pW$-module L. Using this we get an exact sequence of groups [52, 3.14–16]

$$1 \to \mathrm{Aut}_{\mathbb{Z}_pW}(L)/Z(W) \to N_{GL(L)}(W)/W \to \mathrm{Out}_{\mathrm{tr}}(W) \tag{2}$$

where the group to the right is the group

$$\mathrm{Out}_{\mathrm{tr}}(W) = \{\alpha \in \mathrm{Out}(W) \mid \forall w \in W \colon \mathrm{tr}(\alpha(w)) = \mathrm{tr}(w)\}$$

of trace preserving outer automorphisms of $W < GL(L)$. Observe that there is a group homomorphism

$$\psi \colon \mathbb{Z}_p^\times \to N_{GL(L)}(W)/W$$

that takes the p-adic unit $u \in \mathbb{Z}_p^\times$ to scalar multiplication, $\psi^u \colon L \to L$, by u on L. The kernel of ψ is the finite subgroup $\mathbb{Z}_p^\times \cap Z(W)$ of $W < GL(L)$.

If (W, L) is irreducible, $\mathrm{Aut}_{\mathbb{Z}_pW}(L) = \mathbb{Z}_p^\times$ consists only of the scalar matrices ψ^u according to Schur's lemma so that (2) takes the form

$$1 \to \mathbb{Z}_p^\times/Z(W) \xrightarrow{\psi} N_{GL(L)}(W)/W \to \mathrm{Out}_{\mathrm{tr}}(W)\,. \tag{3}$$

Moreover, an explicit case-by-case computation shows that the group $\mathrm{Out}_{\mathrm{tr}}(W)$ is trivial for all irreducible $\mathbb{Z}_p$-reflection groups except for a few of the dihedral groups $G(m, m, 2)$ and for the sporadic reflection groups G_5, G_7, and $G_{28} = W(F_4)$, and in these cases it consists of elements that lift to finite order elements τ in $N_{GL(L)}(W)/W$. We conclude that if (W, L) is irreducible then $N_{GL(L)}(W)/W$ consists only of elements of the form $\tau\psi^u$ where τ has finite order.

Theorem 2.2 (Classification of p-compact groups at odd primes [52, 7])**.** *Let p be an odd prime. The assignment $X \rightsquigarrow (W_X, L_X)$ gives a bijective correspondence between isomorphism classes of connected p-compact groups X and isomorphism classes of $\mathbb{Z}_p$-reflection groups (W, L). We have*

$$\mathrm{Out}(X) \cong N_{GL(L)}(W)/W$$

where (W, L) is the reflection group assigned to the connected p-compact group X.

The *irreducible* p-compact groups, which are the p-compact groups corresponding to the irreducible $\mathbb{Z}_p$-reflection groups of the Clark–Ewing classification table [18] (see also [23, 1.5]) are

- Family 1: $BSU(n+1)^\wedge_p$ (the special unitary groups)
- Family 2a: $B\mathbf{X}(m, r, n)$, $(m, r, n) \neq (m, m, 2)$, (the generalized Grassmannians)
- Family 2b: $B\mathbf{X}(m, m, 2)$, $m \geq 3$
- Family 3: $B\hat{S}_p^{2m-1}$ (the Sullivan spheres)
- Sporadic groups: 34 sporadic p-compact groups $B\mathbf{X}_i$, $4 \leq i \leq 37$.

Among the generalized Grassmannians we find

$$B\mathbf{X}(2, 1, n) = BSO(2n+1)^\wedge_p, \quad B\mathbf{X}(2, 2, n) = BSO(2n)^\wedge_p,$$

in family 2b

$$B\mathbf{X}(3, 3, 2) = BPU(3)^\wedge_p, \quad B\mathbf{X}(6, 6, 2) = (BG_2)^\wedge_p\,,$$

and among the sporadic cases we find

$$B\mathbf{X}_{28} = (BF_4)^\wedge_p, \quad B\mathbf{X}_{35} = (BE_6)^\wedge_p, \quad B\mathbf{X}_{36} = (BE_7)^\wedge_p, \quad B\mathbf{X}_{37} = (BE_8)^\wedge_p\,.$$

Any simply connected p-compact group splits as a product of irreducible p-compact groups [26, 54], and, in general, any connected p-compact group is locally isomorphic to the product of finitely many irreducible simply connected p-compact groups and a p-compact torus [48, 2.8].

If $H^*(BX;\mathbb{F}_p)$ is a polynomial $\mathbb{F}_p$-algebra, we say that BX is a *polynomial* p-compact group. Observe that all the irreducible p-compact groups are either polynomial or of the form $BG_p^\wedge$ where G is an irreducible compact connected Lie group [52, 7.4].

The polynomial irreducible p-compact groups, which include all irreducible p-compact groups that are exotic, can be constructed as homotopy colimits of diagrams whose nodes are the p-subgroups of the Weyl group [52, 7.8]. We mention these special cases for later reference:

- Clark–Ewing p-compact groups: The p-compact groups corresponding to the reflections groups (W,L) where the order of W is prime to p [18]. They have the form
$$BX = (B(\check{T} \rtimes W))_p^\wedge$$
where $\check{T} \rtimes W$ is the semi-direct product for the action of the Weyl group on the discrete maximal torus $\check{T} = (L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)/L \cong (\mathbb{Z}/p^\infty)^r$ where r is the rank. The Sullivan spheres (family 3)
$$B\hat{S}_p^{2m-1} \equiv B(\mathbb{Z}/p^\infty \rtimes C_m)_p^\wedge\,,$$
where $m|(p-1)$, are special cases of this construction. Also family 2b for $p>3$ is included here.
- Aguadé p-compact groups: The four p-compact groups, $\mathbf{X}_{12}$ at $p=3$, $\mathbf{X}_{29}$ at $p=5$, $\mathbf{X}_{31}$ at $p=5$, and $\mathbf{X}_{34}$ at $p=7$ constructed by Aguadé [1] in a uniform way as homotopy colimits of diagrams
$$\begin{tikzcd} BSU(r+1) \arrow[loop left, "Z(W)^{\mathrm{op}}"] & & BT \arrow[ll, "\Sigma_{r+1}^{\mathrm{op}}\backslash W^{\mathrm{op}}"'] \arrow[loop right, "W^{\mathrm{op}}"] \end{tikzcd}$$
with two nodes where $r=2,4,4,6$, respectively, is the rank and $Z(W)$, cyclic of order $2,4,4,6$, respectively, is the center of the Weyl group W. In all four cases p divides the order of the Weyl group exactly once. The two cases $\mathbf{X}_{12}$, $\mathbf{X}_{31}$ had been constructed by Zabrodsky using different methods [69].
- Generalized Grassmannians: The p-compact groups $\mathbf{X}(m,r,n)$ corresponding to the reflection groups $G(m,r,n)$ where $r|m|(p-1)$. The cases $r=1$ where constructed by Quillen as p-completed classifying spaces of general linear groups over suitable infinite fields for characteristic prime to p. The cases with $r>1$ where later obtained by Oliver, see Notbohm [57]. See also [52, 7.10].

Theorem 2.2 describes $\mathrm{Out}(X)$, the group of invertible elements of the monoid $[BX,BX]$ of unpointed homotopy classes of self-maps of BX, in purely algebraic terms as the 'Weyl group of the Weyl group', $N_{GL(L)}(W)/W$. In particular, we may regard the automorphism ψ^u of (W,L) as the homotopy class of a self-homotopy equivalence of BX. The map $\psi^u\colon BX \to BX$ is called an *unstable Adams operation* of exponent $u \in \mathbb{Z}_p^\times$.

Classically, unstable Adams operations were first defined by Sullivan [65] on $BU(n)$, for $q \in \mathbb{Z}$, $(p,q)=1$, $q>n$, as restrictions of Adams operations defined on BU. Then extended by Wilkerson to all compact Lie groups [66]. In [37] it is shown that p-completed classifying spaces of compact connected Lie groups admit unstable Adams operations ψ^q of exponent a p-adic unit $q \in \mathbb{Z}_p^*$. This is extended to p-compact groups for odd primes p in [52].

3. p-local finite groups

The concept of *p-local finite group* has been introduced in [13] (see also [14]). A p-local finite group is a triple $(S, \mathcal{F}, \mathcal{L})$ where S is a finite p-group, $\mathcal{F}$ a saturated fusion system over S, and $\mathcal{L}$ a centric linking system associated to $\mathcal{F}$. We will state here again all necessary definitions for the convenience of the reader.

A *fusion system* over a finite group S consists of a set $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ of monomorphisms for every pair of subgroups P, Q of S, such that it contains at least those monomorphisms induced by conjugation by elements of S and all together form a category where every morphism factors as an isomorphism followed by an inclusion. A fusion system is *saturated* if it satisfies certain additional axioms formulated by L. Puig (see [13, §1] or the original source [58]). Two subgroups P, P' of S are called *$\mathcal{F}$-conjugate* if there is an isomorphism between them in $\mathcal{F}$.

Definition 3.1. Let $\mathcal{F}$ be a fusion system over a p-group S.

(1) A subgroup $P \leq S$ is *fully centralized in $\mathcal{F}$* if $|C_S(P)| \geq |C_S(P')|$ for all $P' \leq S$ which is $\mathcal{F}$-conjugate to P.
(2) A subgroup $P \leq S$ is *fully normalized in $\mathcal{F}$* if $|N_S(P)| \geq |N_S(P')|$ for all $P' \leq S$ which is $\mathcal{F}$-conjugate to P.
(3) $\mathcal{F}$ is a *saturated fusion system* if the following two conditions hold:
   (i) For each $P \leq S$ which is fully normalized in $\mathcal{F}$, P is fully centralized in $\mathcal{F}$ and $\mathrm{Aut}_S(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$.
   (ii) If $P \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ are such that φP is fully centralized, and if we set
   $$N_\varphi = \{g \in N_S(P) \mid \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(\varphi P)\},$$
   then there is $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\bar{\varphi}|_P = \varphi$.

A subgroup P of S is *$\mathcal{F}$-centric* if $C_S(P') \leq P'$ for every subgroup $P' \leq S$ which is $\mathcal{F}$-conjugate to P. $\mathcal{F}^c$ denotes the full subcategory whose objects are the *$\mathcal{F}$-centric* subgroups of S.

A subgroup $P \leq S$ is *$\mathcal{F}$-radical* if $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$ is p-reduced, namely, it does not contain non-trivial normal p-subgroups.

Definition 3.2. Let $\mathcal{F}$ be a fusion system over the p-group S. A *centric linking system associated to* $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of S, together with a functor
$$\pi\colon \mathcal{L} \longrightarrow \mathcal{F}^c,$$
and *distinguished* monomorphisms $\delta_P\colon P \longrightarrow \mathrm{Aut}_{\mathcal{L}}(P)$ for each $\mathcal{F}$-centric subgroup $P \leq S$, which satisfy the following conditions.

(A) π is the identity on objects and surjective on morphisms. More precisely, for each pair of objects $P, Q \in \mathcal{L}$, $Z(P)$ acts freely on $\mathrm{Mor}_{\mathcal{L}}(P, Q)$ by composition (upon identifying $Z(P)$ with $\delta_P(Z(P)) \leq \mathrm{Aut}_{\mathcal{L}}(P)$), and π induces a bijection
$$\mathrm{Mor}_{\mathcal{L}}(P, Q)/Z(P) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{F}}(P, Q).$$
(B) For each $\mathcal{F}$-centric subgroup $P \leq S$ and each $g \in P$, π sends $\delta_P(g) \in \mathrm{Aut}_{\mathcal{L}}(P)$ to $c_g \in \mathrm{Aut}_{\mathcal{F}}(P)$.

(C) For each $f \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ and each $g \in P$, the following square commutes in $\mathcal{L}$:

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\delta_P(g)"'] & Q \arrow[d, "\delta_Q(\pi(f)(g))"] \\ P \arrow[r, "f"] & Q\,. \end{tikzcd}$$

The *classifying space* of the p-local finite group $(S,\mathcal{F},\mathcal{L})$ is defined as the p-completion $|\mathcal{L}|^\wedge_p$ of the nerve of the category $\mathcal{L}$. The classifying space determines the p-local finite group in the sense that two p-local finite groups are isomorphic if and only if they have homotopy equivalent classifying spaces. Actually, the complete structure of a p-local finite group can be recovered from its classifying space by homotopy theoretic methods.

Finite groups are the main source of examples and motivation for p-local finite group theory.

Example 3.3 (The p-local finite group $(S,\mathcal{F}_S(G),\mathcal{L}^c_S(G))$ of a finite group G)**.** If G is a finite group and S a Sylow p-subgroup, the monomorphisms from $P \leq S$ to $Q \leq S$ induced by conjugation in G, $\mathrm{Hom}_G(P,Q) \cong N_G(P,Q)/C_G(P)$, where $N_G(P,Q) = \left\{ x \in G \,\middle|\, xPx^{-1} \leq Q \right\}$, form a saturated fusion system over S, $\mathcal{F}_S(G)$. The $\mathcal{F}_S(G)$-centric subgroups of S are the subgroups $P \leq S$ which are p-centric in G. A p-subgroup $P \leq G$ is p-centric if its center, $Z(P)$, is the Sylow p-subgroup of $C_G(P)$, or, equivalently, if the centralizer splits as the product of the center of P and a group $C'_G(P)$ of order prime to p, $C_G(P) = Z(P) \times C'_G(P)$.

Now, we define $\mathcal{L}^c_S(G)$ as the category with objects all subgroups of S which are p-centric in G, and morphisms $\mathrm{Mor}_{\mathcal{L}}(P,Q) \cong N_G(P,Q)/C'_G(P)$, where $C'_G(P)$ is the p'-complement in $C_G(P)$ of the center of P, which is well defined because P is p-centric. $\mathcal{L}^c_S(G)$ is a centric linking system associated to $\mathcal{F}_S(G)$, and $(S,\mathcal{F}_S(G),\mathcal{L}^c_S(G))$ is a p-local finite group with classifying space $|\mathcal{L}^c_S(G)|^\wedge_p \simeq BG^\wedge_p$ (see [12, 13]).

A p-subgroup P of G is called p-radical if it is the maximal normal p-subgroup of $N_G(P)$, $P = O_p(N_G(P))$, or, equivalently, if $N_G(P)/P$ is p-reduced [34], whereas being $\mathcal{F}_S(G)$-radical means that $\mathrm{Out}_{\mathcal{F}_S(G)}(P) \cong N_G(P)/PC_G(P) = \mathrm{Out}_G(P)$ is p-reduced. However, if $P \leq S$ is $\mathcal{F}_S(G)$-centric and $\mathcal{F}_S(G)$-radical, then it is p-centric and p-radical in G: Assume that P is not p-radical in G, then there is another p-subgroup Q with $P \triangleleft Q \triangleleft N_G(P)$ and $Q \neq P$. Since P is p-centric, $C_G(P) = Z(P) \times C'_G(P)$, where $C'_G(P)$ is a p'-group, hence also $C'_G(P) \cap Q = 1$, so, therefore $P \triangleleft Q \triangleleft N_G(P)/C'_G(P)$ and $Q/P \triangleleft N_G(P)/PC'_G(P) = N_G(P)/PC_G(P)$, hence $\mathrm{Out}_G(P)$ is not p-reduced. The converse it is not always true.

Alperin's fusion theorem for saturated fusion systems [13, A.10] establishes that morphisms in a saturated fusion system $\mathcal{F}$ are composites of automorphisms of fully normalized, $\mathcal{F}$-*centric*, and $\mathcal{F}$-*radical* subgroups of the system, or restrictions of those. Hence in order to describe a saturated fusion system $\mathcal{F}$ over a finite p-group S it is enough to describe $\mathrm{Aut}_{\mathcal{F}}(Q_i)$ for a set $Q_1,\dots,Q_r$ of fully normalized representatives of $\mathcal{F}$-conjugacy classes of $\mathcal{F}$-*centric*, $\mathcal{F}$-*radical* subgroups of S in $\mathcal{F}$. This motivates the next construction.

If $\mathcal{F}_0$ is a fusion system over S, and $Q_1,\dots,Q_r$ are subgroups of S, and Δ_i a group of automorphisms such that $\mathrm{Inn}(Q_i) \leq \Delta_i \leq \mathrm{Aut}(Q_i)$, for each i, then we denote by $\mathcal{F}_{Q_i}(\Delta_i)$ the fusion system over Q_i whose morphisms are restrictions of elements of Δ_i, and define

$$\mathcal{F} = \langle \mathcal{F}_0; \mathcal{F}_{Q_1}(\Delta_1),\dots,\mathcal{F}_{Q_r}(\Delta_r)\rangle$$

the fusion system over S whose morphisms are composites of morphisms belonging to any of the generating fusion systems (cf. [13, §9]).

R	$N_{GL_p(q)}(R)$	$\mathrm{Out}_{GL_p(q)}(R)$
Z_ℓ	$GL_p(q)$	1
T_ℓ^p	$(\mathbb{F}_q^*)^p \rtimes \Sigma_p$	Σ_p
$\bar{S}$	$(\mathbb{F}_q^*)^p \rtimes (\mathbb{Z}/p \rtimes \mathbb{Z}/p-1)$	$\mathbb{Z}/p-1$
Γ_ℓ	$(\mathbb{F}_q^*) \cdot \Gamma_\ell \cdot SL_2(p)$	$SL_2(p)$
$U_{\ell+1}$	$\mathbb{F}_{q^p}^* \rtimes \mathbb{Z}/p$	$\mathbb{Z}/p$

TABLE 1. p-radical subgroups of $GL_p(q)$ for $q \equiv 1 \bmod p$

Thus, in particular, if $\mathcal{F}$ is a saturated fusion system over a finite p-group S and $Q_1, \ldots, Q_r$ is a set of fully normalized representatives of $\mathcal{F}$-conjugacy classes of $\mathcal{F}$-*centric*, $\mathcal{F}$-*radical* subgroups of S in $\mathcal{F}$, then

$$\mathcal{F} = \langle \mathcal{F}_S(\mathrm{Aut}_{\mathcal{F}}(S)); \mathcal{F}_{Q_1}(\mathrm{Aut}_{\mathcal{F}}(Q_1)), \ldots, \mathcal{F}_{Q_r}(\mathrm{Aut}_{\mathcal{F}}(Q_r)) \rangle .$$

We now describe the fusion systems of $GL_p(q)$ and $SL_p(q)$ over the respective Sylow p-subgroups, where p is a prime number and q is a prime power $q \equiv 1 \bmod p$. This will be useful in later sections.

Example 3.4 (The fusion system of $GL_p(q)$)**.** We will describe the fusion system of $GL_p(q)$ over a Sylow p-subgroup, for p a prime and q a prime power such that $q \equiv 1 \bmod p$. We can use the Alperin–Fong description of p-radical subgroups of general linear groups [4]. The elements

$$B = \mathrm{diag}(1, \zeta, \zeta^2, \ldots, \zeta^{p-1}), \qquad C = \begin{pmatrix} 0 & 0 & \ldots & & 1 \\ 1 & 0 & & & 0 \\ 0 & 1 & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}$$

where ζ be a primitive pth root of unity in $\mathbb{F}_q^*$, generate an extraspecial subgroup $\Gamma_1 = \langle B, C\rangle \leq GL_p(q)$ of order p^3 and exponent p.

The p-primary part of the multiplicative group of units $\mathbb{F}_q^*$ is isomorphic to $\mathbb{Z}/p^\ell$ where $\ell = \nu_p(1-q)$. Let $T_\ell^p \cong (\mathbb{Z}/p^\ell)^p$, the maximal finite torus, be the group of diagonal matrices of p-power order. Then $\bar{S} = T_\ell^p \rtimes \langle C\rangle \cong \mathbb{Z}/p^\ell \wr \mathbb{Z}/p$ is a Sylow p-subgroup of $GL_p(q)$.

Define the subgroup $\Gamma_\ell = Z_\ell \circ \Gamma_1 \leq GL_p(q)$ to be the central product over the center of Γ_1 of the center $Z_\ell \cong \mathbb{Z}/p^\ell$ of $GL_p(q)$ and Γ_1.

There is an standard inclusion $\mathbb{F}_{q^p}^* \subseteq GL_p(q)$, obtained by letting $\mathbb{F}_{q^p}^*$ act on $\mathbb{F}_{q^p}$ by multiplication and considering $\mathbb{F}_{q^p}$ as $\mathbb{F}_q$-vector space. We define $U_{\ell+1}$ as the image in $GL_p(q)$ of the cyclic group $\mathbb{Z}/p^{\ell+1} \leq \mathbb{F}_{q^p}^*$ of all roots of unity of p-power order in $\mathbb{F}_{q^p}$.

With this notation and according to [4], if R is a p-radical subgroup of $GL_p(q)$ then R is conjugate to one of the subgroups displayed in Table 1.

It is now easy to extract from Table 1 the $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups of $\bar{S}$ in the fusion system $\mathcal{F} = \mathcal{F}_{\bar{S}}(GL_p(q))$ of $GL_p(q)$ over $\bar{S}$. Notice that Z_ℓ is clearly not $\mathcal{F}$-centric and $U_{\ell+1}$ clearly not $\mathcal{F}$-radical. This leads to Table 2.

Example 3.5 (The fusion system of $SL_p(q)$)**.** We proceed now by describing the fusion system of $SL_p(q)$ over a Sylow p-subgroup, for p a prime and q a prime power such that $q \equiv 1 \bmod p$. Let $\ell = \nu_p(1-q)$ as in the previous example.

R	$\mathrm{Out}_{GL_p(q)}(R)$
T_ℓ^p	Σ_p
$\bar{S}$	$\mathbb{Z}/p-1$
Γ_ℓ	$SL_2(p)$

TABLE 2. $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups in the fusion system of $GL_p(q)$

P	$\mathrm{Out}_{SL_p(q)}(P)$	Conditions
$T_\ell^{(p-1)}$	Σ_p	$p > 3$
S	$\mathbb{Z}/p-1$	
$\Gamma_1(\xi^r)$	$SL_2(p)$	$r = 0$ if $\ell = 1$, $p = 3$; $r = 0, 1, \dots, p-1$ if $\ell > 1$ or $p > 3$,

TABLE 3. $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups in the fusion system of $SL_p(q)$

We first show that every p-radical subgroup of $SL_p(q)$ is the intersection $Q \cap SL_p(q)$ of a p-radical subgroup Q of $GL_p(q)$ with $SL_p(q)$. For a given p-radical p-subgroup P of $SL_p(q)$ define $Q = O_p(N_{GL_p(q)}(P))$. $Q \cap SL_p(q)$ is a normal subgroup of $N_{SL_p(q)}(P)$ and since P is the maximal normal p-subgroup of $N_{SL_p(q)}(P)$, we have $Q \cap SL_p(q) \leq P$. Same argument with $N_{GL_p(q)}(P)$ shows that $P \leq Q$ and therefore $Q \cap SL_p(q) \leq P$.

Every element $g \in GL_p(q)$ normalizes $SL_p(q)$, so if g normalizes Q it also normalizes $Q \cap SL_p(q) \leq P$, so $N_{GL_p(q)}(Q) \leq N_{GL_p(q)}(P)$. But, by definition of Q, this is normal in $N_{GL_p(q)}(P)$, hence we actually have $N_{GL_p(q)}(Q) = N_{GL_p(q)}(P)$. So, therefore, $Q = O_p(N_{GL_p(q)}(Q))$ is p-radical.

Fix the Sylow p-subgroup $S = \bar{S} \cap SL_p(q)$ of $SL_p(q)$, and let $\mathcal{F} = \mathcal{F}_S(SL_p(q))$ be the fusion system of $SL_p(q)$ over S. Assume that $P \leq S$ is $\mathcal{F}$-centric and $\mathcal{F}$-radical. Then P is p-centric and p-radical in $SL_p(q)$. In particular $P = Q \cap SL_p(q)$ where Q is p-radical in $GL_p(q)$, hence conjugate by an element $g \in GL_p(q)$ to a p-subgroup in the Table 1. Among those intersections, only $S = \bar{S} \cap SL_p(q)$, $T_\ell^{(p-1)} = S \cap T_\ell^p$, and $\Gamma_1 = S \cap \Gamma_\ell$ are also p-centric. Hence, the complete list of conjugacy classes of p-centric and p-radical subgroups of $SL_p(q)$, is obtained by conjugating these three subgroups by elements $g \in GL_p(q)$: where $\Gamma_1(\xi^r)$, $r = 0, 1, \dots, (p-1)$ are subgroups of $SL_p(q)$, defined as the conjugates of Γ_1 in $GL_p(q)$, $\Gamma_1(\xi^r) = x_r \Gamma_1 x_r^{-1}$, where $x_r = \mathrm{diag}(\xi^r, 1, \dots, 1) \in GL_p(q)$, ξ a $(q-1)$st root of unity. Notice that for $g \in GL_p(q)$, gSg^{-1} lies in S if and only if it is exactly S and the same happens with $T_\ell^{(p-1)}$. In the case of Γ_1 we just need to check which of the subgroups $\Gamma_1(\xi^r)$ are conjugate in $SL_p(q)$. In fact, Alperin's fusion theorem [13, A.10], together with the list of p-radical p-centric subgroups that we have obtained so far, tells us that if two subgroups $\Gamma_1(\xi^r)$ and $\Gamma_1(\xi^s)$ are conjugate in $SL_p(q)$ they are already conjugate in $N_{SL_p(q)}(S)$, hence we obtain the Table 3 by direct calculation as a list of p-centric and p-radical subgroups but, by inspection, this coincides with the list of $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups.

An p-local finite group that is not of the form $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ for any finite group G is called *exotic*. Examples of exotic p-local finite groups are already shown in [13]. Recently, Levi and Oliver have obtained a family of exotic 2-local finite groups, $B\,Sol(q)$ [39], based on fusion systems originally described by Solomon [63].

Definition 3.6. (a) For any saturated fusion system $\mathcal{F}$ over a p-group S, and any $P \leq S$, fully centralized in $\mathcal{F}$, the *centralizer* fusion system $C_{\mathcal{F}}(P)$ over $C_S(P)$ is defined by setting

$$\mathrm{Hom}_{C_{\mathcal{F}}(P)}(Q,Q') = \big\{(\varphi|_Q) \,\big|\, \varphi \in \mathrm{Hom}_{\mathcal{F}}(PQ,PQ'),\ \varphi(Q) \leq Q',\ \varphi|_P = Id_P\big\}$$

for all $Q, Q' \leq C_S(P)$.

(b) For a p-local finite group $(S,\mathcal{F},\mathcal{L})$ and $P \leq S$ fully centralized in $\mathcal{F}$, we define the category $C_{\mathcal{L}}(P)$ whose objects are $C_{\mathcal{F}}(P)$-centric subgroups $Q \leq C_S(P)$ and where

$$\mathrm{Mor}_{C_{\mathcal{L}}(P)}(Q,Q') = \left\{\, \varphi \in \mathrm{Hom}_{\mathcal{L}}(PQ,PQ') \;\middle|\; \pi(\varphi)|_P = Id_P,\ \pi(\varphi)(Q) \leq Q' \,\right\}.$$

It is proved in [13, §2] that if $(S,\mathcal{F},\mathcal{L})$ is a p-local finite group and $P \leq S$ is fully centralized in $\mathcal{F}$, then $(C_S(P), C_{\mathcal{F}}(P), C_{\mathcal{L}}(P))$ is a p-local finite group.

In [39] Levi and Oliver have obtained necessary and sufficient conditions for a fusion system to be saturated. We reproduce here their result for the convenience of the reader. We will write $C_{\mathcal{F}}(x) = C_{\mathcal{F}}(\langle x\rangle)$ for $x \in S$.

Proposition 3.7. [39] *Let $\mathcal{F}$ be any fusion system over a p-group S. Then $\mathcal{F}$ is saturated if and only if there is a set $\mathfrak{X}$ of elements of order p in S such that the following conditions hold:*

(a) *Each $x \in S$ of order p is $\mathcal{F}$-conjugate to some element of $\mathfrak{X}$.*
(b) *If x and y are $\mathcal{F}$-conjugate and $y \in \mathfrak{X}$, then there is some $\psi \in \mathrm{Hom}_{\mathcal{F}}(C_S(x), C_S(y))$ such that $\psi(x) = y$.*
(c) *For each $x \in \mathfrak{X}$, $C_{\mathcal{F}}(x)$ is a saturated fusion system over $C_S(x)$.*

4. Recognition of classifying spaces of p-local finite groups

In [13] it is shown that a p-local finite group can be completely recovered from its classifying space by homotopy theoretic methods. Also, a recognition principle for classifying spaces of p-local finite groups is provided in [13, Thm. 7.5]. We will briefly describe these methods and derive an inductive method that will be useful in our situation.

We will first recall how a fusion system $\mathcal{F}_{(S,f)}(X)$ and a linking system $\mathcal{L}_{(S,f)}(X)$ are attached to a space X equipped with a map $f\colon BS \to X$, where S is a finite p-group.

If (S,f) is a p-subgroup of a space X we can define a fusion system over S, $\mathcal{F}_{(S,f)}(X)$, by declaring

$$\mathrm{Hom}_{\mathcal{F}_{(S,f)}(X)}(P,Q) = \big\{\, \varphi \in \mathrm{Hom}(P,Q) \,\big|\, f|_{BP} \simeq f|_{BQ} \circ B\varphi \,\big\}$$

for all $P, Q \leq S$, where $f|_{BP}$ denotes the composition $BP \xrightarrow{Bi_P} BS \xrightarrow{f} X$. Next, we define the category $\mathcal{L}_{(S,f)}(X)$ that has objects the subgroups of S and

$$\begin{aligned}\mathrm{Mor}_{\mathcal{L}_{(S,f)}(X)}(P,Q) = \big\{\, (\varphi,[H]) \,\big|\ &\varphi \in \mathrm{Hom}(P,Q) \text{ and}\\ &[H] \text{ is the homotopy class of a homotopy from } f|_{BP} \text{ to } f|_{BQ} \circ B\varphi \,\big\},\end{aligned}$$

and the full subcategory $\mathcal{L}^c_{(S,f)}(X)$ whose objects are $\mathcal{F}_{(S,f)}(X)$-centric subgroups $P \leq S$.

The important question and the aim of the rest of this section is to find sufficient conditions on a space X and a p-subgroup (S,f) under which

$$(S, \mathcal{F}_{(S,f)}(X), \mathcal{L}^c_{(S,f)}(X))$$

is a p-local finite group and X is its classifying space $|\mathcal{L}^c_{(S,f)}(X)|^\wedge_p \simeq X$.

One first important case is that of $X = |\mathcal{L}|^\wedge_p$, the classifying space itself of a given p-local finite group $(S, \mathcal{F}, \mathcal{L})$. The distinguished homomorphism $\delta_S\colon S \to \mathrm{Aut}_{\mathcal{L}}(S)$ provides a functor $\mathcal{B}S \to \mathcal{L}$, where $\mathcal{B}S$ denotes the category that has one object and its group of automorphisms is S. In turn, this functor induces a map between the respective nerves $|\mathcal{B}S| \to |\mathcal{L}|$. Finally, composing with the p-completion of $|\mathcal{L}|$ we obtain a canonical map for $(S, \mathcal{F}, \mathcal{L})$:

$$\theta_S\colon |\mathcal{B}S| \to |\mathcal{L}|^\wedge_p\,,$$

where we can identify $|\mathcal{B}S| \simeq BS$. It turns out that $(S, \mathcal{F}_{(S,\theta_S)}(|\mathcal{L}|^\wedge_p), \mathcal{L}^c_{(S,\theta_S)}(|\mathcal{L}|^\wedge_p))$ is isomorphic to the original $(S, \mathcal{F}, \mathcal{L})$ [13, 7.3]. This is how a p-local finite group is completely recovered from its classifying space.

The basic tool in order to show that these systems define a p-local finite group with classifying space X is [13, Thm. 7.5]. In order to apply this theorem in our situation we face two main difficulties, namely, to show that the p-completed nerve of $\mathcal{L}_{(S,f)}(X)$ is homotopy equivalent to X and to show that $\mathcal{F}_{(S,f)}(X)$ is a saturated fusion system. In order to overcome these difficulties, we develop in this section an inductive method mainly based on the centralizer decomposition of p-local finite groups.

Definition 4.1. Given spaces X and Y, we say that a map $\alpha\colon X \to Y$ is a homotopy monomorphism at p if the homotopy fibre of α, F, over any connected component of Y, is p-quasi-finite; that is, the inclusion $F \to \mathrm{Map}(B\mathbb{Z}/p, F)$ as constant maps is a weak homotopy equivalence.

Given two maps $f\colon X \to Y$ and $g\colon Y \to Z$, where g is a homotopy monomorphism at p, it is not hard to prove f is also a homotopy monomorphism at p if and only if the composition $g \circ f$ is so.

Definition 4.2. Let X be a space. A finite p-subgroup of X is a pair (P, f), where P is a finite p-group and $f\colon BP \to X$ is a homotopy monomorphism at p. A p-subgroup (S, f) of X is called a Sylow p-subgroup of X if for any other p-subgroup (Q, g) of X, $g\colon BQ \to X$ factors through $f\colon BS \to X$, up to homotopy. If (P, f) is a p-subgroup of X, then we denote $BC_X(P, f) = \mathrm{Map}(BP, X)_f$.

Our basic example comes from p-local finite groups. If $(S, \mathcal{F}, \mathcal{L})$ is a p-local finite group, then (S, θ_S) is a Sylow p-subgroup of $|\mathcal{L}|^\wedge_p$. The map $\theta_S\colon |\mathcal{B}S| \to |\mathcal{L}|^\wedge_p$ satisfies the required conditions by [13, Thm. 4.4].

We will need later the next technical lemma.

Lemma 4.3. *Assume that X and Y are spaces for which $\mathrm{Map}(B\mathbb{Z}/p, X)_{ct} \simeq X$ and $\mathrm{Map}(B\mathbb{Z}/p, Y)_{ct} \simeq Y$. Let $f\colon X \to Y$ be a homotopy monomorphism at p and $\mu\colon BP \to X$ a finite p-subgroup of X, then each map in the diagram*

$$\begin{tikzcd} BC_X(P,\mu) \arrow[r, "ev"] \arrow[d, "f_\sharp"'] & X \arrow[d, "f"] \\ BC_Y(P, f\circ\mu) \arrow[r, "ev"'] & Y \end{tikzcd} \tag{4}$$

is a homotopy monomorphism at p.

Proof. Let F be the homotopy fibre of the evaluation map

$$BC_X(P,\mu) = \mathrm{Map}(BP,X)_\mu \xrightarrow{ev} X\,.$$

There is an induced fibration

$$\mathrm{Map}(B\mathbb{Z}/p,F) \to \mathrm{Map}(B\mathbb{Z}/p,\mathrm{Map}(BP,X)_\mu)_{\tilde{ct}} \to \mathrm{Map}(B\mathbb{Z}/p,X)_{ct}$$

where $\tilde{ct}$ stands for all components mapping down to the component of the constant map in $\mathrm{Map}(B\mathbb{Z}/p,X)$. Since $\mathrm{Map}(B\mathbb{Z}/p,X)_{ct} \simeq X$, also

$$\mathrm{Map}(B\mathbb{Z}/p,\mathrm{Map}(BP,X)_\mu)_{\tilde{ct}} \simeq \mathrm{Map}(BP,\mathrm{Map}(B\mathbb{Z}/p,X)_{ct})_{\tilde{\mu}} \simeq \mathrm{Map}(BP,X)_\mu\,,$$

and therefore $\mathrm{Map}(B\mathbb{Z}/p,F) \simeq F$; that is, F is p-quasi finite and $ev\colon C_X(P,\mu) \to X$ is a homotopy monomorphism at p. Similarly, $ev\colon BC_Y(P,f\circ\mu) \to Y$ is a homotopy monomorphism at p. Finally, since all other maps in diagram (4) are homotopy monomorphisms at p, then, also $f_\sharp$ is a homotopy monomorphism at p. □

The next is a useful result that provides conditions on the space X and a Sylow p-subgroup (S,f) under which the fusion system $\mathcal{F}_{(S,f)}(X)$ is saturated. An element $x \in S$ of order p determines a homomorphism $i_x\colon \mathbb{Z}/p \to S$ an then a map $f\circ Bi_x\colon B\mathbb{Z}/p \to X$. We write $BC_X(x) = \mathrm{Map}(B\mathbb{Z}/p,X)_x$, the connected component that contains the map $f\circ Bi_x$, and $f_x\colon BC_S(x) \to BC_X(x)$ the map induced by f.

Proposition 4.4. *Let X be a space, (S,f) a Sylow p-subgroup of X, and $\mathfrak{X}$ a set of elements of order p in S. Assume that:*

(1) $\mathrm{Map}(B\mathbb{Z}/p,X)_{ct} \simeq X$.
(2) *For all $x\in\mathfrak{X}$, the natural map $f_x\colon BC_S(x) \to BC_X(x)$ is a Sylow p-subgroup for $BC_X(x)$.*
(3) *For all $x\in\mathfrak{X}$, $\mathcal{F}_{(C_S(x),f_x)}(BC_X(x))$ is a saturated fusion system over $C_S(x)$.*
(4) *For all $x \in S$ of order p, there is $\varphi \in \mathrm{Hom}_{\mathcal{F}_{(S,f)}(X)}(\langle x\rangle,S)$ such that $\varphi(x)\in\mathfrak{X}$.*

Then $\mathcal{F}_{(S,f)}(X)$ is a saturated fusion system over S and $C_{\mathcal{F}_{(S,f)}(X)}(x)$ coincides with $\mathcal{F}_{(C_S(x),f_x)}(BC_X(x))$ as fusion systems over $C_S(x)$, for all $x\in\mathfrak{X}$.

Proof. Write $\mathcal{F} = \mathcal{F}_{(S,f)}(X)$ for short. Clearly, $\mathcal{F}$ is a fusion system over S. Condition (a) of Proposition 3.7 holds by (4); and it remains to show that conditions (b) and (c) of 3.7 hold.

Condition (b) of 3.7: Fix $x,y\in S$ of order p such that $y\in\mathfrak{X}$, and such that there is $\psi_0\in\mathrm{Hom}_{\mathcal{F}}(\langle x\rangle,\langle y\rangle)$ with $\psi_0(x)=y$. We must show that ψ_0 extends to some $\psi\in\mathrm{Hom}_{\mathcal{F}}(C_S(x),C_S(y))$.

Since x and y are $\mathcal{F}$-conjugate,

$$[f\circ Bi_x] = [f\circ Bi_y] \in [B\mathbb{Z}/p,X],$$

so $\mathrm{Map}(B\mathbb{Z}/p,X)_x = \mathrm{Map}(B\mathbb{Z}/p,X)_y$. Since $C_S(y)$ is a Sylow p-subgroup of $\mathrm{Map}(B\mathbb{Z}/p,X)_y$ by (2), the natural map $BC_S(x) \to \mathrm{Map}(B\mathbb{Z}/p,X)_x$ factors through $BC_S(y)$. In other words, there is some $\psi\in\mathrm{Hom}(C_S(x),C_S(y))$ such that the following square commutes up to homotopy

$$\begin{tikzcd} BC_S(x)\times B\mathbb{Z}/p \arrow[r, "f\circ B(incl\times i_x)"] \arrow[d, "B\psi\times Id"'] & X \arrow[d, equal] \\ BC_S(y)\times B\mathbb{Z}/p \arrow[r, "f\circ B(incl\times i_y)"] & X\,. \end{tikzcd} \tag{5}$$

Thus $\psi \in \mathrm{Hom}_{\mathcal{F}}(C_S(x), C_S(y))$. If $\rho, \rho' \in \mathrm{Hom}(C_S(x) \times \mathbb{Z}/p, S)$ denote the homomorphisms $\rho(g,t) = gx^t$ and $\rho'(g,t) = \psi(g)y^t$, then $f \circ B\rho \simeq f \circ B\rho'$ by (5), and hence $\mathrm{Ker}(\rho) = \mathrm{Ker}(\rho')$ by [13, Proposition 5.4(d)] (and point (1)). And this implies that $\psi(x) = y$.

Condition (c) of 3.7: Fix some $x \in \mathfrak{X}$; we must show that $C_{\mathcal{F}}(x)$ is a saturated fusion system. By (3), the fusion system $\mathcal{F}' \overset{\text{def}}{=} \mathcal{F}_{(C_S(x), f_x)}(BC_X(x))$ is saturated, so it suffices to show that these two fusion systems over $C_S(x)$ are equal.

To see this, fix $P, Q \leq C_S(x)$, and let $\varphi \in \mathrm{Hom}(P, Q)$ be any monomorphism. Set $\overline{P} = P\cdot\langle x\rangle$ and $\overline{Q} = Q\cdot\langle x\rangle$. Let $\rho \in \mathrm{Hom}(P \times \mathbb{Z}/p, S)$ and $\rho' \in \mathrm{Hom}(Q \times \mathbb{Z}/p, S)$ be defined by $\rho(g,t) = gx^t$ and $\rho'(g,t) = gx^t$. Then $\varphi \in \mathrm{Hom}_{\mathcal{F}'}(P, Q)$ if and only if the following square commutes up to homotopy

$$\begin{tikzcd} BP \times B\mathbb{Z}/p \arrow[r, "f\circ B\rho"] \arrow[d, "B\varphi\times Id"'] & X \arrow[d, equal] \\ BQ \times B\mathbb{Z}/p \arrow[r, "f\circ B\rho'"] & X\,. \end{tikzcd} \tag{6}$$

By (1) and [13, Proposition 5.4(d)], this holds if and only if $K \overset{\text{def}}{=} \mathrm{Ker}(\rho) = \mathrm{Ker}(\rho' \circ (\varphi \times Id))$ and the induced maps from $B((P \times \mathbb{Z}/p)/K)$ to X are homotopic. The kernels are equal if and only if φ extends to a monomorphism $\bar{\varphi}$ from $\overline{P}$ to $\overline{Q}$ which sends x to itself. And in this case, the induced maps on $B((P \times \mathbb{Z}/p)/K)$ are homotopic if and only if $f|_{B\overline{P}} \simeq f|_{B\overline{Q}} \circ B\bar{\varphi}$, if and only if $\varphi \in \mathrm{Hom}_{C_{\mathcal{F}}(x)}(P, Q)$.

Now, Proposition 3.7 implies that $\mathcal{F}_{(S,f)}(X)$ is a saturated fusion system over S and the argument for condition (c) already contains the proof that $C_{\mathcal{F}}(x)$ coincides with $\mathcal{F}' = \mathcal{F}_{(C_S(x), f_x)}(BC_X(x))$ as fusion systems over $C_S(x)$. □

We derive now another characterization that will be useful in the specific cases in which we are interested or more generally in cases in which there is a good knowledge of elementary abelian p-subgroups of X and of their centralizers.

Theorem 4.5. *Let X be a p-complete space and (S, f) a p-subgroup of X. Assume that*

(1) $\mathrm{Map}(B\mathbb{Z}/p, X)_{ct} \simeq X$*, and*
(2) *for each non-trivial element $x \in S$ of order p*
   (a) *$BC_X(x)$ is the classifying space of a p-local finite group, and*
   (b) *if (H, g) is a Sylow p-subgroup for $BC_X(x)$, there is a group homomorphism $\rho\colon H \to S$ that makes the diagram*

$$\begin{tikzcd} BH \arrow[r, "B\rho"] \arrow[d, "g"'] & BS \arrow[d, "f"] \\ BC_X(x) \arrow[r, "ev"] & X \end{tikzcd}$$

commutative up to homotopy,

then, (S, f) is a Sylow p-subgroup for X and

$$(S, \mathcal{F}_{(S,f)}(X), \mathcal{L}^c_{(S,f)}(X))$$

is a p-local finite group.

Furthermore, $X \simeq |\mathcal{L}_{(S,f)}(X)|^\wedge_p$ if and only if the natural map induced by evaluation

$$\operatorname*{hocolim}_{\mathcal{F}^e_{(S,f)}(X)^{op}} \mathrm{Map}(BE, X)_{f|_{BE}} \longrightarrow X$$

is a mod p homology equivalence. Here $\mathcal{F}^e_{(S,f)}(X)$ denotes the full subcategory of $\mathcal{F}_{(S,f)}(X)$ consisting of non-trivial fully centralized elementary abelian p-subgroups of S.

Proof. The proof is divided in five steps. First, we prove that (S, f) is a Sylow p-subgroup of X. Next, that the fusion system of X over (S, f), $\mathcal{F}_{(S,f)}(X)$ is saturated. In the third step we show that for each $\mathcal{F}_{(S,f)}(X)$-centric subgroup $P \leq S$ the map $f|_{BP}$ is centric. A map $g\colon BP \to X$ is called centric if the induced map $f_\sharp\colon \mathrm{Map}(BP, BP)_{Id} \to \mathrm{Map}(BP, X)_g$ is a weak homotopy equivalence.

These two last steps are the hypothesis (a) and (c) of [13, Theorem 7.5]. According to the remarks after the proof of this theorem in [13], this suffices in order to conclude that $(S, \mathcal{F}_{(S,f)}(X), \mathcal{L}^c_{(S,f)}(X))$ is a p-local finite group. This is the first part of the theorem.

The second part states that $X \simeq |\mathcal{L}_{(S,f)}(X)|^\wedge_p$ if and only if the natural map induced by evaluation $\mathrm{hocolim}_{\mathcal{F}^e_{(S,f)}(X)^{op}} \mathrm{Map}(BE, X)_{f|_{BE}} \longrightarrow X$ is a mod p homology equivalence. This is proved in steps 4 and 5. Notice that $X \simeq |\mathcal{L}_{(S,f)}(X)|^\wedge_p$ is condition (b) in [13, Theorem 7.5].

Step 1: (S, f) is a Sylow p-subgroup for X. Let (P, μ) be a finite p-subgroup of X. Choose a central element x of order p in P. It determines a homomorphism $i_x\colon \mathbb{Z}/p \to P$ for which $C_P(\mathbb{Z}/p) = P$, and a map $\mu \circ Bi_x\colon B\mathbb{Z}/p \to X$. According to our hypothesis, $BC_X(x)$ is the classifying space of a p-local finite group, and if (H, g) is its Sylow p-subgroup, there are homomorphisms $\rho\colon H \to S$ and $\varphi\colon C_P(\mathbb{Z}/p) \to H$ that make the diagram

$$\begin{tikzcd}
BC_P(\mathbb{Z}/p) \arrow[r, "\mu_\sharp"] \arrow[d, "ev", "\simeq"'] \arrow[rr, bend left, "B\varphi"] & BC_X(\mathbb{Z}/p, \mu \circ Bi_x) \arrow[d, "ev"] & BH \arrow[l, "g"'] \arrow[d, "B\rho"] \\
BP \arrow[r, "\mu"] & X & BS \arrow[l, "f"']
\end{tikzcd}$$

commutative up to homotopy. Hence, $\rho \circ \varphi\colon P = C_P(\mathbb{Z}/p) \to S$ provides the factorization of (P, μ) through (S, f).

Step 2: The fusion system of X over (S, f), $\mathcal{F}_{(S,f)}(X)$ is saturated. This part of the proof will be based on Proposition 4.4. Define

$$\mathfrak{X} = \left\{ x \in S \,\middle|\, x \text{ of order } p \text{ and } f_x\colon BC_S(x) \to BC_X(x) \right.$$
$$\left. \text{is a Sylow } p\text{-subgroup for } BC_X(x) \right\}.$$

Notice now that conditions (1) and (2) of Proposition 4.4 are satisfied by our hypothesis and by definition of the class $\mathfrak{X}$. Condition (3) is easily verified, too. In fact, by hypothesis, for each $x \in \mathfrak{X}$, $BC_X(x)$ is the classifying space of a p-local finite group and since $f_x\colon BC_S(x) \to BC_X(x)$ is a Sylow p-subgroup for $BC_X(x)$, the fusion system $\mathcal{F}_{(C_S(x), f_x)}(BC_X(x))$ is saturated.

It remains to verify condition (4); that is, that every element $x \in S$ of order p is $\mathcal{F}_{(S,f)}(X)$-conjugate to an element of the class $\mathfrak{X}$.

Assume that $x \in S$ has order p. It gives a homomorphism $i_x\colon \mathbb{Z}/p \to S$ and a map $f \circ Bi_x\colon B\mathbb{Z}/p \to X$. There is an evaluation map $ev\colon B\mathbb{Z}/p \times BC_X(x) \to X$. Let (H, g) be a Sylow p-subgroup of $BC_X(x)$. Since (S, f) is a Sylow p-subgroup of X, there is a homomorphism $\rho\colon \mathbb{Z}/p \times H \to S$ making the diagram

$$\begin{tikzcd} B\mathbb{Z}/p \times BH \arrow[r, "B\rho"] \arrow[d, "1\times g"'] & BS \arrow[d, "f"] \\ B\mathbb{Z}/p \times BC_X(x) \arrow[r, "ev"] & X \end{tikzcd}$$

commutative up to homotopy.

Let $\varphi = \rho|_{\mathbb{Z}/p}$ the restriction of ρ to the first component $\mathbb{Z}/p$. From the above diagram we deduce that $\varphi \in \mathrm{Hom}_{\mathcal{F}_{(S,f)}(X)}(\mathbb{Z}/p, S)$. Let $y = \varphi(x)$.

Then, ρ induces

$$BH \xrightarrow{B\tilde{\rho}} BC_S(y) \xrightarrow{f_y} BC_X(y) \xrightarrow{ev} X$$

where all maps are homotopy monomorphisms at p. The first one because $\tilde{\rho}$ is a monomorphism, the others by Lemma 4.3.

Now, φ induces a homotopy equivalence $BC_X(y) \simeq BC_X(x)$, hence also an isomorphism between the respective Sylow p-subgroups. Since (H, g) is a Sylow p-subgroup for $C_X(x)$, it follows from the above sequence of maps that $(C_S(y), f_y)$ is a Sylow p-subgroup for $C_X(y)$. Hence $y = \varphi(x) \in \mathfrak{X}$.

Step 3: $f|_{BP}$ is a p-centric map for each $\mathcal{F}_{(S,f)}(X)$-centric subgroup $P \le S$. Suppose that $P \le S$ is $\mathcal{F}_{(S,f)}(X)$-centric. Choose a central element $x \in S$ or order p. Since P is $\mathcal{F}_{(S,f)}(X)$-centric, $x \in P$ and we have a sequence of homotopy monomorphisms at p

$$BP \xrightarrow{Bincl} BS \xrightarrow{f_x} BC_X(x) \xrightarrow{ev} X\,.$$

By hypothesis, $BC_X(x)$ is the classifying space of a p-local finite group, and from the above sequence of maps we easily obtain that (S, f_x) is a Sylow p-subgroup for $BC_X(x)$. Furthermore, P is also $\mathcal{F}_{(S,f_x)}(BC_X(x))$-centric, and then $f_x|_{BP}$ is a p-centric map. There is a sequence of equivalences

$$\begin{aligned} \mathrm{Map}(BP, BP)_{Id} &\simeq \mathrm{Map}(BP, BC_X(x))_{f_x|_{BP}} \\ &\simeq \mathrm{Map}(BP \times B\mathbb{Z}/p, X)_{f|_{BP}\circ Bm} \simeq \mathrm{Map}(BP, X)_{f|_{BP}} \end{aligned} \tag{7}$$

where $m\colon P \times \mathbb{Z}/p \to P$ denotes multiplication by x, the generator of $\mathbb{Z}/p = \langle x \rangle$. The last equivalence is implied by the Zabrodsky's lemma (cf. [21, Proposition 3.5]) applied to the fibration $B\mathbb{Z}/p \to BP \times B\mathbb{Z}/p \xrightarrow{Bm} BP$. The homotopy equivalence (7) shows that $f|_{BP}$ is a p-centric map.

Step 4: There is a map $\kappa\colon |\mathcal{L}_{(S,f)}(X)|^\wedge_p \to X$ that induces homotopy equivalences

$$\kappa_P\colon \mathrm{Map}(BP, |\mathcal{L}_{(S,f)}(X)|^\wedge_p)_{|\delta_S||_{BP}} \xrightarrow{\simeq} \mathrm{Map}(BP, X)_{f|_{BP}}\,,$$

for each non-trivial subgroup $P \le S$. The construction of the map $\kappa\colon |\mathcal{L}^c_{(S,f)}(X)| \to X$ requires some technical constructions and will be explained in Proposition 4.6. Indeed, it will be shown that there is a homotopy commutative diagram

$$\begin{tikzcd} & BS \arrow[dl, "\theta_S"'] \arrow[dr, "f"] & \\ |\mathcal{L}^c_{(S,f)}(X)| \arrow[rr, "\kappa"] & & X \end{tikzcd} \tag{8}$$

where we have identified $BS \simeq |\mathcal{B}S|$.

We will show that the induced map

$$\kappa_P\colon \operatorname{Map}(BP, |\mathcal{L}^c_{(S,f)}(X)|^\wedge_p)_{\theta_S|_{BP}} \longrightarrow \operatorname{Map}(BP, X)_{f|_{BP}}\,. \tag{9}$$

is a homotopy equivalence by induction on the order of the group P.

If $P = \langle x\rangle$, for some $x \in S$ of order p, then $BC_X(x) = \operatorname{Map}(BP, X)_{f|_{BP}}$ is the classifying space of a finite p-local group, by hypothesis. According to Step 2 above, we can assume without loss of generality that $x \in \mathfrak{X}$, and so, the induced map $f_x\colon BC_S(x) \to BC_X(x)$ is the inclusion of a Sylow p-subgroup, and the fusion system $\mathcal{F}_{(C_S(x),f_x)}(BC_X(x))$ coincides with $C_{\mathcal{F}_{(S,f)}(X)}(x)$ by Proposition 4.4.

Now, diagram (8) induces the new homotopy commutative diagram

$$\begin{tikzcd} & BC_S(x) \arrow[dl, "\theta_\sharp"'] \arrow[dr, "f_x"] & \\ \operatorname{Map}(BP, |\mathcal{L}^c_{(S,f)}(X)|^\wedge_p)_{\theta|_{BP}} \arrow[rr, "\kappa_P"'] & & BC_X(x) \end{tikzcd} \tag{10}$$

where, according to [13, 6.3], the map $\theta_\sharp$ is the inclusion of a Sylow p-subgroup of the mapping space $\operatorname{Map}(BP, |\mathcal{L}^c_{(S,f)}(X)|^\wedge_p)_{\theta|_{BP}}$ which is the classifying space of a centralizer p-local finite group with fusion system $C_{\mathcal{F}_{(S,f)}(X)}(x)$. Furthermore, κ_P induces an equivalence of fusion systems, and therefore a homotopy equivalence.

For an arbitrary non-trivial subgroup $P \leq S$, we fix an element x of order p in the center of P. Again, we can assume that x belongs to $\mathfrak{X}$. There is a diagram

$$\begin{tikzcd} \operatorname{Map}(BP, |\mathcal{L}^c_{(S,f)}(X)|)_{\theta|_{BP}} \arrow[r] \arrow[d, "\kappa_P"'] & \operatorname{Map}(BP \times B\langle x\rangle\,, |\mathcal{L}^c_{(S,f)}(X)|)_{\theta|_{BP}\circ Bm} \arrow[r] \arrow[d, "\kappa_{P\times\langle x\rangle}"'] & {} \\ \operatorname{Map}(BP, X)_{f|_{BP}} \arrow[r] & \operatorname{Map}(BP \times B\langle x\rangle\,, X))_{f|_{BP}\circ Bm} \arrow[r] & {} \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[r] & \operatorname{Map}(BP, \operatorname{Map}(B\langle x\rangle\,, |\mathcal{L}^c_{(S,f)}(X)|)_{Bincl})_{\theta|_{BP}} \arrow[d, "\operatorname{Map}(1,\kappa_{\langle x\rangle})"] \\ {} \arrow[r] & \operatorname{Map}(BP, \operatorname{Map}(B\langle x\rangle\,, X)_x)_{f|_{BP}} \end{tikzcd}$$

where horizontal arrows are homotopy equivalences, by adjunction and by Zabrodsky's lemma (cf. [21, Proposition 3.5]) applied to the fibration $B\mathbb{Z}/p \to BP \times B\langle x\rangle \xrightarrow{Bm} BP$, where we identify $\mathbb{Z}/p$ with the kernel of the multiplication homomorphism $m\colon P \times \langle x\rangle \xrightarrow{m} P$. Also, $\operatorname{Map}(1, \kappa_{\langle x\rangle})$ is a homotopy equivalence. That concludes the proof that κ_P in equation (9) is a natural mod p homology equivalence for subgroups $P \leq S$.

Step 5: $X \simeq |\mathcal{L}_{(S,f)}(X)|^\wedge_p$ if and only if the natural map

$$\operatorname*{hocolim}_{\mathcal{F}^e_{(S,f)}(X)^{op}} \mathrm{Map}(BE, X)_{f|_{BE}} \longrightarrow X$$

induced by evaluation is a mod p homology equivalence. Diagram (8) induces an isomorphism of fusion systems over S: $\mathcal{F}_{(S,\theta)}(|\mathcal{L}^c_{(S,f)}(X)|) = \mathcal{F}_{(S,f)}(X)$. We will consider the full subcategories of non-trivial fully centralized elementary abelian p-subgroups $E \le S$. In order to simplify the notation, we will write $\mathcal{F}^e = \mathcal{F}^e_{(S,\theta)}(|\mathcal{L}^c_{(S,f)}(X)|) = \mathcal{F}^e_{(S,f)}(X)$.

For every elementary abelian subgroup $E \le S$, the map κ_E, as defined in step 4, fits in a commutative diagram

$$\begin{tikzcd} \mathrm{Map}(BE, |\mathcal{L}^c_{(S,f)}(X)|)_{\theta|_{BE}} \arrow[r, "\kappa_E"] \arrow[d, "ev"'] & \mathrm{Map}(BE, X)_{f|_{BE}} \arrow[d, "ev"] \\ |\mathcal{L}^c_{(S,f)}(X)| \arrow[r, "\kappa"] & X \end{tikzcd}$$

where vertical maps are induced by evaluation at the base point. As a consequence, we obtain a map between the corresponding homotopy colimits together with compatible maps induced by evaluation:

$$\begin{tikzcd} \operatorname*{hocolim}_{(\mathcal{F}^e)^{op}} \mathrm{Map}(BE, |\mathcal{L}^c_{(S,f)}(X)|^\wedge_p)_{\theta|_{BE}} \arrow[r, "\widehat{\kappa}"] \arrow[d, "ev"'] & \operatorname*{hocolim}_{(\mathcal{F}^e)^{op}} \mathrm{Map}(BE, X)_{f|_{BE}} \arrow[d, "ev"] \\ |\mathcal{L}^c_{(S,f)}(X)|^\wedge_p \arrow[r, "\kappa"] & X \end{tikzcd} \tag{11}$$

where $\widehat{\kappa} = \mathrm{hocolim}_{(\mathcal{F}^e)^{op}} \kappa_E$ is the induced map between the respective homotopy colimits. It turns out that $\widehat{\kappa}$ is a homotopy equivalence because all κ_E are homotopy equivalences according to step 4. Also, the left vertical map of is a homotopy equivalence by [13, 2.6 and 6.3].

Hence, the right vertical map ev in (11) is a homotopy equivalence if and only if κ is a homotopy equivalence. This proves step 5. □

Notice also, that, reciprocally, if X is the classifying space of a p-local finite group with Sylow p-subgroup (S, f), then all conditions of Theorem 4.5 are satisfied according to [13, §7].

There seems to be no natural way to construct a map between X and $|\mathcal{L}^c_{(S,f)}(X)|$ in either direction. This problem was solved in [12] by means of some auxiliary constructions. For the convenience of the reader we shall reproduce the argument here. For this aim we will introduce a variation of the categories $\mathcal{F}_{(S,f)}(X)$ and $\mathcal{L}_{(S,f)}(X)$, independent of the choice of a Sylow p-subgroup.

For a space X, we denote $\mathcal{F}_p(X)$ the category in which the objects are finite p-subgroups (P, f) of X, and the morphisms are defined

$$\mathrm{Mor}_{\mathcal{F}_p(X)}((P,f),(Q,g)) = \left\{ \varphi \in \mathrm{Hom}(P,Q) \,\middle|\, f \simeq g \circ B\varphi \right\}.$$

Similarly, $\mathcal{L}_p(X)$ is the category in which the objects are the p-subgroups (P, f) of X and morphisms are defined as

$$\begin{aligned} \mathrm{Mor}_{\mathcal{L}_p(X)}((P,f),(Q,g)) = \{ (\varphi, [H]) \mid{}& \varphi \in \mathrm{Hom}(P,Q) \text{ and} \\ & [H] \text{ is the homotopy class of a homotopy from } f \text{ to } g \circ B\varphi \}. \end{aligned}$$

Notice that if (S,f) is a p-subgroup of X, then, there are obvious functors $\mathcal{F}_{(S,f)}(X) \to \mathcal{F}_p(X)$ and $\mathcal{L}_{(S,f)}(X) \to \mathcal{L}_p(X)$, sending and sending an object P of $\mathcal{F}_{(S,f)}(X)$ (resp. $\mathcal{L}_{(S,f)}(X)$) to the map $f|_{BP}\colon BP \to X$ considered as an object of $\mathcal{F}_p(X)$ (resp. $\mathcal{L}_p(X)$). Furthermore, if (S,f) is a Sylow p-subgroup, then these are equivalences of categories.

Proposition 4.6. *Let X be a space, S a finite p-group, and $f\colon BS \to X$ a map. Assume that (S,f) is a Sylow p-subgroup of X and that for each $\mathcal{F}_{(S,f)}(X)$-centric subgroup $P \le S$, $f|_{BP}$ is a centric map, then there is a homotopy equivalence $\kappa_S\colon |\mathcal{B}S| \to BS$ and a map $\kappa_X\colon |\mathcal{L}^c_{(S,f)}(X)| \to X$ such that the diagram*

$$\begin{tikzcd} {|\mathcal{B}S|} \arrow[r, "\kappa_S", "\simeq"'] \arrow[d, "\theta_S"'] & BS \arrow[d, "f"] \\ {|\mathcal{L}^c_{(S,f)}(X)|} \arrow[r, "\kappa_X"] & X\,. \end{tikzcd}$$

is homotopy commutative.

Proof. We will sketch here the necessary constructions in order to obtain the map $\kappa_X\colon |\mathcal{L}^c_{(S,f)}(X)| \to X$. We refer to [12, §4] for full details.

We denote $\mathcal{L}^c_p(X)$ the full subcategory of $\mathcal{L}_p(X)$ whose objects are the p-subgroups (P,f) of X where f is a centric map. The hypothesis on (S,f) and on $\mathcal{F}_{(S,f)}(X)$-centric subgroups imply that the functor $\mathcal{L}_{(S,f)}(X) \to \mathcal{L}_p(X)$ defined above restricts to an equivalence of categories

$$\mathcal{L}^c_{(S,f)}(X) \to \mathcal{L}^c_p(X)\,.$$

In order to connect the nerve of $\mathcal{L}^c_p(X)$ and X, in [12], it is defined the simplicial space $M^c_\bullet(X)$ where n-simplices are maps $\eta\colon \Delta(\mathbf{P}) \to X$, where $\mathbf{P} = (P_0 \overset{\varphi_1}{\to} P_1 \overset{\varphi_2}{\to} \cdots \overset{\varphi_n}{\to} P_n)$ is a sequence of p-subgroups of S and monomorphisms, and $\Delta(\mathbf{P})$ can be regarded as the homotopy colimit of the sequence $BP_0 \xrightarrow{B\varphi_1} BP_1 \xrightarrow{B\varphi_2} \cdots \xrightarrow{B\varphi_n} BP_n$, with the condition that the restriction of η to any BP_i is a centric map.

The inclusion of base points in BP_i provides a map $\iota_{\mathbf{P}}\colon \Delta^n \to \Delta(\mathbf{P})$, and then, an evaluation map

$$ev_X\colon |M^c_\bullet(X)| \to X\,,$$

$ev_X(t,\eta) = \eta(\iota_{\mathbf{P}}(t))$.

For each i, the mapping cylinder of $BP_{i-1} \xrightarrow{\varphi_i} BP_i$ embeds naturally in $\Delta(\mathbf{P})$ and the restriction of η to this mapping cylinder can be interpreted as a homotopy between $\eta|_{BP_{i-1}}$ and $\eta|_{BP_i} \circ B\varphi_i$, thus, a morphism of $\mathcal{L}^c_p(X)$ from $\eta|_{BP_{i-1}} \to X$ to $\eta|_{BP_i} \to X$. In this way, the n-simplex $\eta\colon \Delta(\mathbf{P}) \to X$ determines an n-simplex in $N_\bullet(\mathcal{L}^c_p(X))$ and gives rise to a simplicial map from $M^c_\bullet(X)$ to the nerve of $\mathcal{L}^c_p(X)$, and therefore a map between the respective geometric realizations:

$$\tau_X\colon |M^c_\bullet(X)| \longrightarrow |\mathcal{L}^c_p(X)|\,.$$

Each object $\alpha\colon BP \to X$ of $\mathcal{L}^c_p(X)$ is a centric map. In particular, $\mathrm{Map}(BP,X)_\alpha \simeq \mathrm{Map}(BP,BP)_{Id} \simeq BZ(P)$ is aspherical, and so, according to [12, Lemma 4.2] (see its proof), $\tau_X\colon |M^c_\bullet(X)| \to |\mathcal{L}^c_p(X)|$ is a homotopy equivalence. Then, choosing a homotopy inverse of τ_X we can define $\kappa_X\colon |\mathcal{L}^c_{(S,f)}(X)| \to X$ as the composition

$$|\mathcal{L}^c_{(S,f)}(X)| \xrightarrow{\simeq} |\mathcal{L}^c_p(X)| \underset{\simeq}{\xleftarrow{\tau_X}} |M^c_\bullet(X)| \xrightarrow{ev_X} X\ . \tag{12}$$

In case $X = BS$, Proposition 2.7, Lemma 4.2 and Lemma 4.3 of [12] provide homotopy equivalences

$$|\mathcal{L}_p^c(S)| \xrightarrow{\simeq} |\mathcal{L}_p^c(BS)| \xleftarrow{\simeq} |M_\bullet^c(BS)| \xrightarrow{\simeq} BS \tag{13}$$

hence, the key to finish the proof of the Proposition lies in the naturality properties of this construction with respect to $f\colon BS \to X$. However, in general, a subgroup $P \leq S$ which is centric in S, need not be centric when regarded as a p-subgroup of X by considering the restriction $f|_{BP} : BP \longrightarrow X$ of f to BP. For this reason, we will have to restrict $M_\bullet^c(BS)$ to the subspace $M_\bullet^S(BS)$ of simplices $\eta\colon \Delta(\mathbf{P}) \to BS$ of $M_\bullet^c(BS)$ where every group in the sequence $\mathbf{P}$ is S itself. Accordingly, we call $\mathcal{L}_p^S(BS)$ the full subcategory of $\mathcal{L}_p^c(BS)$ with objects the homotopy equivalences $g\colon BS \to BS$. With this notation we have a diagram of homotopy equivalences

$$\begin{tikzcd}
{|\mathcal{B}S|} \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & {|\mathcal{L}_p^S(BS)|} \arrow[d, "\simeq"'] & {|M_\bullet^S(BS)|} \arrow[l, "\simeq"'] \arrow[r, "ev_{BS}", "\simeq"'] \arrow[d, "\simeq"'] & BS \arrow[d, equal] \\
{|\mathcal{L}_p^c(S)|} \arrow[r, "\simeq"] & {|\mathcal{L}_p^c(BS)|} & {|M_\bullet^c(BS)|} \arrow[l, "\simeq"'] \arrow[r, "ev_{BS}", "\simeq"'] & BS
\end{tikzcd} \tag{14}$$

where same arguments as in [12] for the sequence (13) are used.

Now, for every equivalence $g\colon BS \to BS$, the composition $BS \xrightarrow{g} BS \xrightarrow{f} X$ defines a centric p-subgroup of X, and then f induces a well defined map of simplicial spaces $M_\bullet^S(BS) \to M_\bullet^c(X)$, that makes commutative the diagram

$$\begin{tikzcd}
{|\mathcal{B}S|} \arrow[r, "\simeq"] \arrow[d, "\theta_S"'] & {|\mathcal{L}_p^S(BS)|} \arrow[d, "f_\sharp"] & {|M_\bullet^S(BS)|} \arrow[l, "\simeq"'] \arrow[r, "ev_{BS}", "\simeq"'] \arrow[d, "f_\sharp"] & BS \arrow[d, "f"] \\
{|\mathcal{L}_{(S,f)}^c(X)|} \arrow[r, "\simeq"] & {|\mathcal{L}_p^c(X)|} & {|M_\bullet^c(X)|} \arrow[l, "\tau_X"', "\simeq"] \arrow[r, "ev_X"] & X\,.
\end{tikzcd} \tag{15}$$

Then $\kappa_S\colon |\mathcal{B}S| \xrightarrow{\simeq} BS$ is the composite homotopy equivalence in the top row of the above diagram, and this finishes proof. ☐

5. Homotopy fixed point p-compact groups

Let M be a space and G a discrete group. An action of the group G on the space M is group homomorphism

$$\rho\colon G \to \operatorname{aut}(M)$$

where $\operatorname{aut}(M)$ is the topological monoid of self-homotopy equivalences of M.

Dwyer and Wilkerson introduced [25, §10] the homotopy theoretic notion of proxy actions. A proxy action of G on M is defined as a fibration

$$M \longrightarrow M_{hG} \xrightarrow{p} BG \tag{16}$$

Now, this is classified up to fibre homotopy equivalence by a map

$$BG \to B\operatorname{aut}(M)\,.$$

Any action $\rho\colon G \to \operatorname{aut}(M)$ of G on M determines a proxy action by taking $M_{hG} = M \times_G EG$ to be the Borel construction and the classifying map is $B\rho\colon BG \to B\operatorname{aut}(M)$. Conversely, the proxy action (16) produces a rigid action of G on a space homotopy equivalent to M by turning $M \to M_{hG}$ into a covering space.

We will adopt the more flexible notion of proxy actions throughout this paper and by abuse of language will call just an action to a proxy action. In this setting, the total space M_{hG} of (16) is called the *homotopy quotient space* and the *homotopy fixed point space* is defined as the space M^{hG} of sections of fibration (16). In this section we use obstruction theory to develop some basic structure results for M^{hG}, and we apply them in the case where $M = BX$ is the classifying space of a p-compact group and to the proof of Theorem B.

We will show conditions under which M^{hG} is nonempty, and if this is the case, a way to describe the set of path-components. Fibration (16) induces an action of G on the set of path-components of M and $\pi_0(M)^G$ denotes the set of path-components of M that remain fixed under this action. Then, evaluation of a section at the base point $b \in BG$ induces a map

$$\pi_0(M^{hG}) \xrightarrow{\pi_0(\mathrm{ev})} \pi_0(M)^G \tag{17}$$

thus, a necessary condition for M^{hG} being nonempty is that $\pi_0(M)^G$ is nonempty.

Fix now a point $m \in M$ which represents a G-invariant path-component of M, then, there is a short exact sequence

$$1 \to \pi_1(M, m) \to \pi_1(M_{hG}, m) \to \pi_1(BG, b) \to 1 \tag{18}$$

of fundamental groups, where $b = p(m)$. If $m \in \pi_0(M)^G$ happens to be in the image of the evaluation map (17), then $s(b) = m$ for some homotopy fixed point $s \in M^{hG}$ and then the exact sequence (18) does have a section, namely $\pi_1(s)$.

Define $H^1(G; \pi_1(M, m))$ [60] to be the set (possibly empty) of $\pi_1(M, m)$-conjugacy classes of sections $\pi_1(BG, b) \to \pi_1(M_{hG}, m)$ of the exact sequence (18). Then, the argument in the previous paragraph produces a well defined map $\pi_0(\mathrm{ev})^{-1}([m]) \to H^1(G; \pi_1(M, m))$. In next Lemma it will be shown that, under certain conditions, this is a bijection for every $[m] \in \pi_0(M)^G$.

Since $\pi_1(M_{hG}, m)$ acts on the homotopy groups $\pi_i(M, m)$ of the fibre, also $G = \pi_1(BG, b)$ acts on $\pi_i(M, m)$ through $\pi_1(s)$, for a given element $s \in M^{hG}$. We let $\pi_i(M, m)^{s_*G}$, $i \geq 1$, denote the fixed point group for this action.

Lemma 5.1. *Suppose that G is a finite group of order prime to p and that $\pi_i(M, m)$ is a module over the ring $\mathbb{Z}_{(p)}$ of p-local integers for all $i \geq 2$ and all base points $m \in \pi_0(M)^G$. Then the following hold:*

1. *A class $[m] \in \pi_0(M)^G$ is in the image of the evaluation map (17) if and only if the exact sequence (18) splits.*
2. *If $[m] \in \pi_0(M)^G$ is in the image of the evaluation map (17), then there is an exact sequence of pointed sets*
$$* \to H^1(G; \pi_1(M, m)) \to \pi_0(M^{hG}) \xrightarrow{\pi_0(ev)} \pi_0(M)^G$$
where $[m]$ is the base point of $\pi_0(M)^G$.
3. *If $s \in M^{hG}$ is a homotopy fixed point with $s(b) = m$ then*
$$\pi_i(M^{hG}, s) \cong \pi_i(M, m)^{s_*G}$$
for all $i \geq 1$.

Proof. The Postnikov functors P_r, defined as nullification with respect to S^{r-1} (see [19]), determine a tower of fibrations

$$M_{hG} \to \cdots \to P_r M_{hG} \to P_{r-1} M_{hG} \to \cdots \to P_1 M_{hG} \to BG$$

so that M^{hG} is the homotopy inverse limit of a sequence

$$\cdots \to (P_rM)^{hG} \to (P_{r-1}M)^{hG} \to \cdots \to (P_1M)^{hG}$$

of Postnikov homotopy fixed point spaces.

Note that $\pi_0(P_1M_{hG}) = \pi_0(M_{hG})$ and that each path-component of P_1M_{hG} is aspherical with fundamental group $\pi_1(P_1M_{hG}, m) = \pi_1(M_{hG}, m)$ for all $m \in P_1M$. It is now easy to see that $H^1(G; \pi_1(M, m))$ is indeed the fibre over $[m] \in \pi_0(P_1M)^G = \pi_0(M)^G$ of the evaluation map $\pi_0(P_1M^{hG}) \to \pi_0(M)^G$ and also that $\pi_1(P_1M^{hG}, s) = \pi_1(M, m)^{s_*G}$ for any $s \in P_1M^{hG}$ with $s(b) = m$, cf. [46, §6]. Obstruction theory implies that $\pi_0(M^{hG}) = \pi_0(P_1M^{hG})$. This proves the first two items.

For the third item, suppose that the homotopy fixed point space is nonempty and let $s \in M^{hG}$ be a homotopy fixed point. Then the component $\big(M^{hG}, s\big)$ containing s is the homotopy inverse limits of the corresponding components

$$\cdots \to \big(P_rM^{hG}, s_r\big) \to \big(P_{r-1}M^{hG}, s_{r-1}\big) \to \cdots \to \big(P_1M^{hG}, s_1\big)$$

of the Postnikov homotopy fixed point spaces. To finish the proof, observe [46, 3.1] that the fibre of $\big(P_rM^{hG}, s_r\big) \to \big(P_{r-1}M^{hG}, s_{r-1}\big)$ is the Eilenberg-Mac Lane space $K(\pi_r(M, m)^{s_*G}, r)$. □

Theorem 5.2. *Let M be any simply connected p-complete space, G a finite group of order prime to p, and*

$$M \to M_{hG} \to BG$$

an action of G on M. Then the homotopy fixed point space M^{hG} is nonempty, $\pi_i(M^{hG}) = \pi_i(M)^G$ for all $i \geq 0$, and there is a homotopy equivalence

$$\Omega M \xrightarrow{\simeq} \Omega(M^{hG}) \times \mathrm{Fib}(M^{hG} \to M)$$

In particular, the fibre $\mathrm{Fib}(M^{hG} \to M)$ of the evaluation map $M^{hG} \to M$ is an H-space.

Proof. The space of sections M^{hG} is nonempty, connected, and $\pi_*(M^{hG}) = \pi_*(M)^G$ according to Lemma 5.1 since M is simply connected and p-complete. We will show first how to turn this action with a homotopy fixed point into an honest action of G on a space homotopy equivalent to M and with a fixed point. The pullback diagram

$$\begin{tikzcd} M \arrow[r, shift left] \arrow[d] & EG \arrow[l, dashed, shift left] \arrow[d] \\ M_{hG} \arrow[r, shift left] & BG \arrow[l, dashed, shift left] \end{tikzcd}$$

realizes $M \to M_{hG}$ as a regular covering space with G acting on M. Liftings of sections $BG \to M_{hG}$ provide G-equivariant maps $EG \to M$. Let $M/EG = M \cup C(EG)$ be the homotopy cofibre of any such G-map. Then $M \to M/EG$ is a G-equivariant homotopy equivalence and the G-action on M/EG has a fixed point.

Now, we can assume that there is an honest G-action on M with a fixed point. Let ΩM denote the loop space based at any G-fixed point. There is a fibration sequence

$$\cdots \to \Omega M^{hG} \to \Omega M \to \mathrm{Fib}(M^{hG} \to M) \to M^{hG} \to M$$

and it suffices to construct a homotopy left inverse for $\Omega M^{hG} \to \Omega M$.

Define $\mathrm{tr}\colon \Omega M \to \Omega M$ to be the map that takes any loop ω to the product $\prod g\omega$ of the loops $g\omega$ where g runs through the elements of G in some fixed order. The image of the induced map $\mathrm{tr}_*\colon \pi_*(\Omega M) \to \pi_*(\Omega M)$, which takes a homotopy class α to $\sum_{g\in G} g_*\alpha$, is contained in the fixed group $\pi_*(\Omega M)^G$ and the composition $\pi_*(\Omega M)^G \to \pi_*(\Omega M) \xrightarrow{\mathrm{tr}_*} \pi_*(\Omega M)^G$ is an isomorphism. This implies that the composition $\Omega M^{hG} \to \Omega M \to T$, where T is the mapping telescope of $\Omega M \xrightarrow{\mathrm{tr}} \Omega M \xrightarrow{\mathrm{tr}} \cdots$, is a (weak) homotopy equivalence and we have the left inverse we were looking for. □

Let (X, BX, e) be a p-compact group or, more generally, a loop space. The above arguments suggest the following definition of a (proxy) action of a discrete group G on X.

Definition 5.3. Let (X, BX, e) be a loop space and G a group. A *proxy* action of G on (X, BX, e) is a fibration

$$BX \xrightarrow{i} BX_{hG} \underset{s}{\overset{p}{\rightleftarrows}} BG\,. \tag{19}$$

with a section, fixed up to vertical homotopy.

When it is clear from the context that we refer to an action in the sense of this definition, we will simply say that G acts on the loop space X. The section in (19) guarantees an induced action of G on the space X, compatible with the loop structure. In fact, the homotopy quotient for this action on X is defined as the pullback space in the diagram

$$\begin{tikzcd} X_{hG} \arrow[r, "\bar p"] \arrow[d, "\bar p"'] & BG \arrow[d, "s"] \\ BG \arrow[r, "s"'] & BX_{hG}\,. \end{tikzcd} \tag{20}$$

This diagram turns out to be a diagram of spaces over BG. The homotopy fibre of $\bar p$ is X, and it has a canonical section $\bar s$ defined by the pullback diagram (20) that we can interpret as the *homotopy constant loop*

$$X \xrightarrow{\bar i} X_{hG} \underset{\bar s}{\overset{\bar p}{\rightleftarrows}} BG\,.$$

The action of G on X depends on the section $s\colon BG \to BX_{hG}$, and for this action we obtain that the homotopy fixed point space X^{hG} is a loop space with classifying space $B(X^{hG}) \simeq (BX)^{hG}_s$, the connected component of $(BX)^{hG}$ with base point the section s. Furthermore, the evaluation map $X^{hG} \to X$ is seen to be the loop map of the evaluation map $(BX)^{hG}_s \to BX$, thus we have a sequence of fibrations

$$X^{hG} \xrightarrow{ev} X \longrightarrow X/X^{hG} \longrightarrow (BX)^{hG}_s \xrightarrow{ev} BX$$

where we write X/X^{hG} for the homotopy fibre of the evaluation map $(BX)^{hG}_s \to BX$.

In section 2 we have introduced $\mathrm{Out}(X)$ as the group of invertible elements of the monoid $[BX, BX]$ of unbased homotopy classes of unbased self-maps of BX. By analogy with discrete group theory, we call *outer* action of G on X to a homomorphism of groups $\rho\colon G \to \mathrm{Out}(X)$. Since $\mathrm{Out}(X)$ is well understood (see Theorem 2.2), outer actions will be a source for group actions on p-compact groups provided we can lift outer actions to actions in the sense of Definition 5.3. Theorem B solves the problem in case of finite groups of order prime to p.

Proof of Theorem B. Fix a finite group G of order prime to p and $\rho\colon G \to \mathrm{Out}(X)$ an outer action of G on a connected p-compact group X. Recall that we have a fibration sequence

$$B^2Z(X) \to B\mathrm{aut}(BX) \to B\mathrm{Out}(X)$$

and that the center of X, $Z(X)$, is p-local. By obstruction theory we obtain a unique lifting of ρ to a map $\varphi\colon BG \to B\operatorname{aut}(BX)$, that determines an action $BX \to BX_{hG} \to BG$. Furthermore, since $\pi_1(BX) = 1$, Lemma 5.1.(2) implies that $\pi_0(BX^{hG}) = *$; that is, there is a unique section

$$BX \longrightarrow BX_{hG} \rightleftarrows BG\,. \tag{21}$$

up to fibre homotopy equivalence; in other words, ρ lifts to a unique action of G on X.

This is part (1) of the Theorem. Now, Theorem 5.2 provides the splitting $X \simeq X^{hG} \times X/X^{hG}$. It follows that X/X^{hG} is an $\mathbb{F}_p$-finite H-space, X^{hG} is a loop space with classifying space BX^{hG} and it is also $\mathbb{F}_p$-finite. Furthermore, BX^{hG} is p-complete because BX is p-complete [25, 11.13], hence X^{hG} is a connected p-compact group.

The rational cohomology algebra $H^*(BY;\mathbb{Q}_p)$ is polynomial for any connected p-compact group Y and it follows that the Hurewicz homomorphism induces an isomorphism

$$QH^*(BY;\mathbb{Q}_p) \to \pi_*(BY)^\vee \otimes \mathbb{Q}$$

between the indecomposables and the rationalized dual ($\pi^\vee = \operatorname{Hom}_{\mathbb{Z}_p}(\pi,\mathbb{Z}_p)$) of the homotopy groups of the simply connected space BY [5, Theorem 3.2.3]. For the connected fixed point p-compact group BX^{hG}, in particular, we have

$$QH^*(BX^{hG};\mathbb{Q}_p) \cong \pi_*(BX^{hG})^\vee \otimes \mathbb{Q} \cong \big(\pi_*(BX)^\vee \otimes \mathbb{Q}\big)_G \cong \big(QH^*(BX;\mathbb{Q}_p)\big)_G$$

for $\pi_*(BX^{hG}) = \pi_*(BX)^G$ as the order of G is prime to p. This proves points (2) and (3).

We finish by proving point (4). Assume p is odd. If X is a polynomial p-compact group

$$H^*(X;\mathbb{F}_p) \cong H^*(X^{hG};\mathbb{F}_p) \otimes H^*(X/X^{hG};\mathbb{F}_p)$$

is an exterior algebra, hence $H^*(X^{hG};\mathbb{F}_p)$ is an exterior algebra, too. Therefore, $H^*(BX^{hG};\mathbb{F}_p)$ is a polynomial algebra. □

Example 5.4. At any odd prime, let C_2 act on E_6 through the unstable Adams operation ψ^{-1}. Since the fixed point p-compact group $BE_6^{hC_2}$ is the p-compact group BF_4 (A.12), there is a splitting

$$E_6 \simeq F_4 \times E_6/F_4$$

of homogeneous spaces. This splitting is due to Harris [35]. Also, $BPE_6^{hC_2} \simeq BF_4$, where PE_6 is the adjoint form of E_6, (A.12), thus there is also a splitting $PE_6 \simeq F_4 \times PE_6/F_4$.

Let p be an odd prime and m a divisor of $p-1$ so that the cyclic group C_m of order m acts on $BSU(mn+s)$, $0 \le s < m$, through unstable Adams operations. Since the fixed point p-compact group $BSU(mn+s)^{hC_m}$ is (A.9) the generalized Grassmannian $BX(m,1,n)$ with polynomial cohomology $H^*(BX(m,1,n);\mathbb{F}_p) = \mathbb{F}_p[x_m,\dots,x_{nm}]$, $|x_{im}| = 2im$, there is a splitting

$$SU(mn+s) \simeq X(m,1,n) \times SU(mn+s)/X(m,1,n)$$

of homogeneous spaces. This splitting is originally due to Mimura, Nishida, and Toda [44]), although the recognition of $X(m,1,n)$ as a loop space is due to Quillen [59] (see also [64, 69, 17]). The case $m = 2$ is the classical splitting $SU(2n) \simeq Sp(n) \times SU(2n)/Sp(n)$. Similar splittings for central quotients of $SU(n)$ can be worked out.

Similarly, at $p = 5$, let C_4 act on BE_8 through unstable Adams operations. The fixed point p-compact group $BE_8^{hC_4}$ is the p-compact group $B\mathbf{X}_{31}$ corresponding to reflection group number 31 on the Clark-Ewing list (see A.12), $H^*(B\mathbf{X}_{31};\mathbb{F}_p) = \mathbb{F}_p[x_{16},x_{24},x_{40},x_{48}]$ where subscripts indicate degrees, there is a splitting

$$E_8 \simeq \mathbf{X}_{31} \times E_8/\mathbf{X}_{31}$$

of homogeneous spaces, that was obtained in [66].

At $p = 3$, BF_4 admits an exceptional isogeny of order 2 and the fixed point group ${BF_4}^{hC_2}$ is [15] the p-compact group BDI_2 whose cohomology realizes the Dickson algebra $\mathbb{F}_3[x_{12}, x_{16}]$. The corresponding splitting

$$F_4 \simeq DI_2 \times F_4/DI_2$$

was first obtained in [36]. Later proofs of this splitting were obtained independently by Wilkerson and by Kono, using Friedlander's exceptional isogeny of F_4 localized away from two.

In these last two cases, it was Zabrodsky [69, 4.3], who first recognized the factors $\mathbf{X}_{12} = DI_2$ and $\mathbf{X}_{31}$ as loop spaces. Later, Aguadé gave a nice uniform construction of a family of modular p-compact groups including these cases [1].

6. Homotopy fixed point spaces of twisted unstable Adams operations

In this section we proof Theorem E. Part (1) of the Theorem follows from Proposition 6.2 and Remark 6.3, while Part (2) is Proposition 6.5.

Let X be a connected p-compact group and set $\alpha\colon X \to X$ a p-compact group automorphism. The homotopy pullback diagram

$$\begin{tikzcd} BX^{h\alpha} \arrow[r, "\iota"] \arrow[d, "\iota"'] & BX \arrow[d, "\Delta"] \\ BX \arrow[r, "{(1,\alpha)}"] & BX \times BX \end{tikzcd} \tag{22}$$

serves as the definition of the space $BX^{h\alpha}$. If α is homotopic to α', then $BX^{h\alpha} \simeq BX^{h\alpha'}$.

In the special case where $\alpha = \tau\psi^q$ is a twisted unstable Adams operation with $q \in \mathbb{Z}_p$, $q \neq 1$, and $q \not\equiv 0 \bmod p$, we also write $B^\tau X(q) = BX^{h\,\tau\psi^q}$, or just $BX(q)$, if $\tau = 1$. For $q = 1$ we trivially obtain $BX(1) \simeq \Lambda(BX)$, the free loop space.

Assume that α represents an element of finite order r in $\mathrm{Out}(X)$, with r prime to p, and X is a connected p-compact group. According to Theorem B, it defines an action of the cyclic group C_r on X. The next proposition shows that the natural map $\Lambda(BX^{hC_r}) \to BX^{h\alpha}$ is a homotopy equivalence.

Proposition 6.1. *Assume that X is a connected p-compact group. If $\beta\colon BX \to BX$ represents an element of* $\mathrm{Out}(X)$ *of finite order r, prime to p, then $BX^{h\beta}$ is homotopy equivalent to the space of free loops on BX^{hC_r}, where the action of the cyclic group C_r on BX is given by β.*

Proof. According to Theorem B, β defines an action of C_r on X,

$$\begin{tikzcd} BX \arrow[r, "i"] & BX_{hC_r} \arrow[r, shift left, "p"] & BC_r \arrow[l, shift left, "s"] \end{tikzcd} .$$

Evaluation at the base point of BC_r induces a map $\mathrm{ev}\colon BX^{hC_r} \to BX_{hC_r}$ that makes the triangle

$$\begin{tikzcd}
& BX \arrow[dd, "\beta"] \\
BX^{hC_r} \arrow[ur, "\mathrm{ev}"] \arrow[dr, "\mathrm{ev}"'] & \\
& BX
\end{tikzcd}$$

commutative up to homotopy. Therefore, we can form a homotopy commutative diagram

$$\begin{tikzcd}
\Lambda(BX^{hC_r}) \arrow[rr] \arrow[dr] \arrow[dd] & & BX^{hC_r} \arrow[dr, "\Delta"] \arrow[dd] & \\
& BX^{hC_r} \arrow[rr, "\Delta"] \arrow[dd] & & BX^{hC_r} \times BX^{hC_r} \arrow[dd] \\
BX^{h\alpha} \arrow[rr] \arrow[dr] & & BX \arrow[dr, "\Delta"] & \\
& BX \arrow[rr, "{(1,\beta)}"] & & BX \times BX\,.
\end{tikzcd} \tag{23}$$

We will show that $\Lambda(BX^{hC_r}) \to BX^{h\alpha}$ is a homotopy equivalence. According to Theorem 5.2, BX^{hC_r} is the classifying space of a connected p-compact group and by Lemma 5.1 the map $\mathrm{ev}\colon BX^{hC_r} \to BX$ induces an identification of the homotopy groups of BX^{hC_r} with the invariant elements in the homotopy groups of BX by the action of C_r: $\pi_i(BX^{hC_r}) \cong \pi_i(BX)^{C_r} \hookrightarrow \pi_i(BX)$. There is a Mayer–Vietoris long exact sequence for the homotopy groups of $BX^{h\beta}$,

$$\ldots \to \pi_i(BX^{h\beta}) \to \pi_i(BX) \xrightarrow{1-\beta_*} \pi_i(BX) \to \pi_{i-1}(BX^{h\beta}) \to \ldots$$

and for the homotopy groups of the free loop space,

$$\ldots \to \pi_i(\Lambda(BX^{hC_r})) \to \pi_i(BX^{hC_r}) \xrightarrow{0} \pi_i(BX^{hC_r}) \to \pi_{i-1}(\Lambda(BX^{hC_r})) \to \ldots$$

Both long exact sequences together give

$$\begin{tikzcd}
0 \arrow[r] & \pi_{i+1}(BX)^{C_r} \arrow[r] \arrow[d] & \pi_i(\Lambda(BX^{hC_r})) \arrow[r] \arrow[d] & \pi_i(BX)^{C_r} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Coker}\{1-\beta_*\} \arrow[r] & \pi_i(BX^{h\beta}) \arrow[r] & \mathrm{Ker}\{1-\beta_*\} \arrow[r] & 0\,.
\end{tikzcd}$$

Now, $\mathrm{Ker}(1-\beta_*) = \pi_i(BX)^{C_r}$ and $\mathrm{Coker}(1-\beta_*) = \pi_{i+1}(BX)_{C_r}$. Since r is prime to p, and the homotopy groups $\pi_i(BX)$ are $\mathbb{Z}_{(p)}$-modules for every $i \geq 2$, the composition $\pi_{i+1}(BX)^{C_r} \to \pi_{i+1}(BX) \to \pi_{i+1}(BX)_{C_r}$ is an isomorphism. Hence also the middle vertical map $\pi_i(\Lambda(BX^{hC_r})) \to \pi_i(BX^{h\alpha})$ is an isomorphism. □

Our next result contains Proposition 6.1 as a special case and it will reduce, in many cases, the question of describing $BX^{h\alpha}$ to two separate steps. The computation of the homotopy fixed point space BX^{hC_r}, for elements α of order r prime to p, and the case in which $\alpha = \psi^q$ is an unstable Adams operation of exponent $q \equiv 1 \bmod p$ (see Theorem 2.2 and formula (3) in Section 2). It is one of the two claims of Theorem E.

Proposition 6.2. *Let X be a connected p-compact group. If α is an automorphism of X that factors $\alpha = \psi^q \beta$ with*

(1) *$q \equiv 1 \bmod p$, and $(\psi^q)^* \colon H^*(X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p)$ is the identity, and*
(2) *β is an automorphism of X that represents an element of finite order r, prime to p, in $\mathrm{Out}(X)$,*

then $BX^{h\alpha} \simeq BX^{hC_r}(q)$ where $C_r = \langle \beta \rangle \subset \mathrm{Out}(X)$ is the cyclic group of order r generated by the homotopy class of β.

Proof. Let $BY = BX^{hC_r}$ denote the homotopy fixed point p-compact group for the action of the cyclic group $C_r = \langle \beta \rangle \subset \mathrm{Out}(X)$ and $i \colon BY \to BX$ the evaluation map. Now, β restricts trivially to BY, an in the proof of Proposition 6.1, and then, since ψ^q commutes with β, up to homotopy, we have a homotopy commutative diagram

$$\begin{tikzcd}
BY \arrow[r, "\psi^q"] \arrow[d, "i"'] & BY \arrow[d, "i"] \\
BX \arrow[r, "\alpha=\psi^q\beta"] & BX
\end{tikzcd}$$

that extends to

$$\begin{tikzcd}
BY(q) \arrow[rr] \arrow[dr] \arrow[dd] & & BY \arrow[dr, "\Delta"] \arrow[dd] & \\
& BY \arrow[rr, "{(1,\psi^q)}"] \arrow[dd] & & BY \times BY \arrow[dd] \\
BX^{h\alpha} \arrow[rr] \arrow[dr] & & BX \arrow[dr, "\Delta"] & \\
& BX \arrow[rr, "{(1,\alpha)}"] & & BX \times BX
\end{tikzcd}$$

where the top and bottom faces are homotopy pullback diagrams, and the front face commutes up to homotopy. Consequently, the homotopy fibres of the vertical maps form another homotopy pullback diagram:

$$\begin{tikzcd}
(X/Y)^{h\alpha} \arrow[r] \arrow[d] & X/Y \arrow[d, "\Delta"] \\
X/Y \arrow[r, "{(1,\alpha)}"] & X/Y \times X/Y
\end{tikzcd}$$

with $(X/Y)^{h\alpha} \simeq \mathrm{hofib}(BY(q) \to BX^{h\alpha})$, and where we still denote by α the self-equivalence of X/Y induced by $\alpha \colon BX \to BX$. Theorem B says that X/Y is a connected H-space and then we can also describe $(X/Y)^{h\alpha}$ as the homotopy fibre of $1 - \alpha \colon X/Y \to X/Y$. It also implies that the map $(\psi^q)^* \colon H^*(X/Y; \mathbb{F}_p) \to H^*(X/Y; \mathbb{F}_p)$ can be read off the map $(\psi^q)^*$ defined on $H^*(X; \mathbb{F}_p)$, which by hypothesis is the identity. This fact easily implies that $(1-\alpha)^* = (1-\beta)^*$ on $H^*(X/Y; \mathbb{F}_p)$.

According to Proposition 6.1, the homotopy fibre, $(X/Y)^{h\beta}$, of $1-\beta$ is contractible, hence $(1-\beta)^*$ is an automorphism of $H^*(X/Y; \mathbb{F}_p)$. Thus, a spectral sequence argument shows that $(X/Y)^{h\alpha}$ is mod p acyclic. Finally, it is easy to see that $(X/Y)^{h\alpha}$ is p-complete, hence contractible, and therefore $BY(q) \simeq BX^{h\alpha}$. $\square$

Remark 6.3. If X polynomial, the effect of $\Omega\psi^q$, $q \equiv 1 \bmod p$, on mod p cohomology of X is determined by the effect of ψ^q on $H^*(BX, \mathbb{F}_p)$ and this is in turn determined by the effect of ψ^q on $H^*(BT_X;\mathbb{F}_p)$ which is multiplication by q, hence the identity. For $X = F_4$, E_6, E_7, E_8 at the prime 3 or $X = E_8$ at the prime 5, we also obtain that $\Omega\psi^q$, $q \equiv 1 \bmod p$, acts trivially on $H^*(X;\mathbb{F}_p)$. In order to check this, we can look at the Serre spectral sequence for the path-loop fibration $X \to PBX \to BX$. It turns out that the generators for $H^*(X;\mathbb{F}_p)$ either transgress to elements detected in the maximal torus of BX, or are linked to such elements by Steenrod operations (cf. [45, Ch7]). In particular, 6.2 applies to all 1-connected p-compact groups, p odd, according to the classification theorem [7].

In particular, $BX(\zeta q) = BX^{h\langle\zeta\rangle}(q)$ when ζ is a $(p-1)$th root of unity and $q \equiv 1 \bmod p$ satisfies the conditions of Proposition 6.2. If $q = 1$ we obtain Proposition 6.1, $BX(\zeta) = BX^{h\langle\zeta\rangle}(1) = \Lambda(BX^{h\langle\zeta\rangle})$, as a special case.

For the next result, we need to interpret $BX^{h\alpha}$ as homotopy fixed point set by the action of $\mathbb{Z}$ generated by $\alpha \in \mathrm{Out}(X)$. In fact, given $\alpha \in \mathrm{Out}(X)$, we denote again by α a representative homotopy equivalence $\alpha\colon BX \to BX$. The mapping torus is defined $BX_{h\alpha} = BX \times I/{\sim}$, where $I = [0,1]$ is the unit interval and $(x,0) \sim (\alpha(x),1)$. There is a fibration, up to homotopy,

$$BX \to BX_{h\alpha} \to S^1$$

given by projection onto the second component. This fibration is classified by a loop $\omega_\alpha\colon S^1 \to B\,\mathrm{aut}(BX)$ that represents $\alpha \in \pi_1(B\,\mathrm{aut}(BX)) = \mathrm{Out}(X)$.

The space of sections for this fibration clearly coincides with $BX^{h\alpha}$ as defined in diagram (22), so we can interpret $BX_{h\alpha}$ and $BX^{h\alpha}$ as the homotopy quotient space $BX_{h\mathbb{Z}}$ and the homotopy fixed point space $BX^{h\mathbb{Z}}$, respectively, for the action of $\mathbb{Z}$ on BX determined by $\alpha \in \mathrm{Out}(X)$. Notice that since X is connected, so is $BX^{h\alpha} \simeq BX^{h\mathbb{Z}}$ and therefore, there is a unique lifting, up to equivalence, of the action of $\mathbb{Z}$ on BX to an action of $\mathbb{Z}$ on X, in the sense of Definition 5.3.

We will use this point of view in order to proof the second claim of Theorem E. We will see that an action of $\mathbb{Z}$ on BX generated by an unstable Adams operation ψ^q of exponent $q \equiv 1 \bmod p$, extends to an action of $\mathbb{Z}_p$ on BX, and that this implies that the homotopy type of the homotopy fixed point space $BX(q) = BX^{h\mathbb{Z}}$ depends only on the p-adic valuation $\nu_p(1-q)$.

Lemma 6.4. *Suppose that $BX_{h\mathbb{Z}_p} \to B\mathbb{Z}_p$ is a fibration over $B\mathbb{Z}_p$ with fibre BX. The p-completion map $\ell\colon B\mathbb{Z} \to B\mathbb{Z}_p$ induces a homotopy equivalence $BX^{h\mathbb{Z}_p} \to BX^{h\mathbb{Z}}$ of spaces of sections.*

Proof. The maps $B\mathbb{Z} \longrightarrow B\mathbb{Z}_p \longleftarrow BX_{h\mathbb{Z}_p}$ determine a commutative diagram

$$\begin{tikzcd} \mathrm{Map}(B\mathbb{Z}_p, BX_{h\mathbb{Z}_p}) \arrow[r] \arrow[d] & \mathrm{Map}(B\mathbb{Z}, BX_{h\mathbb{Z}_p}) \arrow[d] \\ \mathrm{Map}(B\mathbb{Z}_p, B\mathbb{Z}_p) \arrow[r, "\simeq"'] & \mathrm{Map}(B\mathbb{Z}, B\mathbb{Z}_p) \end{tikzcd}$$

which is a pullback diagram since $\ell\colon B\mathbb{Z} \to B\mathbb{Z}_p$ is an $\mathbb{F}_p$-equivalence [9, 12.2]. Thus the fibre of the left fibration over the identity map of $B\mathbb{Z}_p$, $BX^{h\mathbb{Z}_p}$, is homotopy equivalent to the fibre of the right fibration over the p-completion map $\ell\colon B\mathbb{Z} \to B\mathbb{Z}_p$, $BX^{h\mathbb{Z}}$. $\square$

Using the description of $\mathrm{Out}(X)$ in Section 2 we will see that actions of $\mathbb{Z}$ on connected p-compact groups given by Adams operations ψ^q extend to the p-adics precisely when $q \equiv$

$1 \bmod p$. The inclusion of Adams operations in $\mathrm{Out}(X)$, described as $q \in \mathbb{Z}_p^\times \mapsto \psi^q \in \mathrm{Out}(X)$ induces a diagram of group homomorphisms

$$\begin{tikzcd} \mathrm{Hom}(\mathbb{Z}_p, \mathbb{Z}_p^\times) \arrow[r, "\mathrm{res}"] \arrow[d] & \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_p^\times) \arrow[d] \\ \mathrm{Hom}(\mathbb{Z}_p, \mathrm{Out}(X)) \arrow[r, "\mathrm{res}"] & \mathrm{Hom}(\mathbb{Z}, \mathrm{Out}(X)) \end{tikzcd}$$

where the horizontal homomorphisms are given by restriction.

Recall that, for an odd prime p, $\mathbb{Z}_p^\times \cong \mathbb{Z}/p-1 \times \mathbb{Z}_p$, where $\mathbb{Z}/p-1$ corresponds to the subgroup of $\mathbb{Z}_p^\times$ of roots of unity and $\mathbb{Z}_p$ is identified with the subgroup of elements $q \in \mathbb{Z}_p^\times$, with $q \equiv 1 \bmod p$, via the exponential map:

$$a \in \mathbb{Z}_p \mapsto \exp(pa) \in \mathbb{Z}_p^\times$$

(exp defined by the usual expansion $\exp(pa) = 1 + pa + \ldots$). Since there are no non-trivial homomorphisms $\mathbb{Z}_p \to \mathbb{Z}/p-1$, an action of $\mathbb{Z}$ on BX determined by an Adams operation ψ^q can only be the restriction of an action of $\mathbb{Z}_p$ if $q \equiv 1 \bmod p$. On the other hand, if $q \equiv 1 \bmod p$, then, we can write $q = 1 + pm_q$ ($m_q = \frac{1}{p}\log(q)$), and the homomorphism $\omega_q \colon \mathbb{Z} \to \mathbb{Z}_p^\times$ that maps 1 to q is clearly the restriction to $\mathbb{Z}$ of the homomorphism $\bar{\omega}_q \colon \mathbb{Z}_p \to \mathbb{Z}_p^\times$ defined $\omega_q(x) = \exp(xpm_q)$.

Now, we can prove the second claim of Theorem E.

Proposition 6.5. *If $q, q' \in \mathbb{Z}_p^\times$, $q \equiv q' \bmod p$, both are of order $r \bmod p$, and $\nu_p(1-q^r) = \nu_p(1-q'^r)$, then $BX(q) \simeq BX(q')$, for any connected p-compact group X.*

Proof. The proof is divided in two steps. First, we will consider the case $q \equiv q' \equiv 1 \bmod p$ ($r = 1$). In these cases, the actions of $\mathbb{Z}$ given by ψ^q and $\psi^{q'}$, respectively, extend to actions of the p-adics described by $m_q = \frac{1}{p}\log(q)$ and $m_{q'} = \frac{1}{p}\log(q')$, respectively. The homotopy fixed point space $BX^{h\mathbb{Z}_p}$ depends only of the image of the action $\mathbb{Z}_p \to \mathrm{Out}(X)$. The image of the two actions are clearly the same if and only if m_q and $m_{q'}$ differ by a p-adic unit; that is, if and only if $\nu_p(m_q) = \nu_p(m_{q'})$, if and only if $\nu_p(1-q) = \nu_p(1-q')$, in which case, we have

$$BX(q) \simeq BX^{h\psi^q} \simeq BX^{h\mathbb{Z}_p} \simeq BX^{h\psi^{q'}} \simeq BX(q').$$

In the general case, we can decompose $q = \zeta \cdot q_0$ and $q' = \zeta \cdot q_0'$, where ζ is a primitive rth of unity and $q_0 \equiv q_0' \equiv 1 \bmod p$, thus

$$BX(q) \simeq BX^{h\zeta}(q_0) \simeq BX^{h\zeta}(q_0') \simeq BX(q'). \qquad \square$$

Remark 6.6. If q is a p-adic unit, we can find a prime number q_0 such that $q \equiv q_0 \bmod p$ and $\nu_p(1-q^r) = \nu_p(1-q_0^r)$, where r is the order of $q \bmod p$, and then,

$$BX(q) \simeq BX(q_0)$$

by Proposition 6.5.

In fact, we can assume that q is an integer, otherwise change it by the sum of enough first terms in its p-adic expansion. Then, by Dirichlet's theorem there is a prime number q_0 of the form $q_0 = p^N c + q$, with $N > \nu_p(1-q^r)$, satisfying the above conditions.

7. General structure of Chevalley p-local finite groups

In this section we will study some general properties of the spaces $BX(q)$, obtained as homotopy fixed point spaces for the action of unstable Adams operations on classifying spaces of connected p-compact groups. The main results being the identification of the maximal finite torus, the Weyl group, and the fusion category of elementary abelian p-subgroups of $BX(q)$.

Proposition 7.1. *Let X be a connected p-compact group and α a self homotopy equivalence of X. Then*
(1) *$BX^{h\alpha}$ is connected and p-complete.*
(2) *$\iota\colon BX^{h\alpha} \to BX$ is a homotopy monomorphism at p.*
(3) *For any finite p-group P, $\mathrm{Map}(BP, BX^{h\alpha})_c \simeq BX^{h\alpha}$.*

Proof. From the definition we obtain a fibration $X \to BX^{h\alpha} \xrightarrow{\iota} BX$ where X and BX are p-complete, X is connected and BX is simply-connected. It follows that $BX^{h\alpha}$ is connected and p-complete.

For any finite p-group P, $\mathrm{Map}(BP, BX)_c \simeq BX$, and $\mathrm{Map}_*(BP, X) \simeq X$ for any choice of base point. It then follows that $\iota\colon BX^{h\alpha} \to BX$ is a homotopy monomorphism at p, and from the induced fibration

$$\mathrm{Map}(BP, X) \to \mathrm{Map}(BP, BX^{h\alpha})_c \to \mathrm{Map}(BP, BX)_c$$

it follows that $\mathrm{Map}(BP, BX^{h\alpha})_c \simeq BX^{h\alpha}$. □

Lemma 7.2. *Let X be a p-compact group, α a self homotopy equivalence of BX, and (P, ν) an object of $\mathcal{F}_p(BX)$ fixed by α up to homotopy; that is, $\nu \simeq \alpha \circ \nu$. If $C_X(P, \nu)$ is connected, then there is a unique lifting of $\nu\colon BP \to BX$ to a homotopy monomorphism $g\colon BP \to BX^{h\alpha}$, and*

$$\begin{tikzcd}
\mathrm{Map}(BP, BX^{h\alpha})_g \arrow[r] \arrow[d] & \mathrm{Map}(BP, BX)_\nu \arrow[d, "\Delta"] \\
\mathrm{Map}(BP, BX)_\nu \arrow[r, "1\times\alpha_\sharp"] & \mathrm{Map}(BP, BX)_\nu \times \mathrm{Map}(BP, BX)_\nu
\end{tikzcd}$$

is a homotopy pullback diagram.

Proof. Since (22) is a homotopy pullback diagram, there is at least a lifting of ν, $g\colon BP \to BX^{h\alpha}$.

The homotopy fibre of $\mathrm{Map}(BP, BX)_\nu \xrightarrow{\Delta} \mathrm{Map}(BP, BX)_\nu \times \mathrm{Map}(BP, BX)_\nu$ is $C_X(P, \nu) = \Omega\,\mathrm{Map}(BP, BX)_\nu$, hence pulling back along $1 \times \alpha_\sharp$ we obtain a fibration, up to homotopy,

$$C_X(P, \nu) \to \mathrm{Map}(BP, BX^{h\alpha})_{\hat{\nu}} \xrightarrow{\iota_\sharp} \mathrm{Map}(BP, BX)_\nu$$

where $\mathrm{Map}(BP, BX^{h\alpha})_{\hat{\nu}}$ consists of all possible liftings of ν up to homotopy. The base space consists of just one connected component, hence if we assume that the fibre $C_X(P, \nu)$ is also connected, then the total space must be connected, and therefore any other lifting of ν is homotopic to g. □

The following lemma will help us determine the restriction of α to the centralizers.

Lemma 7.3. *Let X be a connected p-compact group and α a self-equivalence of BX. Let $T(\alpha)$ be a given restriction of α to the maximal torus $T = T_X$, and (P, ν) an object of $\mathcal{F}_p(BX)$.*

Suppose that $\nu\colon BP \to BX$ admits a factorization $\mu\colon BP \to BT$ through the maximal torus $j\colon BT \to BX$. Then, the object (P,ν) is fixed by α if and only if $T(\alpha)\mu = w\mu$ for an element w of the Weyl group. If this is the case, the restriction to the maximal torus of the induced self homotopy equivalence $\alpha|_{C_X(P,\nu)}$ of the centralizer $C_X(P,\nu)$ is $T(\alpha|_{C_X(P,\nu)}) = w^{-1}\circ T(\alpha)$.

Proof. (P,ν) is fixed by α means that $\nu \simeq \alpha\circ B\nu$, and if ν factors as $j\circ\mu$, that is to say, $j\circ B\mu \simeq \alpha\circ j\circ\mu \simeq j\circ T(\alpha)\circ\mu$, and according to [53, 4.1], [48, 3.4], this is equivalent to the existence of w, in the Weyl group of X, such that $w\circ\mu \simeq BT(\alpha)\circ\mu$.

Now assuming the existence of such element w, we read from the commutative diagram

$$\begin{tikzcd}
BT \arrow[r, "T(\alpha)"] & BT & BT \arrow[l, "w"'] \\
\mathrm{Map}(BP,BT)_\mu \arrow[u, "\simeq", "ev"'] \arrow[r, "T(\alpha)_\sharp"] \arrow[d, "\simeq"', "j_\sharp"] & \mathrm{Map}(BP,BT)_{w\mu} \arrow[u, "\simeq", "ev"'] \arrow[d, "\simeq"', "j_\sharp"] & \mathrm{Map}(BP,BT)_\mu \arrow[u, "\simeq", "ev"'] \arrow[l, "w_\sharp"'] \arrow[dl, "\simeq", "j_\sharp"'] \\
\mathrm{Map}(BP,BX)_\nu \arrow[r, "\alpha_\sharp"] & \mathrm{Map}(BP,BX)_\nu &
\end{tikzcd}$$

that the restriction of $\alpha|_{C_X(P,\nu)} = \alpha_\sharp$ to the maximal torus of $C_X(V,\nu)$ is $w^{-1}\circ T(\alpha)$. □

If the centralizer $C_X(V,\nu)$ is connected, this determines the restriction $\alpha|_{C_X(V,\nu)}$ (see Section 2).

Corollary 7.4. *Let X be a p-compact group and $\nu\colon BV \to BX$ a toral elementary abelian p-subgroup such that its centralizer $C_X(V,\nu)$ is connected. If ψ^q is an unstable Adams operation of exponent $q \equiv 1 \bmod p$, $q \neq 1$, then*

(a) *there is a unique lift of ν to $g\colon BV \to BX(q)$,*
(b) *$\psi^q|_{C_X(V,\nu)}$ is an unstable Adams operation of exponent q, and*
(c) *the centralizer of (V,g) in $X(q)$ is $C_{X(q)}(V,g) \cong C_X(V,\nu)(q)$.*

Proof. In particular, when $\nu\colon BV \to BX$ is a toral elementary abelian p-group in X and $\alpha = \psi^q$ is an Adams operation of exponent $q \equiv 1 \bmod p$, then we can write $T(\psi^q) = \psi^q$, the qth power map in the maximal torus $T = T_X$ and $\psi^q\circ\mu \simeq \mu$, where $\mu\colon BV \to BT$ is a lift to BT of $\nu\colon BV \to BX$, so, by Lemma 7.3, there is a commutative diagram

$$\begin{tikzcd}
BT \arrow[r, "T(\psi^q)=\psi^q"] \arrow[d] & BT \arrow[d] \\
BC_X(V,\nu) \arrow[r, "\psi^q|_{C_X(V,\nu)}"] \arrow[d] & BC_X(V,\nu) \arrow[d] \\
BX \arrow[r, "\psi^q"] & BX
\end{tikzcd}$$

this proves (b), namely, $\psi^q|_{BC_X(V,\nu)}$ is, as well, an unstable Adams map ψ^q.

Now, (a) and (c) follow from Lemma 7.2. □

We will now restrict our attention to cases with $q \equiv 1 \bmod p$, $q \neq 1$. According to Proposition 6.2, the general case can be reduced to this one, in the cases that are of interest to us (see Remark 6.3). Hence, essentially, there will be no loss of generality in our assumption.

Proposition 7.5. *Let X be a connected p-compact group, p an odd prime, and ψ^q an unstable Adams operation of exponent $q \in \mathbb{Z}_p^*$, with $q \equiv 1 \bmod p$, $q \neq 1$. Then the inclusion $\nu\colon Bt_X \to BX$ of the subgroup of elements of order p in the maximal torus T_X has a unique lift to $g\colon Bt_X \to BX(q)$ and its centralizer is*

$$C_{X(q)}(t_X, g) = T_X(q)\,.$$

Proof. Since $C_X(t_X,\nu) = T_X$ [48, 3.2] and $\psi^q|_{T_X} = T(\psi^q) = \psi^q$ this follows from 7.3 (see 7.4). □

The group $T_X(q) \cong T_\ell^n \cong (\mathbb{Z}/p^\ell)^n$, where n is the rank of X and $\ell = \nu_p(q-1)$, established in Proposition 7.5, embeds in $BX(q)$

$$i\colon BT_\ell^n \to BX(q)$$

as a subgroup (T_ℓ^n, i) that will be referred to as the *maximal finite torus* of $X(q)$. When no confusion is possible we will simply write T_ℓ^n for the maximal finite torus of $BX(q)$. Notice that T_ℓ^n is self-centralizing in $BX(q)$. Then, we define the Weyl group of $BX(q)$ as the automorphism group

$$W_{X(q)}(T_\ell^n, i) = \mathrm{Aut}_{\mathcal{F}_p(BX(q))}(T_\ell^n) = \left\{ \varphi \in \mathrm{aut}(T_\ell^n) \,\middle|\, i \simeq i \circ B\varphi \right\}$$

of (T_ℓ^n, i) in the category $\mathcal{F}_p(BX(q))$. It comes with a faithful representation $W_{X(q)}(T_\ell^n, i) \to GL_n(\mathbb{Z}/p^\ell)$ afforded by T_ℓ^n.

The Weyl group of $BX(q)$ can also be interpreted as the set of connected components of $\mathrm{Map}_*(BT_\ell^n, BT_\ell^n)$ that lie over the connected component of $i\colon BT_\ell^n \to BX(q)$ through the map $\mathrm{Map}_*(BT_\ell^n, BT_\ell^n) \to \mathrm{Map}(BT_\ell^n, BX(q))$. The *normalizer* of the maximal finite torus of $BX(q)$, $BN_{X(q)}(T_\ell^n)$, is defined by its classifying space, the Borel construction for the action of $W_{X(q)}(T_\ell^n)$ on $\mathrm{Map}(BT_\ell^n, BX(q))_i$, together with the inclusion

$$\bar{i}\colon BN_{X(q)}(T_\ell^n) = \big(\mathrm{Map}(BT_\ell^n, BX(q))_i\big)_{hW_{X(q)}(T_\ell^n)} \longrightarrow BX(q)$$

induced by evaluation at the base point of BT_ℓ^n.

Proposition 7.6. *Let X be a connected p-compact group, p an odd prime, and ψ^q an unstable Adams operation of exponent $q \in \mathbb{Z}_p^*$, with $q \equiv 1 \bmod p$, $q \neq 1$. If (T_ℓ^n, i) is the maximal finite torus of $BX(q)$, then its Weyl group is*

$$W_{X(q)}(T_\ell^n) \cong W_X$$

the Weyl group of X, with action on T_ℓ^n given by the mod p^ℓ reduction of the p-adic representation of W_X. The normalizer of the maximal finite torus is the split extension $N_{X(q)}(T_\ell^n) = T_\ell^n \rtimes W_{X(q)}(T_\ell^n)$, and its classifying space fits in a homotopy commutative diagram

$$\begin{tikzcd} BN_{X(q)}(T_\ell^n) \arrow[r, "\tilde{\iota}"] \arrow[d, "\bar{i}"'] & BN \arrow[d, "j"] \\ BX(q) \arrow[r, "\iota"] & BX\,. \end{tikzcd} \tag{24}$$

where $j\colon BN \to BX$ is the inclusion of the maximal torus normalizer of X.

Proof. We will first see that the automorphism in $\mathcal{F}_p(BX)$ of T_ℓ^n as a subgroup of BX via the composition $\iota \circ i\colon BT_\ell^n \to BX$, $W_X(T_\ell^n)$, coincides with the Weyl group W_X. In fact, an element $w \in W_X$ is a homotopy equivalence of BT over BX. Its restriction to BT_ℓ^n, factors again to give a homotopy equivalence $\bar{w}$ of BT_ℓ^n and a homotopy commutative diagram

$$\begin{tikzcd} BT_\ell^n \arrow[r] \arrow[d, "\bar{w}"'] & BT \arrow[d, "w"'] \arrow[dr, "j"] & \\ BT_\ell^n \arrow[r] & BT \arrow[r, "j"] & BX \end{tikzcd} \tag{25}$$

where $BT \simeq K(\mathbb{Z}_p^n, 2)$ and the map $BT_\ell^n \to BT$ classifies the estension class of the exact sequence $(\mathbb{Z}_p)^n \xrightarrow{p^\ell} (\mathbb{Z}_p)^n \to (\mathbb{Z}/p^\ell)^n$. Hence, if w is represented by a certain matrix in $GL_n(\mathbb{Z}_p)$, then $\bar{w}$ is represented by its mod p^ℓ reduction in $GL_n(\mathbb{Z}/p^\ell)$. We have produced a homomorphism $W_X(T_\ell^n) \to W_X$ which is injective because W_X is finite and mod p^ℓ reduction has torsion free kenel in $GL_n(\mathbb{Z}_p)$. Furthermore, since $\mathrm{Map}(BT_\ell^n, BX)_{\iota\circ i} \cong BT$, it turns out that every homotopy equivalence of BT_ℓ^n over BX can be extended to a diagram like (25) and therefore we actually have an isomorphism $W_X(T_\ell^n) \cong W_X$.

Next, we compare $W_{X(q)}(T_\ell^n)$ and $W_X(T_\ell^n)$. By composition with $\iota\colon BX(q) \to BX$, every homotopy equivalence $\bar{w}$ of BT_ℓ^n over $BX(q)$ can also be considered over BX

$$\begin{tikzcd} BT_\ell^n \arrow[d, "\bar{w}"'] \arrow[dr, "i"] & & \\ BT_\ell^n \arrow[r, "i"] & BX(q) \arrow[r, "\iota"] & BX \end{tikzcd}$$

what gives an inclusion $W_{X(q)}(T_\ell^n) \hookrightarrow W_X(T_\ell^n)$. Now, Lemma 7.2 implies that this is actually an isomorphism.

Finally, the natural maps

$$\mathrm{Map}(BT_\ell^n, BX(q))_i \xrightarrow{\iota_\sharp} \mathrm{Map}(BT_\ell^n, BX(q))_{\iota\circ i} \xleftarrow{\simeq} \mathrm{Map}(BT_\ell^n, BX(q))_j$$

induced by composition with $\iota\colon BX(q) \to BX$ and with the inclusion $BT_\ell^n \to BT$, respectively, are equivariant for the respective actions of $W_{X(q)}(T_\ell^n)$, $W_X(T_\ell^n)$ and W_X, respectively, induced by the natural actions on the first component. Applying the Borel construction, we obtain a map

$$BN_{X(q)}(T_\ell^n) \xrightarrow{\tilde{\iota}} \left(\mathrm{Map}(BT_\ell^n, BX(q))_{\iota\circ i}\right)_{hW_X(T_\ell^n)} \simeq BN$$

and diagram (24) is induced by evaluation at base points. Moreover this maps extends the map between classifying spaces of tori to give a diagram of fibrations

$$\begin{tikzcd} T \arrow[r, "\simeq"] \arrow[d] & T \arrow[d] & \\ BT_\ell^n \arrow[r] \arrow[d] & BN_{X(q)}(T_\ell^n) \arrow[r] \arrow[d, "\tilde{\iota}"] & BW_{X(q)}(T_\ell^n) \arrow[d, "\simeq"] \\ BT \arrow[r] & BN \arrow[r] & BW_X \end{tikzcd}$$

where $T \simeq \Omega BT \simeq K((\mathbb{Z}_p)^n, 1)$. By [6, 1.2] the bottom row fibration has a section and by [6, 3.3] this section lifts to a section of the fibration in the middle row. It follows that $N_{X(q)}(T_\ell^n)$ is a split extension. $\square$

For X a p-compact group and α a self equivalence, the *inclusion* $\iota\colon BX^{h\alpha} \to BX$ induces a functor between the respective fusion categories

$$\iota_\sharp\colon \mathcal{F}_p(BX^{h\alpha}) \longrightarrow \mathcal{F}_p(BX)$$

and Lemma 7.2 above gives some useful information in order to compare the morphism sets. Thus, for instance,

$$\mathrm{Mor}_{\mathcal{F}_p(BX^{h\alpha})}((P,g),(Q,h)) \longrightarrow \mathrm{Mor}_{\mathcal{F}_p(BX)}((P,\iota\circ g),(Q,\iota\circ h)) \tag{26}$$

is a bijection provided $C_X(P,\iota\circ g)$ is connected. It rarely happens that those centralizers are connected for a general p-group P, but it is not so unusual if we restrict to some particular classes of small groups. For a space Y, we denote $\mathcal{F}_p^e(Y)$ the full subcategory of $\mathcal{F}_p(Y)$ whose objects are the elementary abelian subgroups of Y.

Corollary 7.7. *Let p be an odd prime. If X is a connected polynomial p-compact group and α a self homotopy equivalence, then the functor*

$$\iota_\sharp\colon \mathcal{F}_p^e(BX^{h\alpha}) \longrightarrow \mathcal{F}_p^e(BX)$$

is both full and faithful.

Proof. If X is a connected polynomial p-compact group, then centralizers of elementary abelian p-subgroups are connected and Lemma 7.2 applies. In fact, if (E,ν) is an elementary abelian p-subgroup of X, then the centralizer $C_X(E,\nu)$ is also a polynomial p-compact group, hence $H^1(BC_X(E,\nu);\mathbb{F}_p)=0$ and therefore $C_X(E,\nu)$ is connected (see [28, 1.3]) and the map (26) is a bijection for every elementary abelian p-subgroups (P,g) and (Q,h) of $BX^{h\alpha}$. □

Corollary 7.8. *Let p be an odd prime. If X is a connected polynomial p-compact group and ψ^q an unstable Adams operation of exponent $q\in\mathbb{Z}_p^*$, with $q\equiv 1 \bmod p$, then*

$$\iota_\sharp\colon \mathcal{F}_p^e(BX(q)) \longrightarrow \mathcal{F}_p^e(BX)$$

is an equivalence of categories.

Proof. By Corollary 7.7 we only have to check that $\iota_\sharp$ induces in this case a bijection between isomorphism classes of objects, and this follows from Proposition 7.5, because in a polynomial p-compact group every elementary abelian subgroup is toral. □

Let X be a polynomial p-compact group with trivial center and $q\in\mathbb{Z}_p^*$ a p-adic unit with $q\equiv 1 \bmod p$, $q\neq 1$. Putting $BC_{X(q)}(V,g)=\mathrm{Map}(BV,BX(q))_g$ for any object (V,g) of $\mathcal{F}_p^e(BX(q))$ we get a functor from $\mathcal{F}_p^e(BX(q))^{op}$ to topological spaces. There is natural map

$$\operatorname*{hocolim}_{\mathcal{F}_p^e(BX(q))^{op}} BC_{X(q)} \to BX(q)$$

from the homotopy colimit of this functor. When $C_X(V,g)$ is connected, we have

$$BC_{X(q)}(V,g)\simeq BF(\psi^q|_{C_X(V,g)})(C_X(V,\iota\circ g))\simeq BC_X(V,\iota\circ g)(q)$$

according to Lemma 7.3 and Remark 7.4.

Let T_X be the maximal torus and W_X the Weyl group of a p-compact group X, p odd. As usually, we denote by t_X the group of all elements of order p in T_X, and $g\colon Bt_X\to X(q)$ the inclusion. For any nontrivial elementary abelian p-subgroup $E\leq T$, write $W_X(E)$ for the point-wise stabilizer subgroup of E.

Proposition 7.9. *Let X be a polynomial p-compact group with trivial center, p odd, and $q \in \mathbb{Z}_p^*$ a p-adic unit with $q \equiv 1 \bmod p$, $q \neq 1$. Assume that*

$$H^*(BX(q); \mathbb{F}_p) \cong H^*(BT_X(q); \mathbb{F}_p)^{W_X}$$

and that

$$H^*(BC_{X(q)}(E, g|_{BE}); \mathbb{F}_p) \cong H^*(BT_X(q); \mathbb{F}_p)^{W_X(E)}$$

for any nontrivial, subgroup E of t_X. Then, the natural map

$$\operatorname*{hocolim}_{\mathcal{F}_p^e(BX(q))^{op}} BC_{X(q)} \to BX(q) \tag{27}$$

is an $\mathbb{F}_p$-equivalence.

A similar statement holds with $\mathcal{F}_p^e(BX(q))$ replaced by the full subcategory generated by all objects of the form $(t_X)^P$ where P runs through the subgroups of a Sylow p-subgroup of W_X.

Proof. The functor from $\mathcal{F}_p^e(BX(q)) = \mathcal{F}_p^e(BX)$ to the category of $\mathbb{F}_p$-vector spaces that takes an object (V, g) to $H^*(BC_{X(q)}(V, g); \mathbb{F}_p) = H^*(BT_X(q); \mathbb{F}_p)^{W_X(E)}$ is acyclic in the sense that $\lim^0 = H^*(BT_X(q); \mathbb{F}_p)^{W_X}$ and the higher limits vanish [24, 8.1]. Therefore, the Bousfield–Kan spectral sequence for the cohomology of the homotopy colimit

$$\operatorname*{hocolim}_{\mathcal{F}_p^e(BX(q))^{op}} BC_{X(q)}$$

(see [10, XII.4.5]) collapses at E_2-term, and then, it shows that (27) is an $\mathbb{F}_p$-equivalence. The same conclusion holds if we replace the category $\mathcal{F}_p^e(BX(q))$ by its full subcategory generated by all objects of the form $(t_X)^P$ where P runs through the subgroups of a Sylow p-subgroup of W_X [52, 2.16]. $\square$

This result motivates the research in next sections of the cohomology rings $H^*(BX(q); \mathbb{F}_p)$ and the invariant rings $H^*(BT_X(q); \mathbb{F}_p)^{W_X}$.

8. Cohomology rings

This section is devoted to the proof of Theorem F. The Eilenberg-Moore spectral sequence is used in order to get a hold of the cohomology rings of the spaces $BX(q)$ of fixed points of unstable Adams operations acting on polynomial p-compact groups BX. We follow the arguments of [62] that already contain the first part of the theorem.

Proof of Theorem F. Part (1) is due to L. Smith [62]. We will sketch his arguments here and then will continue with the proof of the second part of the theorem.

There is an Eilenberg-Moore spectral sequence associated to the pullback diagram

$$\begin{tikzcd} BX(q) \arrow[r, "\iota"] \arrow[d, "\iota"] & BX \arrow[d, "\Delta"] \\ BX \arrow[r, "1\times\psi^q"] & BX \times BX\,. \end{tikzcd} \tag{28}$$

This is a second quadrant spectral sequence with

$$E_2^{s,t} \cong \mathrm{Tor}^{s,t}_{H^*(BX;\mathbb{F}_p)^{\otimes 2}}(H^*(BX; \mathbb{F}_p), H^*(BX; \mathbb{F}_p)) \Longrightarrow H^{s+t}(BX(q); \mathbb{F}_p)$$

converging to a graded ring associated of $H^*(BX(q); \mathbb{F}_p)$.

For simplicity, we will write $P[x_i] = P[x_1, \ldots, x_n] \cong H^*(BX; \mathbb{F}_p)$. The Koszul complex

$$\mathcal{E}(x_i) = P[x_i] \otimes P[x_i] \otimes E[sx_1, \ldots sx_n]$$

with $\mathrm{bideg}(sx_i) = (-1, 2d_i)$ and $d(sx_i) = x_i \otimes 1 - 1 \otimes x_i$, is a free resolution of $P[x_i]$ as $(P[x_i] \otimes P[x_i])$-module, with module structure given by the multiplication $m = \Delta^*$. Then, $\mathrm{Tor}^{**}_{P[x_i]\otimes P[x_i]}(P[x_i], P[x_i])$ is the homology of the complex

$$P[x_i] \otimes_{P[x_i]\otimes P[x_i]} \mathcal{E}(x_i) \cong P[x_i] \otimes E[sx_1, \dots sx_n]$$

where now the action of $P[x_i] \otimes P[x_i]$ on the left hand side term $P[x_i]$ in given by the algebra map $(1 \times \psi^q)^*$, hence one obtains the expression $d(sx_i) = x_i - q^{d_i} x_i$ for the differential, but since $q \equiv 1 \bmod p$, we actually have $d(sx_i) = 0$ for all $i = 1, \dots, n$. This yields

$$E_2^{**} \cong \mathrm{Tor}^{**}_{P[x_i]\otimes P[x_i]}(P[x_i], P[x_i]) \cong P[x_1, \dots, x_n] \otimes E[sx_1, \dots, sx_n]$$

and, since the algebra generators appear in filtration degrees 0 and -1, the spectral sequence collapses at the E_2-page and then we can find elements y_i in $H^*(BX(q); \mathbb{F}_p)$ representing sx_i in the graded associated ring, with

$$H^*(BX(q); \mathbb{F}_p) \cong P[x_1, \dots, x_n] \otimes E[y_1, \dots, y_n]\,.$$

Let T_X be the maximal torus of X and W_X the Weyl group. Since X is polynomial, the mod p cohomology ring of BX coincides with the invariants by the action of the Weyl group on the mod p cohomology of BT_X, $H^*(BT_X; \mathbb{F}_p)^{W_X} \cong H^*(BX; \mathbb{F}_p) \cong P[x_1, \dots, x_n]$.

According to 7.4, 7.5, the classifying space of maximal finite torus of $X(q)$ is $BT(q) \cong BT^n_\ell$ and it is obtained from a pullback diagram

$$\begin{tikzcd} BT^n_\ell \arrow[r, "\iota"] \arrow[d, "\iota"] & BT \arrow[d, "\Delta"] \\ BT \arrow[r, "1\times\psi^q"] & BT \times BT\,. \end{tikzcd} \tag{29}$$

Furthermore, the Weyl group is W_X (7.6) hence, the restriction map

$$i^*\colon H^*(BX(q); \mathbb{F}_p) \to H^*(BT^n_\ell; \mathbb{F}_p)$$

has image in the invariant subring by the action of the Weyl group, W_X. It remains to show that this restriction map is injective.

The pullback diagram (29) yields another Eilenberg-Moore spectral sequence:

$$\overline{E}_2^{s,t} \cong \mathrm{Tor}^{s,t}_{H^*(BT;\mathbb{F}_p)^{\otimes 2}}(H^*(BT; \mathbb{F}_p), H^*(BT; \mathbb{F}_p)) \Longrightarrow H^{s+t}(BT^n_\ell; \mathbb{F}_p)\,.$$

We will pay special attention to the map between the two spectral sequences $i^*\colon E_r^{**} \to \overline{E}_r^{**}$ induced by the natural map from diagram (29) to diagram (28) given by inclusion of the maximal torus. In order to describe the induced map at the level of E_2-pages, we need some elementary algebraic considerations.

Again for simplicity, we will write $P[t_i] = P[t_1, \dots, t_n] \cong H^*(BT_X; \mathbb{F}_p)$. The kernel of the multiplication $m\colon P[t_i] \otimes P[t_i] \to P[t_i]$ is a Borel ideal

$$\mathrm{Ker}\, m = (t_1 \otimes 1 - 1 \otimes t_1, \dots, t_n \otimes 1 - 1 \otimes t_n)$$

and then we can define derivations

$$\partial_i\colon P[t_i] \to P[t_i]$$

for $i = 1, \dots, n$, in the following way. For any homogeneous polynomial $f \in P[t_i]$, $f \otimes 1 - 1 \otimes f \in \mathrm{Ker}\, m$, hence we can find an expression $f \otimes 1 - 1 \otimes f = \sum_i c_i(f)(t_i \otimes 1 - 1 \otimes t_i)$, with coefficients $c_i(f) \in P[t_i] \otimes P[t_i]$, and then define $\partial_i(f) = m(c_i(f)) \in P[t_i]$. A routine calculation shows:

(1) ∂_i is well defined and does not depend on the choice of coefficients $c_1(f), \ldots, c_n(f)$,
(2) ∂_i is a derivation of $P[t_i]$, and
(3) $\partial_i(t_i) = 1$ and $\partial_i(t_j) = 0$ if $j \neq i$.

These properties show that these are the partial derivatives:

$$\partial_i(f) = \frac{\partial f}{\partial t_i}\,.$$

After these considerations we can easily describe the map between the respective E_2-pages and show that it is injective. In order to compute the $\overline{E}_2^{**}$, we define now the Koszul complex

$$\mathcal{E}(t_i) = P[t_i] \otimes P[t_i] \otimes E[st_1, \ldots st_n]$$

with $\mathrm{bideg}(st_i) = (-1, 2)$ and $d(st_i) = t_i \otimes 1 - 1 \otimes t_i$. As before, we obtain that

$$\overline{E}_2^{**} \cong \mathrm{Tor}^{**}_{P[t_i]\otimes P[t_i]}(P[t_i], P[t_i]) \cong P[t_i] \otimes_{P[t_i]\otimes P[t_i]} \mathcal{E}(t_i) \cong P[t_i] \otimes E[st_1, \ldots st_n] \tag{30}$$

since the differential in this complex turns out to be trivial, again, because $q \equiv 1 \bmod p$. Also as before, the algebra generators of $\overline{E}_2^{**}$ appear in filtration degree 0 and -1 and therefore the spectral sequence $\overline{E}_r^{**}$ collapses at the E_2-page.

Now, the inclusion $i^*\colon P[x_i] \to P[t_i]$ extends to a map of Koszul complexes

$$i^*\colon \mathcal{E}(x_i) \to \mathcal{E}(t_i)$$

which is a $P[x_i] \otimes P[x_i]$-module map defined by

$$i^*(sx_i) = \sum_j c_i(x_i) \otimes st_j$$

on generators. Then the induced map

$$\begin{aligned} i^*\colon\ \mathrm{Tor}^{**}_{P[x_i]\otimes P[x_i]}(P[x_i], P[x_i]) &\cong P[x_i] \otimes E[sx_1, \ldots sx_n] \\ &\longrightarrow \mathrm{Tor}^{**}_{P[t_i]\otimes P[t_i]}(P[t_i], P[t_i]) \cong P[t_i] \otimes E[st_1, \ldots st_n] \end{aligned}$$

is determined by

$$i^*(sx_i) = \sum_j \partial_j(x_i) \otimes st_j = \sum_j \frac{\partial x_i}{\partial t_j} \otimes st_j\,.$$

Now, i^* is injective because the jacobian determinant is non-trivial,

$$J = \det\left(\frac{\partial x_i}{\partial t_j}\right) \neq 0,$$

by [67]. Since both spectral sequences collapse at the E_2-page, it follows that the induced homomorphism $i^*\colon H^*(BX(q); \mathbb{F}_p) \to H^*(BT_\ell^n; \mathbb{F}_p)$ is also injective. □

Remark 8.1. The argument with the Eilenberg-Moore spectral sequence used in the proof of part (1) of the above Theorem applies more generally to the case of any unstable Adams operation ψ^q of arbitrary exponent $q \in \mathbb{Z}_p^*$ acting on a polynomial p-compact group (see [62]). Under these more general hypothesis we obtain that if $H^*(BX) \cong P[x_1, \ldots, x_n]$ then the cohomology of $BX(q)$ is

$$H^*(BX(q); \mathbb{F}_p) \cong P[x_{i_1}, \ldots, x_{i_k}] \otimes E[y_{i_1}, \ldots, y_{i_k}]$$

where the polynomial generators x_{i_j} correspond to those x_i with degree $2d_i = \deg x_i$ where $m|d_i$, if m is the order of $q \bmod p$, and $2d_i - 1 = \deg y_i$.

Notice that we can write $q = \zeta q'$ where ζ is an m-root of one in $\mathbb{Z}_p$ and $q' \equiv 1 \bmod p$. Hence $\psi^q = \psi^{q'} \circ \psi^\zeta$, and ψ^ζ has finite order m as automorphism of the p-compact group X. It follows from 6.2, 6.3, that $BY(q') \simeq BX(q)$ if $BY = BX^{h\psi^\zeta}$. Moreover, by Theorem B, $Y = X^{h\psi^\zeta}$ is again a polynomial p-compact group. According to Theorem F the cohomology of BY must be

$$H^*(BY;\mathbb{F}_p) \cong P[x_{i_1}, \dots, x_{i_k}]\,.$$

9. Invariant theory

Let X be a polynomial p-compact group of rank n and let q be a p-adic unit, $q \equiv 1 \bmod p$, $q \neq 1$, and $\ell = \nu_p(1-q)$. In the second part of Theorem F we obtained a monomorphism $i^*\colon H^*(BX(q);\mathbb{F}_p) \hookrightarrow H^*(BT^n_\ell;\mathbb{F}_p)^{W_X}$, where T^n_ℓ is the maximal finite torus of $BX(q)$ and W_X the Weyl group (see 7.5, 7.6). Whether or not i^* is an isomorphism, $H^*(BX(q);\mathbb{F}_p) \cong H^*(BT^n_\ell;\mathbb{F}_p)^{W_X}$, is now a question of invariant theory and this is the subject of this section.

We recollect the necessary results from invariant theory and apply that in a case by case discussion, based on the Clark-Ewing list, and restricted our cases of interest, namely:

(1) Non-modular groups. This consists of groups represented in a characteristic p that does not divide the order of the group. (Example 9.2.)
(2) Family 1 in the Clark–Ewing list. These are the symmetric groups Σ_{n+1} represented as Weyl groups of $SU(n+1)$. (Examples 9.3 and 9.4.)
(3) Family 2a en the Clark–Ewing list. The groups $G(m,r,n)$, $r|m|(p-1)$, $m > 1$, $(m,r,n) \neq (m,m,2)$. (Example 9.5.)
(4) Family 2b en the Clark–Ewing list. Dihedral groups $D_{2m} = G(m,m,2)$, $m \geq 3$. (Example 9.6.)
(5) The Aguadé family. These are the groups G_{12}, G_{29}, G_{31}, and G_{34} in the Clark-Ewing list (see [1]). (Example 9.7.)

We obtain that $i^*\colon H^*(BX(q);\mathbb{F}_p) \hookrightarrow H^*(BT^n_\ell;\mathbb{F}_p)^{W_X}$ is an isomorphism in all cases except for Σ_3 at the prime 3 (included in class (2) above) and W_{G_2}, the Weyl group of G_2 and G_{12}, at the prime 3. It is also excluded the case 2b with $m = 3$ and $p = 3$, that corresponds to $PU(3)$ at prime 3.

From here one easily derives the structure of $BX(q)$ for Clark–Ewing p-compact groups and this is done in Theorem 9.8. The Aguadé family and 2a family are our cases of main interest and the discussion is postponed to sections 10 and 11, respectively. All of the other cases correspond to compact Lie groups.

At the end of the section we illustrate this methods with some examples going from 9.9 to 9.13.

Continuing with the notation of the preceding section we write $V = t_X$ for the elements of order p in the maximal finite torus and identify the dual vector space with the two dimensional primitive elements in the cohomology of BT^n_ℓ, $V^* \cong PH^2(BT^n_\ell;\mathbb{F}_p)$. The Bockstein operations provide a vector space isomorphism $PH^2(BT^n_\ell;\mathbb{F}_p) \cong H^1(BT^n_\ell;\mathbb{F}_p)$, that we will denote as $d\colon V^* \to dV^*$, of degree (-1). If $P(V^*)$ is the symmetric algebra on V^* and $E(dV^*)$ the exterior algebra on dV^*, we can describe the algebra structure of $H^*(BT^n_\ell;\mathbb{F}_p)$ as

$$K(V^*) = P(V^*) \otimes E(dV^*) = P[x_1, \dots, x_n] \otimes E[dx_1, \dots, dx_n]\,,$$

and d extends to an algebra derivation on $K(V^*)$. Moreover, any subgroup $G \leq GL(V)$ of linear substitutions acts on $K(V^*)$ in a natural way that commutes with the derivation d, hence $K(V^*)^G$ is still a differential algebra.

Assume that the ring of invariants, $P(V^*)^G = P[\rho_1,\dots,\rho_n]$ is a polynomial algebra; in particular, G is a pseudoreflection group. Then $d\rho_1,\dots,d\rho_n$ are also invariant under the action of G. The purpose of the next theorem is to establish the cases in which $\{\rho_1,\dots,\rho_n,d\rho_1,\dots,d\rho_n\}$ is a free system of generators for $K(V^*)^G$.

An element f of $P(V^*)$ is *invariant relative to* $\det^{-1}$ if $g\cdot f = \det^{-1}(g)f$ for all $g\in G\le GL(V)$. The subspace of relative invariant elements, $P(V^*)^G_{\det^{-1}}$, is a module over the invariant ring $P(V^*)^G$. In fact, $P(V^*)^G_{\det^{-1}} = f_{\det^{-1}}\cdot P(V^*)^G$ is a free module on one generator $f_{\det^{-1}}\in P(V^*)$, unique up to an invertible of $\mathbb{F}_p$ [16]. For instance, if we write $d\rho_i = \sum_{j=1}^n a_{ij}dx_j$, then the jacobian $J=\det(a_{ij})\in P(V^*)$, of degree $\deg J = \sum_{i=1}^n(\deg\rho_i - 2)$, is invariant relative to $\det^{-1}$. In particular, $f_{\det^{-1}}$ divides J in $P(V^*)$ and $\deg f_{\det^{-1}} \le \sum_{i=1}^n(\deg\rho_i-2)$

Theorem 9.1 ([11])**.** *Let V be a vector space of dimension n over a field of characteristic $p\neq 2$. Assume that $G\le GL(V)$ is a group of linear substitutions such that $P(V^*)^G = P[\rho_1,\dots,\rho_n]$ is a polynomial algebra, then*

$$K(V^*)^G = P[\rho_1,\dots,\rho_n]\otimes E[d\rho_1,\dots,d\rho_n]$$

if and only if $f_{\det^{-1}}$ has degree $\deg f_{\det^{-1}} = \sum_{i=1}^n(\deg\rho_i-2)$.

Proof. Since $P(V^*)^G = P[\rho_1,\dots,\rho_n]$ is a polynomial ring of invariants, the Jacobian is nonzero, $J\neq 0$ (see [67]), and this implies that the homomorphism $P[\rho_1,\dots,\rho_n]\otimes E[d\rho_1,\dots,d\rho_n]\to K(V^*)$ defined from the free anticommutative algebra to the subalgebra of $K(V^*)^G$ by mapping the variable ρ_i to the polynomial ρ_i of $P(V^*)^G$ and $d\rho_i$ to the differential of ρ_i in $K(V^*)$ is injective.

If $I=(i_1,\dots,i_k)$ is an ordered sequence of integers $1\le i_1<\cdots<i_k\le n$, we write $d\rho_I = d\rho_{i_1}d\rho_{i_2}\dots d\rho_{i_k}$ and also $dx_I = dx_{i_1}dx_{i_2}\dots dx_{i_k}$. Let $FP(V^*)$ be the graded field of fractions of $P(V^*)$. Then, $FK(V^*) = FP(V^*)\otimes_{P(V^*)}K(V^*)$ is a vector space over $FP(V^*)$ spanned by $\{dx_I\}_I$. And then, $\{d\rho_I\}_I$ is also a base of $FK(V^*)$.

Assume that $\deg f_{\det^{-1}} = \sum_{i=1}^n(\deg\rho_i-2)$. This is the degree of the Jacobian J, hence $J=f_{\det^{-1}}$, up to an invertible of $\mathbb{F}_p$. Let $w\in K(V^*)^G$ be an arbitrary element. We can write $w=\sum_I w_I d\rho_I$, with $w_I\in FP(V^*)$ and then we will show that actually, for each index I, $w_I\in P(V^*)$. We choose I_0 of minimal length such that $w_{I_0}\neq 0$. Let I_0' be the complementary sequence, then

$$w\,d\rho_{I_0'} = w_{I_0}d\rho_{I_0}d\rho_{I_0'} = \pm w_{I_0}d\rho_1\dots d\rho_n = \pm w_{I_0}Jdx_1\dots dx_n$$

is an element of $K(V^*)^G$, and, since $dx_1\dots dx_n$ is invariant relative to det, $w_{I_0}J\in P(V^*)^G_{\det^{-1}} = f_{\det^{-1}}P(V^*)^G$. So, our assumption implies that $w_{I_0}\in P(V^*)^G$. Now we can repeat the argument with $w - w_{I_0}d\rho_{I_0}\in K(V^*)^G$. It follows that each w_I belongs to $P(V^*)^G$ and then $w\in P[\rho_1,\dots,\rho_n]\otimes E[d\rho_1,\dots,d\rho_n]$.

Assume otherwise that $\deg f_{\det^{-1}}\neq\sum_{i=1}^n(\deg\rho_i-2)$; that is, $J=\iota f_{\det^{-1}}$ for some element $\iota\in P(V^*)^G$ of positive degree, then

$$w = \frac{d\rho_1\dots d\rho_n}{\iota} = f_{\det^{-1}}dx_1\dots dx_n$$

is an element of $K(V^*)^G$ which does not belong to $P[\rho_1,\dots,\rho_n]\otimes E[d\rho_1,\dots,d\rho_n]$. $\square$

In the examples below, we explore the invariants $K(V^*)^G$ for all groups G in the Clark-Ewing list that have polynomial invariants. We proceed by looking at the families.... and isolate the cases.... the only ones where $K(V^*)^G$ is not free graded anticommutative algebra.

Example 9.2 (G a non-modular group [3])**.** If $G \leq GL(V)$ is a pseudoreflection group of order not divisible by p, then it is known that $P(V^*)^G = P[\rho_1, \ldots, \rho_n]$ is a polynomial algebra and also that the degree of $f_{\det^{-1}}$ is twice the number of pseudoreflections of G. On the other hand, the number of pseudoreflection is G is $\sum_{i=1}^n(\frac{\deg \rho_i}{2} - 1)$. Hence $\deg f_{\det^{-1}} = \sum_{i=1}^n(\deg \rho_i - 2)$ and then Theorem 9.1 implies

$$K(V^*)^G = P[\rho_1, \ldots, \rho_n] \otimes E[d\rho_1, \ldots, d\rho_n] \,.$$

For a group $G \leq GL(V)$ we denote $[x] = \{gx \mid g \in G\}$ the orbit of an element $x \in V^*$. The coefficients c_i of the polynomial $\prod_{y \in [x]}(X - y) = X^m + c_1 X^{m-1} + \cdots + c_{m-1}X + c_m$ are the Chern classes of the orbit $[x]$ and belong to $P(V^*)^G$. The element $c_m = \prod_{y\in[x]} y$ is also called the Euler element of $[x]$. If we choose just one element $z_L \in L \cap [x]$ for each 1-dimensional vector subspace L of V^* that intersects the orbit $[x]$ non trivially, $E[x] = \prod z_L$ is the pre-Euler element of the orbit $[x]$, defined up to a non-zero escalar. This is a relative invariant respect a linear character χ of G that we can associate to the orbit $[x]$ by the equation $g(E[x]) = \chi(g) \cdot E[x]$, for all $g \in G$. (See [11, 16].)

Example 9.3 (Family 1 in the Clark-Ewing list: Σ_{n+1}, except Σ_3 at $p = 3$)**.** The symmetric group Σ_{n+1} acts on the integral lattice of $SU(n+1)$ that we can describe as $V = \mathbb{Z}\{(\hat{t}_1 - \hat{t}_{n+1}), (\hat{t}_2 - \hat{t}_{n+1}), \ldots, (\hat{t}_n - \hat{t}_{n+1})\}$ where Σ_{n+1} permutes the letters $\hat{t}_1, \ldots \hat{t}_{n+1}$. Dually, V^* is generated by classes $t_1, t_2, \ldots, t_n$, and Σ_{n+1} permutes $t_1, t_2, \ldots, t_n, t_{n+1}$ with the relation $t_1 + t_2 + \cdots + t_n + t_{n+1} = 0$.

The orbit of t_1 is $[t_1] = \{t_1, t_2, \ldots, t_n, t_{n+1}\}$, and the Chern classes of this orbit, obtained as the coefficients of the polynomial $\prod_{i=1}^{n+1}(X - t_i)$, are, up to a sign, the generators c_i of the invariant ring $P(V^*)^{\Sigma_{n+1}} = P[c_2, \ldots, c_{n+1}]$.

The orbit of $t_1 - t_2$ is

$$\begin{aligned}
[t_1 - t_2] &= \{(t_i - t_j) \mid 1 \leq i, j \leq n+1, i \neq j\} \\
&= \{\pm(t_i - t_j) \mid 1 \leq i \lneq j \leq n+1\} \\
&= \{\pm(t_i - t_j) \mid 1 \leq i \lneq j \leq n\} \cup \{\pm(t_1 + \cdots + 2t_i + \cdots + t_n) \mid 1 \leq i \leq n\} \,,
\end{aligned}$$

thus the pre-Euler element associated to this orbit is

$$E = E[t_1 - t_2] = \prod_{1 \leq i \lneq j \leq n} (t_i - t_j) \prod_{1 \leq i \leq n} (t_1 + \cdots + 2t_i + \cdots + t_n) \,.$$

Notice here the exception $n = 2$ at $p = 3$, in which case $E[t_1 - t_2] = (t_1 - t_2)$. With this exception, we can check that the linear character associated to the pre-Euler element is precisely the determinant ($\det = \det^{-1}$ in this case) and also that the degree of E, $n^2 + n$, coincides with the degree of the jacobian J. Thus for $(n, p) \neq (2, 3)$, we have

$$K(V^*)^{\Sigma_{n+1}} = P[c_2, \ldots, c_{n+1}] \otimes E[dc_2, \ldots, dc_{n+1}] \,.$$

Example 9.4 (Σ_3 at the prime 3)**.** The integral lattice of $SU(3)$ is $\pi_2(T_{SU(3)}) = \mathbb{Z}\{(\hat{t}_1 - \hat{t}_3), (\hat{t}_2 - \hat{t}_3)\}$ with the action of Σ_3 that permutes $\hat{t}_1$, $\hat{t}_2$, and $\hat{t}_3$. If Σ_3 is generated by the 3-cycle σ and the transposition τ, the representation afforded by $\pi_2(BT_{SU(3)})$ is determined by

$$\sigma \mapsto \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} , \quad \tau \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

The dual action in mod 3 cohomology $V^* = H^2(BT_{SU(3)};\mathbb{F}_3) = \mathbb{F}_3\{t_1,t_2\}$ gives $P(V^*)^{\Sigma_3} \cong P[x_4,x_6]$, where $x_4 = {t_1}^2 + t_1t_2 + {t_2}^2$ and $x_6 = t_1t_2(t_1+t_2)$. This is the particular case of Example 9.3 with $n=2$ at the prime 3.

The action extends to $K(V^*) = P[t_1,t_2]\otimes E[dt_1,dt_2]$ where we obtain invariant elements

$$\begin{aligned} y_3 &= dx_4 = (t_2-t_1)dt_1 + (t_1-t_2)dt_2 \\ y_5 &= dx_6 = ({t_2}^2 - t_1t_2)dt_1 + ({t_1}^2 - t_1t_2)dt_2 \end{aligned}$$

and

$$y_4 = (t_2-t_1)dt_1dt_2$$

so that

$$y_3y_5 = ({t_1}^2+t_1t_2+{t_2}^2)(t_2-t_1)dt_1dt_2 = x_4y_4\,.$$

These elements together with the polynomial invariants generate the invariant ring $K(V^*)^{\Sigma_3}$:

$$K(V^*)^{\Sigma_3} \cong \frac{P[x_4,x_6]\otimes E[y_3,y_4,y_5]}{(y_3y_5 - x_4y_4,\ y_3y_4,\ y_4y_5)}\,. \tag{31}$$

The proof follows the method of Theorem 9.1. In this particular case $1, dt_1, dt_2, dt_1dt_2$ is a basis of $K(V^*)$ as a free $P(V^*)$-module, while $1, y_3, y_5, y_3y_5$ or $1, y_3, y_4, y_5$ are basis of $FK(V^*)$ as graded $FP(V^*)$ vector spaces.

Assume that w is an element of $K(V^*)^{\Sigma_3}$ of even degree. We can write $w = w_0 + w_1y_4$, with $w_0, w_1 \in FP(V^*)$. First, multiply the equality by y_4: $wy_4 \in K(V^*)^{\Sigma_3}$ and $wy_4 = w_0y_4 = w_0(t_2-t_1)dt_1dt_2$. Then, $w_0(t_2-t_1) \in P(V^*)^{\Sigma_3}_{\det^{-1}} = (t_2-t_1)P(V^*)^{\Sigma_3}$, hence $w_0 \in P(V^*)^{\Sigma_3}$. Now, we also have $w_1y_4 \in K(V^*)^{\Sigma_3}$, hence the same argument implies that $w_1 \in P(V^*)^{\Sigma_3}$.

Next, assume that w is an element of $K(V^*)^{\Sigma_3}$ of odd degree. In this case, $w = w_2y_3 + w_3y_5$ with $w_2, w_3 \in FP(V^*)$. If we multiply this equality by $y_5 \in K(V^*)^{\Sigma_3}$ we get $wy_5 \in K(V^*)^{\Sigma_3}$ and $wy_5 = w_2y_3y_5 = w_2x_4y_4$, and then again the equality $w_2x_4y_4 = w_2x_4(t_2-t_1)dt_1dt_2 \in K(V^*)^{\Sigma_3}$ implies that $w_2x_4 \in P(V^*)^{\Sigma_3}$. Since $P(V^*)^{\Sigma_3} = P[x_4,x_6]$, we can write $w_2 = q_2 + \lambda\frac{x_6^r}{x_4}$, $q_2 \in P(V^*)^{\Sigma_3}$ and $\lambda \in \mathbb{F}_3$, $r \geq 0$. A similar argument, in which we multiply w by y_3, implies that $w_3 = q_3 + \mu\frac{x_6^s}{x_4}$, $q_3 \in P(V^*)^{\Sigma_3}$ and $\mu \in \mathbb{F}_3$, $s \geq 0$. If we substitute these expressions in $w = w_2y_3 + w_3y_5$ we can easily check that this element can only belong to $K(V^*)$ provided $\lambda = \mu = 0$. It follows that $w_2 = q_2 \in P(V^*)^{\Sigma_3}$ and $w_3 = q_3 \in P(V^*)^{\Sigma_3}$. This proves the isomorphism (31).

Example 9.5 (Family 2a in the Clark-Ewing list: $G = G(m,r,n)$, $r|m|p-1$, [11])**.** $G(m,r,n)$ is the subgroup of $GL_n(\mathbb{Z}_p)$ generated by the permutation matrices and the diagonal matrices $\mathrm{diag}(\theta_1,\ldots,\theta_n)$, where $\theta_i^m = 1$ and $(\theta_1\ldots\theta_n)^{\frac{m}{r}} = 1$. In particular, $G(m,1,n)$ is isomorphic to the semidirect product $(\mathbb{Z}/m)^n \rtimes \Sigma_n$. In this case we clearly have $P(V^*)^{G(m,1,n)} = P[\rho_1,\ldots,\rho_n]$, where $1+\rho_1+\cdots+\rho_n = \prod_{i=1}^n(1+x_i^m)$, if we write $P(V^*) = P[x_1,\ldots,x_n]$. Now, $\rho_n = (x_1\ldots x_n)^m$ is the Euler element associated to the orbit of x_1, $[x_1]$. The pre-Euler element is $E_1 = E[x_1] = x_1\ldots x_n$. It carries an associated linear character χ_1, defined by $\chi_1(\mathrm{diag}(\theta_1,\ldots,\theta_n)) = \theta_1\ldots\theta_n$ and $\chi_1(\sigma) = 1$ if $\sigma\in\Sigma_n$ is a permutation matrix. Notice that $G(m,r,n) = \operatorname{Ker}\chi_1^{\frac{m}{r}}$ and

$$P(V^*)^{G(m,r,n)} = P[\rho_1,\ldots,\rho_{n-1},E_1^{\frac{m}{r}}]\,.$$

The orbit of (x_1-x_2) is $[x_1-x_2] = \{\,\theta_1x_i - \theta_2x_j \,|\, \theta_1^m = \theta_2^m = 1\,, i<j\,\}$ and its pre-Euler element is $E_2 = \prod_{i<j}(x_i^m - x_j^m)$. In this case the associated character is χ_2 defined by

$\chi_2(\mathrm{diag}(\theta_1,\dots,\theta_n)) = 1$ and $\chi_2(\sigma) = \mathrm{sg}(\sigma)$ is the sign of the permutation. We clearly have $\det = \chi_1\chi_2$ and then $\det^{-1} = \chi_1^{\frac{m}{r}-1}\chi_2$. It follows that $f_{\det^{-1}} = E_1^{\frac{m}{r}-1}E_2$. Counting degrees, we obtain $\sum_{i=1}^{n-1}(\deg\rho_i - 2) + \deg(E_1^{\frac{m}{r}}) - 2 = \sum_{i=1}^{n-1}(2im-2) + 2n\frac{m}{r} - 2 = n(n-1)m + 2n(\frac{m}{r}-1) = \deg f_{\det^{-1}}$. Hence, Theorem 9.1 implies

$$K(V^*)^{G(m,r,n)} = P[\rho_1,\dots,\rho_{n-1},E_1^{\frac{m}{r}}] \otimes E[d\rho_1,\dots,d\rho_{n-1},d(E_1^{\frac{m}{r}})]\,.$$

Example 9.6 (D_{12} at the prime 3)**.** In family 2b there are two modular cases at odd primes, namely, D_6 and D_{12} at $p = 3$. The first one is the Weyl group of $PU(3)$ which is not polynomial at $p = 3$, the second case corresponds to the Weyl group of the exceptional Lie group G_2. The action of D_{12} on $\pi_2(BT_{G_2})$ gives a representation

$$\omega \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad \tau \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The dual action in mod 3 cohomology $V^* = H^2(BT_{G_2};\mathbb{F}_3) = \mathbb{F}_3\{t_1,t_2\}$ gives $P(V^*)^{D_{12}} \cong P[x_4,x_{12}]$, where $x_4 = {t_1}^2 + t_1t_2 + {t_2}^2$ and $x_{12} = (t_1t_2(t_1+t_2))^2$.

The extension of this action to $K(V^*)$ gives now

$$K(V^*)^{D_{12}} \cong \frac{P[x_4,x_{12}] \otimes E[y_3,y_{10},y_{11}]}{(y_3y_{11} - x_4y_{10},\ y_3y_{10},\ y_{10}y_{11})}\,. \tag{32}$$

with elements $y_3 = dx_4$ and $y_{11} = dx_{12}$, so that $y_3y_{11} = ({t_1}^2 + t_1t_2 + {t_2}^2)t_1t_2({t_1}^2 - {t_2}^2)dt_1dt_2 = x_4y_{10}$, which serves as definition for y_{10}. The isomorphism (32) is proved with same arguments of Example 9.4.

Actually, the inclusion of $SU(3)$ as maximal subgroup of G_2, induces an inclusion $\Sigma_3 \hookrightarrow D_{12}$, identifying the generator τ and σ with ω^2. The induced inclusion $K(V^*)^{D_{12}} \hookrightarrow K(V^*)^{\Sigma_3}$ identifies the generators x_4 and y_3 and maps x_{12} to ${x_6}^2$, y_{10} to $-x_6y_4$ and y_{11} to $-x_6y_5$.

Example 9.7 (G_{12}, G_{29}, G_{31}, and G_{34} in the Clark–Ewing list at modular primes)**.** The groups G_{12} (rank 2, p=3), G_{29} (rank 4, p=5), G_{31} (rank 4, p=5), and G_{34} (rank 6, p=7), of the Clark-Ewing list have polynomial invariants [1, 2, 68].

We obtain by direct calculation that the generator of the $\det^{-1}$-relative invariants $f_{\det^{-1}}$ has the same degree as the corresponding jacobian in cases G_{29}, G_{31}, and G_{34}, and then Theorem 9.1 applies.

The case $G_{12} = GL(2,3)$ is special. Notice that all those groups contain a copy of the symmetric group of the same rank affording the representation of Example 9.3. G_{12} contains Σ_3 as described in Example 9.4. The invariant ring $K(V^*)^{GL(2,3)}$ was computed by Mui [55] (alternatively, use the arguments in Example 9.4):

$$K(V^*)^{GL(2,3)} \cong \frac{P[x_{12},x_{16}] \otimes E[y_{10},y_{11},y_{15}]}{(y_{11}y_{15} - x_{16}y_{10},\ y_{10}y_{11},\ y_{10}y_{15})}$$

where

$$x_{12} = \frac{(t_1{t_2}^9 - t_2{t_1}^9)}{(t_1{t_2}^3 - t_2{t_1}^3)}, \quad x_{16} = \mathcal{P}^1(x_{12}) = \frac{({t_1}^3{t_2}^9 - {t_2}^3{t_1}^9))}{(t_1{t_2}^3 - t_2{t_1}^3)},$$

$y_{11} = dx_{12}$, $y_{15} = dx_{16}$, and y_{10} is defined by the relation $y_{11}y_{15} = x_{16}y_{10}$.

We can easlily obtain the description of the inclusion

$$K(V^*)^{GL(2,3)} \hookrightarrow K(V^*)^{\Sigma_3}$$

as

$$R\colon \frac{P[x_{12},x_{16}]\otimes E[y_{10},y_{11},y_{15}]}{(y_{11}y_{15}-x_{16}y_{10},\ y_{10}y_{11},\ y_{10}y_{15})} \longrightarrow \frac{P[x_4,x_6]\otimes E[y_3,y_4,y_5]}{(y_3y_5-x_4y_4,\ y_3y_4,\ y_4y_5)}$$

mapping

$$\begin{aligned} x_{12} &\mapsto {x_4}^3+{x_6}^2\,,\\ x_{16} &\mapsto {x_6}^2x_4\,,\\ y_{15} &\mapsto {x_6}^2y_3-x_4x_6y_5\,,\\ y_{11} &\mapsto -x_6y_5\,,\ \text{and}\\ y_{10} &\mapsto x_6y_4\,. \end{aligned}$$

Let X be a Clark–Ewing p-compact group; that is, a connected p-compact group for which p does not divide the order of the Weyl group. Models for these p-compact groups were constructed by Clark–Ewing [18]. If W_X is the Weyl group of X, the action of W_X on the maximal torus T_X is determined by the induced representation $\rho\colon W_X\to GL_n(\mathbb{Z}_p)$, where n is the rank of X. This representation gives W_X the structure of a pseudo-reflection group, thus product of irreducibles listed in [18]. It turns out that $BX\simeq (BT_{hW_X})^\wedge_p$, where the action of W_X on BT is given by ρ [23]. Our next result is a similar description of $X(q)$, for $q\equiv 1 \bmod p$.

Theorem 9.8. *Let X be a Clark–Ewing p-compact group and $q\equiv 1 \bmod p$, $q\neq 1$, then*

$$BX(q)\simeq \big((BT_X(q))_{hW_X}\big)^\wedge_p \simeq B(T^n_\ell\rtimes W_X)^\wedge_p$$

with $T^n_\ell\cong(\mathbb{Z}/p^\ell)^n$, where n is the rank of X and $\ell=\nu_p(q-1)$.

Proof. In Proposition 7.5 we have obtained a map $BT^n_\ell \xrightarrow{\simeq} \mathrm{Map}(BV,BX(q))_{Bi}\to BX(q)$ and according to Proposition 7.6 we have a factorization

$$BT^n_\ell\simeq \mathrm{Map}(BV,BX(q))_{Bi}\longrightarrow \big(\mathrm{Map}(BV,BX(q))_{Bi}\big)_{hW_X}\longrightarrow BX(q)\,.$$

The induced maps in cohomology are

$$H^*(BX(q);\mathbb{F}_p)\longrightarrow H^*((BT^n_\ell)_{hW_X})\xrightarrow{\cong} H^*(BT^n_\ell)^{W_X}$$

where the second arrow is an isomorphism because the order of W_X is coprime to p and the composition is a monomorphism by Theorem F.

According to theorems F and 9.1, $H^*(BX(q);\mathbb{F}_p)$ and $H^*(BT^n_\ell)^{W_X}$ has the same Poincaré series, hence $H^*(BX(q);\mathbb{F}_p)\cong H^*((BT^n_\ell)_{hW_X})$ and the result follows. ☐

Example 9.9 ($BSU(2)$ at odd primes)**.** The Weyl group of $SU(2)$ is $\mathbb{Z}/2$ that acts on the maximal torus $S^1\subset\mathbb{C}$ by sign change, that is, as ψ^{-1}. Then, Theorem 9.8 applies. All spaces will be considered completed at p.

Let ψ^q be an Adams map of exponent $q\in\mathbb{Z}_p^*$, $q\neq 1$. For $q\equiv 1 \bmod p$, define $\ell=\nu_p(1-q)$, and then $BSU(2)(q)$ has maximal finite torus $\mathbb{Z}/p^\ell$ and Weyl group $\mathbb{Z}/2$, acting by sign change, so

$$BSU(2)(q)\simeq (B\mathbb{Z}/p^\ell)_{h\mathbb{Z}/2}$$

is an equivalence at the prime p. In case $q\equiv -1 \bmod p$, we can write $\psi^q=\psi^{-1}\circ\psi^{-q}$, with ψ^{-1} in the Weyl group and $-q\equiv 1 \bmod p$, in which case we should define $\ell=\nu_p(1+q)$, and the above equivalence holds.

Notice that if $q \not\equiv \pm 1 \bmod p$, then we can write $\psi^q = \psi^\zeta \circ \psi^{q'}$, where ζ is a $(p-1)$th root of 1, different than ± 1, and $q' \equiv 1 \bmod p$. Then, by Proposition 6.2, $BSU(2)(q) \simeq BSU(2)^{h\psi^\zeta}(q')$, and according to Proposition A.5 (see A.8), $BSU(2)^{h\psi^\zeta}$ is trivial, hence $BSU(2)(q)^\wedge_p$ is also trivial.

For q a prime power, coprime to p, $SU(2)(q)$ is equivalent at p to the finite Chevalley group $SU_2(q)$. This agrees with the above calculations, for in any case $\ell = \nu_p(1-q^2)$.

Example 9.10 (Sullivan spheres S^{2m-1}, $m \mid p-1$)**.** This generalizes the previous example. When $m \geq 2$ divides $p-1$, the cyclic group C_m of order m acts on $\mathbb{Z}/p^\infty$. The Sullivan sphere BS^{2m-1} is the p-completion of the classifying space of the semi-direct product $\mathbb{Z}/p^\infty \rtimes C_m$ for this action and $H^*(BS^{2m-1};\mathbb{F}_p) = P[x_{2m}]$. If u is any p-adic unit then

$$BS^{2m-1}(u) = \begin{cases} \Lambda(BS^{2m-1}) & u^m = 1 \\ B(\mathbb{Z}/p^\ell \rtimes C_m) & u^m \neq 1, u^m \equiv 1 \bmod p,\ \ell = \nu_p(u^m-1) \\ * & u^m \not\equiv 1 \bmod p \end{cases}$$

All spaces are understood to be completed at p. To see this, note that $BS^{2m-1}(u)$ is contractible if $u^m \not\equiv 1 \bmod p$ by Theorem B. Otherwise, if $u^m \equiv 1 \bmod p$, then $u = \zeta q$ with $\zeta \in C_m \subset C_{p-1}$, $q \equiv 1 \bmod p$ and $BS^{2m-1}(u) = (BS^{2m-1})^{h\langle\zeta\rangle}(q) = BS^{2m-1}(q) = B(\mathbb{Z}/p^\ell \rtimes C_m)$ by Proposition 6.2, A.8, and Theorem 9.8, because $\nu_p(q-1) = \nu_p(q^m-1) = \nu_p(u^m-1)$.

Example 9.11 ($SU(3)(q)$ at the prime 3)**.** Fix q a 3-adic integer with $0 < \ell = \nu_3(1-q) < \infty$. According to Theorem F

$$H^*(SU(3)(q);\mathbb{F}_3) \cong P[x_4,x_6] \otimes E[y_3,y_5]\,,$$

with $\beta_{(\ell)}(y_3) = x_4$ and $\beta_{(\ell)}(y_5) = x_6$.

According to propositions 7.5 and 7.6, $T^2_\ell \cong (\mathbb{Z}/3^\ell)^2$ is the maximal finite torus of $SU(3)(q)$ with Weyl group Σ_3. Now, the invariant ring

$$H^*(T^2_\ell;\mathbb{F}_3)^{\Sigma_3} \cong \frac{P[x_4,x_6]\otimes E[y_3,y_4,y_5]}{(y_3y_5 - x_4y_4,\ y_3y_4,\ y_4y_5)}$$

computed in Example 9.4, turns out to differ from $H^*(SU(3)(q);\mathbb{F}_3)$. The natural map $H^*(SU(3)(q);\mathbb{F}_3) \hookrightarrow H^*(T^2_\ell;\mathbb{F}_3)^{\Sigma_3}$ (see Theorem F) has cokernel isomorphic to $P[x_6]y_4$.

Example 9.12 (G_2 at the prime 3)**.** The exceptional Lie group G_2 has rank two and the Weyl group is dihedral $D_{12} \cong \Sigma_3 \times C_2$, listed in family 2b for $m = 6$ in the Clark-Ewing list. The category $\mathcal{F}^e_3(G_2)$ of non-trivial elementary abelian 3-subgroups of G_2 has an isomorphism class of rank two elementary abelian 3-subgroups with automorphism group D_{12}, the Weyl group of G_2, and two classes of elementary abelian 3-subgroups of rank one, with automorphism group of order two. It is equivalent to the category $\mathbb{I}(2)$ of Appendix B, with $G = D_{12}$, $H_1 = \Sigma_3$, and $H_2 = \Sigma_2$. The centralizer diagram for elementary abelian 3-subgroups is equivalent to

$$\begin{tikzcd} C_2 \circlearrowright BSU(3) & BT^2 \arrow[l, "(\Sigma_3)^{op}\backslash(D_{12})^{op}"'] \arrow[r, "(\Sigma_2)^{op}\backslash(D_{12})^{op}"'] \arrow[loop below, "(D_{12})^{op}"] & BU(2) \circlearrowleft C_2 \end{tikzcd} .$$

By Corollary 7.8 the categories of non-trivial elementary abelian 3-subgroups of G_2 and $G_2(q)$ coincide: $\mathcal{F}^e_3(G_2(q)) \cong \mathcal{F}^e_3(G_2)$, and furthermore, for every object (E,ν) of $\mathcal{F}^e_p(G_2)$,

$BC_{G_2(q)}(E,\nu) \simeq BC_{G_2}(E,\nu)(q)$, thus the centralizer diagram of elementary abelian subgroups of $G_2(q)$ is equivalent to

$$\begin{tikzcd} C_2 \arrow[loop left] BSU_3(q) & BT^2_\ell \arrow[l, "(\Sigma_3)^{op}\backslash(D_{12})^{op}"'] \arrow[r, "(\Sigma_2)^{op}\backslash(D_{12})^{op}"'] \arrow[loop below, "(D_{12})^{op}"] & BU_2(q) \arrow[loop right] C_2 \end{tikzcd} \tag{33}$$

and there is a natural map $\operatorname{hocolim}_{\mathcal{F}_p^e(G_2(q))^{op}} BC_{G_2(q)} \to BG_2(q)$ which is a a sharp homology decomposition [20]; that is, the Bousfield–Kan spectral sequence for the homotopy colimit collapses at the E_2-term and gives

$$H^*(G_2(q);\mathbb{F}_3) \cong \varprojlim_{\mathcal{F}_p^e(G_2(q))}{}^0 \; H^*(BC_{G_2(q)};\mathbb{F}_3) \cong P[x_4,x_{12}] \otimes E[y_3,y_{11}]\,.$$

This result can also be obtained by direct calculation from Proposition B.1 using the invariant theory calculations in Examples 9.4 and 9.6. Notice that Proposition 7.9 does not apply to $BG_2(q)$ at the prime 3 (see Example 9.11).

Example 9.13 (G_2 at primes $p > 3$)**.** Let p be a prime > 3. The compact Lie group G_2, at the prime p, is a Clark–Ewing p-compact group and $H^*(BG_2;\mathbb{F}_p) = P[x_4,x_{12}]$. The Weyl group has order $|W(G_2)| = 12$ and the center is cyclic of order two. Let $u \neq \pm 1$ be a p-adic unit, and let r denote the order of $u \bmod p$. Then

$$BG_2(u) = \begin{cases} BG_2(u^2) = B(T^2_\ell \rtimes W(G_2)) & r \in \{1,2\},\ \ell = \nu_p(u^2-1) \\ BS^{11}(u^6) = B(\mathbb{Z}/p^\ell \rtimes C_6) & r \in \{3,6\},\ \ell = \nu_p(u^6-1) \\ * & \text{otherwise,} \end{cases}$$

where it is understood that all spaces are completed at p. To see this, write $u = \zeta q$ where ζ is a $(p-1)$th root of unity and $q \equiv 1 \bmod p$. Note first that $BG_2(u) = BG_2(\pm u)$ as the Weyl group of G_2 contains -1. In case $u^2 \equiv 1 \bmod p$ ($u^2 \neq 1$), we have that $u = \pm q$ so that $BG_2(u) = BG_2(\pm u) = BG_2(u^2) = B(T^2_\ell \rtimes W(G_2))$ by 9.8 and 6.5. If $u^2 \not\equiv 1 \bmod p, u^6 \equiv 1 \bmod p$, then $u = \pm\sigma q$ where $\sigma^3 = 1$ and $BG_2(u) = BG_2(\pm\sigma q) = BG_2(\sigma q) = BG_2^{h\langle\sigma\rangle}(q) = BS^{11}(q) = BS^{11}(u^6)$ by 6.2, A.10; the last equality follows from 6.5 since $\nu_p(u^6-1) = \nu_p(q^6-1) = \nu_p(q-1)$. If $u^6 \not\equiv 1 \bmod p$ then $BG_2(u)$ is contractible by Theorem B. It follows that

$$H^*(BG_2(u);\mathbb{F}_p) = \begin{cases} P[x_4,x_{12}] \otimes E(y_3,y_{11}) & r \in \{1,2\} \\ P[x_{12}] \otimes E(y_{11}) & r \in \{3,6\} \\ \mathbb{F}_p & \text{otherwise} \end{cases}$$

with higher order Bocksteins as explained in Theorem F. This provides the geometric explanation of Kleinermann's computation [38, Thm 1-1] of cohomology rings of finite Chevalley groups of type G_2.

10. Chevalley p-local finite groups from Aguadé p-compact groups

In [1], Aguadé constructed the exotic p-compact groups $\mathbf{X}_i$, $i = 12, 29, 31, 34$, with Weyl groups the groups G_{12} (rank 2, $p = 3$), G_{29} (rank 4, $p = 5$), G_{31} (rank 4, $p = 5$), and G_{34} (rank 6, $p = 7$), on the Sheppard–Todd and Clark–Ewing lists, respectively. All four of them are obtained as the homotopy colimit of a diagram that we proceed by describing.

Write G_i to denote one of the groups G_{12}, G_{29}, G_{31}, or G_{34}, and Z its center, namely, $Z \cong \mathbb{Z}/2$ for G_{12}, $Z \cong \mathbb{Z}/4$ for G_{29}, $Z \cong \mathbb{Z}/4$ for G_{31}, $Z \cong \mathbb{Z}/6$ for G_{34}, in all cases represented by diagonal matrices with entries $p-1$ roots of unity. In all four cases we also fix a subgroup isomorphic to Σ_p. Then, the index category is the opposite category of $\mathbb{I}(1)$, with two objects 0 and 1 and

$$\begin{aligned}
&\mathrm{Aut}_{\mathbb{I}(1)}(0) = G_i\,,\\
&\mathrm{Aut}_{\mathbb{I}(1)}(1) = N_{G_i}(\Sigma_p)/\Sigma_p \cong Z\,,\\
&\mathrm{Mor}_{\mathbb{I}(1)}(1,0) = \Sigma_p\backslash G_i\,,\ \text{and}\\
&\mathrm{Mor}_{\mathbb{I}(1)}(0,1) = \emptyset\,.
\end{aligned}$$

The functor assigns BT^{p-1} to 0 and $BSU(p)$ to 1, up to homotopy, where the center of G_i, Z, acts on BSU_p via unstable Adams operations. The diagram is described in the following picture

$$\begin{tikzcd} Z \circlearrowright BSU(p) & BT^{p-1} \circlearrowleft (G_i)^{op} \arrow[l, "(\Sigma_p)^{op}\backslash(G_i)^{op}"'] \end{tikzcd}.$$

Each $\mathbf{X}_i$ is a p-compact group with maximal torus $T_{\mathbf{X}_i} = T^{p-1}$ and Weyl group $W_{\mathbf{X}_i} = G_i$. The respective cohomology rings coincide with the invariant rings $H^*(B\mathbf{X}_i;\mathbb{F}_p) \cong H^*(BT_{\mathbf{X}_i};\mathbb{F}_p)^{G_i}$, and these are the polynomial rings ([1, 2, 68]):

$$\begin{aligned}
H^*(B\mathbf{X}_{12};\mathbb{F}_3) &\cong P[x_{12},x_{16}]\,,\\
H^*(B\mathbf{X}_{29};\mathbb{F}_5) &\cong P[x_{8},x_{16},x_{24},x_{40}]\,,\\
H^*(B\mathbf{X}_{31};\mathbb{F}_5) &\cong P[x_{16},x_{24},x_{40},x_{48}]\,,\\
H^*(B\mathbf{X}_{34};\mathbb{F}_7) &\cong P[x_{12},x_{24},x_{36},x_{48},x_{60},x_{84}]\,.
\end{aligned}$$

Throughout this section we fix an unstable Adams operation ψ^q of exponent $q \in \mathbb{Z}_p^*$ with $q \equiv 1 \bmod p$, $q \neq 1$. We will describe the p-local structure of the spaces $B\mathbf{X}_i(q)$ and will show that they are classifying spaces of p-local finite groups. In particular, cases $i = 29, 34$ provide new exotic examples of p-local finite groups.

The first results on the p-local structure of $B\mathbf{X}_i(q)$ are given by Propositions 7.5 and 7.6. Set $\ell = \nu_p(1-q)$. The maximal elementary abelian p-subgroup of $\mathbf{X}_i$, $(t_{\mathbf{X}_i},\nu)$, factors as a p-subgroup $(t_{\mathbf{X}_i},g)$ of $\mathbf{X}_i(q)$, and the centralizer of this group

$$C_{\mathbf{X}_i(q)}(t_{\mathbf{X}_i},g) \simeq T_\ell^{p-1} \cong (\mathbb{Z}/p^\ell)^{p-1}$$

is the maximal finite torus of $\mathbf{X}_i(q)$. All elementary abelian p-subgroups of $\mathbf{X}_i(q)$ factor through this one. Moreover, the Weyl group is $W_{\mathbf{X}_i(q)}(T_\ell^{p-1}) = G_i$, and the normalizer $N_{\mathbf{X}_i(q)}(T_\ell^{p-1}) = T_\ell^{p-1} \rtimes G_i$ sits in the maximal torus normalizer of $\mathbf{X}_i(q)$, making homotopy commutative the diagram

$$\begin{tikzcd}
BN_{\mathbf{X}_i(q)}(T_\ell^{p-1}) \arrow[r] \arrow[d] & BN_{\mathbf{X}_i}(T^{p-1}) \arrow[d] \\
B\mathbf{X}_i(q) \arrow[r] & B\mathbf{X}_i\,.
\end{tikzcd}$$

Now, we fix the Sylow p-subgroup $S = (\mathbb{Z}/p^\ell)^{(p-1)} \rtimes \mathbb{Z}/p$ of $N_{\mathbf{X}_i(q)}(T_\ell^{p-1})$, generated by $T_\ell^{p-1}$and a p-cycle of $\Sigma_p \leq G_i$. We will denote by $f\colon BS \to B\mathbf{X}_i(q)$ the homotopy monomorphism obtained as the composition $BS \to BN_{\mathbf{X}_i(q)}(T_\ell^{p-1}) \to B\mathbf{X}_i(q)$. Then (S,f) is a p-subgroup of $B\mathbf{X}_i(q)$, and it will play the role of a Sylow p-subgroup.

Since $\mathbf{X}_i$, $i = 12, 29, 31, 34$, are polynomial p-compact groups, according to Corollary 7.8, $\iota\colon B\mathbf{X}_i(q) \to B\mathbf{X}_i$ induces an equivalence of categories

$$\iota_\sharp\colon \mathcal{F}_p^e(B\mathbf{X}_i(q)) \longrightarrow \mathcal{F}_p^e(B\mathbf{X}_i)\,.$$

Thus, we obtain that every elementary abelian p-subgroup (E,μ) of $B\mathbf{X}_i(q)$ factors as a subgroup of $t_{\mathbf{X}_i}$: $E \leq t_{\mathbf{X}_i}$ and $\mu \simeq \nu|_{BE}$. There is a distinguished subgroup $\mathbb{Z}/p \cong Z \leq t_{\mathbf{X}_i}$ such that, $Z \leq t_{\mathbf{X}_i} \leq T_{\mathbf{X}_i} \leq SU_p \cong C_{\mathbf{X}_i}(Z, \nu|_{BZ})$. If $E \leq t_{\mathbf{X}_i}$ is not conjugate to Z in $\mathbf{X}_i$, then the centralizer $C_{\mathbf{X}_i}(E, \nu|_{BE})$ is a p-compact group whose Weyl group, the point-wise stabilizer of $E \leq T_{\mathbf{X}_i}$, $W_{\mathbf{X}_i}(E)$, has order not divisible by p. In $\mathbf{X}_i(q)$, we obtain:

Proposition 10.1. *There is one conjugacy class of elements of order p in $\mathbf{X}_i(q)$, $(Z, g|_{BZ})$, such that the centralizer is*

$$C_{\mathbf{X}_i(q)}(Z, g|_{BZ}) \simeq SU_p(q)$$

and contains (S,f):

$$\begin{tikzcd} BS \arrow[d, "Bincl"'] \arrow[dr, "f"] & \\ BSU_p(q) \arrow[r] & B\mathbf{X}_i(q) \end{tikzcd}$$

as Sylow p-subgroup of $SU_p(q)$.

If $E \leq t_{\mathbf{X}_i}$ represents another conjugacy class of elementary abelian p-subgroups, then

$$C_{\mathbf{X}_i(q)}(E, g|_{BE}) \simeq T_\ell^{p-1} \rtimes W_{\mathbf{X}_i}(E)$$

where the order of $W_{\mathbf{X}_i}(E)$ is not divisible by p. Furthermore, the diagram

$$\begin{tikzcd} BT_\ell^{(p-1)} \arrow[r, "Bincl"] \arrow[d, "Bincl"'] & BS \arrow[d, "f"] \\ BC_{\mathbf{X}_i(q)}(E, g|_{BE}) \arrow[r, "j"] & B\mathbf{X}_i(q) \end{tikzcd}$$

is commutative up to homotopy, where $j\colon BC_{\mathbf{X}_i(q)}(E, g|_{BE}) \to B\mathbf{X}_i(q)$ is the natural map induced by evaluation.

Proof. For $Z \leq t_{\mathbf{X}_i}$, we have $C_{\mathbf{X}_i(q)}(Z, g|_{BZ}) \cong SU_p(q)$ by Corollary 7.4.

If $E \leq t_{\mathbf{X}_i}$ be another subgroup, not conjugate to Z, then the centralizer in $\mathbf{X}_i$ is the Clark–Ewing p-compact group $BC_{\mathbf{X}_i}(E, \nu|_{BE}) \simeq B(T_{\mathbf{X}_i} \rtimes W_{\mathbf{X}_i}(E))^\wedge_p$, and then, first, Corollary 7.4 implies that $BC_{\mathbf{X}_i(q)}(E, g|_{BE}) \simeq BC_{\mathbf{X}_i}(E, \nu|_{BE})(q)$, and secondly, Theorem 9.8 gives $BC_{\mathbf{X}_i}(E, \nu|_{BE})(q) \simeq B(T_\ell^{p-1} \rtimes W_{\mathbf{X}_i}(E))^\wedge_p$.

Finally, we use the inclusions $BE \to Bt_{\mathbf{X}_i} \to BS \xrightarrow{f} B\mathbf{X}_i(q)$ in order to compare the centralizers of E and $t_{\mathbf{X}_i}$ in S and $\mathbf{X}_i(q)$:

$$\begin{tikzcd} BT_\ell^{p-1} \simeq BC_S(t_{\mathbf{X}_i}) \arrow[r, "\simeq"] \arrow[d, "f_\sharp", "\simeq"'] & BC_S(E) \arrow[r] \arrow[d, "f_\sharp"] & BS \arrow[d, "f"] \\ BC_{\mathbf{X}_i(q)}(t_{\mathbf{X}_i}, g) \arrow[r] & BC_{\mathbf{X}_i(q)}(E, g|_{BE}) \arrow[r] & B\mathbf{X}_i(q)\,. \end{tikzcd}$$

$\square$

Proposition 10.2. *For $i = 12, 29, 31, 34$, the natural map*

$$\operatorname*{hocolim}_{\mathcal{F}_p^e(B\mathbf{X}_i(q))^{op}} BC_{\mathbf{X}_i(q)} \to B\mathbf{X}_i(q) \tag{34}$$

is a mod p homology equivalence.

Proof. According to Theorem F the cohomology rings of $B\mathbf{X}_i(q)$ are:

$$\begin{aligned}
&H^*(B\mathbf{X}_{12}(q);\mathbb{F}_3) \cong P[x_{12}, x_{16}] \otimes E[y_{11}, y_{15}]\,,\\
&H^*(B\mathbf{X}_{29}(q);\mathbb{F}_5) \cong P[x_8, x_{16}, x_{24}, x_{40}] \otimes E[y_7, y_{15}, y_{23}, y_{39}]\,,\\
&H^*(B\mathbf{X}_{31}(q);\mathbb{F}_5) \cong P[x_{16}, x_{24}, x_{40}, x_{48}] \otimes E[y_{15}, y_{23}, y_{39}, y_{47}]\,,\\
&H^*(B\mathbf{X}_{34}(q);\mathbb{F}_7) \cong P[x_{12}, x_{24}, x_{36}, x_{48}, x_{60}, x_{84}] \otimes E[y_{11}, y_{23}, y_{35}, y_{47}, y_{59}, y_{83}]\,,
\end{aligned}$$

and they embed in the invariant rings $H^*(B\mathbf{X}_i(q);\mathbb{F}_p) \subseteq H^*(BT_\ell^{p-1};\mathbb{F}_p)^{G_i}$. These invariant rings are described in the Example 9.7. It turns out that the above inclusion is an isomorphism if $i = 29, 31, 34$, but it is not surjective when $i = 12$.

The centralizers of elementary abelian p-subgroups of $B\mathbf{X}_i(q)$ are described in Proposition 10.1. The centralizer, $C_{\mathbf{X}_i(q)}(E, g|_{BE})$, of an elementary abelian p-subgroup $E \le t_{\mathbf{X}_i}$ in $\mathbf{X}_i(q)$ is either $SU_p(q)$ or $C(q)$ where C is a Clark–Ewing p-compact group.

In cases $i = 29, 31, 34$, $H^*(C_{\mathbf{X}_i(q)}(E, g|_{BE});\mathbb{F}_p) \cong H^*(BT_{\mathbf{X}_i};\mathbb{F}_p)^{W(E)}$ is satisfied by Theorem F and examples 9.2 and 9.3, hence we meet the conditions of Proposition 7.9 and the map (34) is a mod p homology equivalence.

In the case $i = 12$, Proposition 7.9 does not apply, so we will need a separate analysis. The p-compact group $\mathbf{X}_{12}$, $p = 3$, is also denoted DI_2, because $G_{12} \cong GL(2,3)$ and $H^*(BDI_2;\mathbb{F}_3) \cong H^*(BT^2;\mathbb{F}_3)^{GL(2,3)} \cong \mathbb{F}_3[x_{12}, x_{16}]$ is the rank two Dickson algebra at $p = 3$. It admits two conjugacy classes of elementary abelian p-subgroups, one of rank one and another of rank two, hence so does $BDI_2(q)$, as well. We have equivalences of categories

$$\mathcal{F}_p^e(BDI_2) \cong \mathcal{F}_p^e(BDI_2(q)) \cong \mathbb{I}(1)$$

with $\mathrm{Aut}_{\mathbb{I}(1)}(0) = GL(2,3)$, $\mathrm{Aut}_{\mathbb{I}(1)}(1) = N_{GL(2,3)}(\Sigma_3)/\Sigma_3 \cong \mathbb{Z}/2$, where $N_{GL(2,3)}(\Sigma_3) = \Sigma_3 \times \mathbb{Z}/2$, and $\mathrm{Mor}_{\mathbb{I}(1)}(1,0) = \Sigma_3\backslash GL(2,3)$, $\mathrm{Mor}_{\mathbb{I}(1)}(0,1) = \emptyset$. The centralizers diagram $BC_{DI_2(q)}$ is described in the picture

$$\mathbb{Z}/2 \circlearrowright BSU_3(q) \xleftarrow{\;\Sigma_3^{op}\backslash GL(2,3)^{op}\;} BT_\ell^2 \circlearrowleft GL(2,3)^{op}\;. \tag{35}$$

The Bousfield–Kan spectral sequence

$$E_2^{i,j} \cong \varprojlim_{\mathbb{I}(1)}{}^i H^j(BC_{DI_2(q)};\mathbb{F}_3) \Longrightarrow H^{i+j}(\operatorname*{hocolim}_{\mathbb{I}(1)^{op}} BC_{DI_2(q)};\mathbb{F}_3)$$

computes the cohomology of the homotopy colimit $\operatorname{hocolim}_{\mathbb{I}(1)^{op}} BC_{DI_2(q)}$.

The computation of the E_2-term follows from Proposition B.1. Since $N_{GL(2,3)}(\Sigma_3) \cong \Sigma_3 \times \mathbb{Z}/2$ and $H^*(GL(2,3);A) \cong H^*(N_{GL(2,3)}(\Sigma_3);A) \cong H^*(\Sigma_3;A)$, for any $GL(2,3)$-module A, there is an exact sequence

$$\begin{aligned}
0 \to \varprojlim_{\mathbb{I}(1)}{}^0 H^*(BC_{DI_2(q)};\mathbb{F}_3) \to H^*(BSU(3)(q);\mathbb{F}_3)^{\mathbb{Z}/2} \oplus H^*(BT_\ell^2;\mathbb{F}_3)^{GL(2,3)}\\
\to H^*(BT_\ell^2;\mathbb{F}_3)^{\Sigma_3\times\mathbb{Z}/2} \to \varprojlim_{\mathbb{I}(1)}{}^1 H^*(BC_{DI_2(q)};\mathbb{F}_3) \to 0\,,
\end{aligned} \tag{36}$$

and $\varprojlim^i_{\mathbb{I}(1)} BC_{DI_2(q)} = 0$ if $i \geq 2$.

The invariant rings $H^*(BT^2_\ell;\mathbb{F}_3)^{GL(2,3)}$ and $H^*(BT^2_\ell;\mathbb{F}_3)^{\Sigma_3}$ as well as the restriction $R\colon H^*(BT^2_\ell;\mathbb{F}_3)^{GL(2,3)} \hookrightarrow H^*(BT^2_\ell;\mathbb{F}_3)^{\Sigma_3}$ have been described in examples 9.4 and 9.7. The cohomology of $BSU(3)(q)$ is identified with the subalgebra $P[x_4,x_6]\otimes E[y_3,y_5]$ of $H^*(BT^2_\ell;\mathbb{F}_3)^{\Sigma_3}$. The cokernel of the inclusion is isomorphic to $P[x_6]y_4$, and then the exact sequence (36) is simplified to

$$0 \to \varprojlim_{\mathbb{I}(1)}{}^0 H^*(BC_{DI_2(q)};\mathbb{F}_3) \to \frac{P[x_{12},x_{16}]\otimes E[y_{10},y_{11},y_{15}]}{(y_{11}y_{15}-x_{16}y_{10},\ y_{10}y_{11},\ y_{10}y_{15})}$$
$$\xrightarrow{\bar{R}} \big(P[x_6]y_4\big)^{\mathbb{Z}/2} \to \varprojlim_{\mathbb{I}(1)}{}^1 H^*(BC_{DI_2(q)};\mathbb{F}_3) \to 0\,,$$

and $\big(P[x_6]y_4\big)^{\mathbb{Z}/2} = P[{x_6}^2](x_6y_4)$ which is in the image of $\bar{R}$. It follows that

$$\varprojlim_{\mathbb{I}(1)}{}^0 H^*(BC_{DI_2(q)};\mathbb{F}_3) \cong P[x_{12},x_{16}]\otimes E[y_{11},y_{15}]$$

and $\varprojlim^i_{\mathbb{I}(1)} BC_{DI_2(q)} = 0$ if $i \geq 1$, so, therefore the Bousfield–Kan spectral sequence collapses to an isomorphism

$$H^*(\operatorname*{hocolim}_{\mathbb{I}(1)^{op}} BC_{DI_2(q)};\mathbb{F}_3) \cong \varprojlim_{\mathbb{I}(1)}{}^0 H^*(BC_{DI_2(q)};\mathbb{F}_3) \cong P[x_{12},x_{16}]\otimes E[y_{11},y_{15}]\,;$$

that is, $\operatorname{hocolim}_{\mathbb{I}(1)^{op}} BC_{DI_2(q)} \to BG_2(q)$ is a sharp homology decomposition at the prime 3 and

$$H^*(DI_2(q);\mathbb{F}_3) \cong \varprojlim_{\mathbb{I}(1)}{}^0 H^*(BC_{DI_2(q)};\mathbb{F}_3) \cong P[x_{12},x_{16}]\otimes E[y_{11},y_{15}]\,. \qquad \square$$

Theorem 10.3. *(S,f) is a Sylow p-subgroup for $B\mathbf{X}_i(q)$, the fusion system $\mathcal{F}_{(S,f)}(B\mathbf{X}_i(q))$ of the space $B\mathbf{X}_i(q)$ over the p-subgroup (S,f) is saturated, and*

$$(S,\mathcal{F}_{(S,f)}(B\mathbf{X}_i(q)),\mathcal{L}_{(S,f)}(B\mathbf{X}_i(q)))$$

is a p-local finite group with classifying space

$$|\mathcal{L}_{(S,f)}(B\mathbf{X}_i(q))|^\wedge_p \simeq B\mathbf{X}_i(q)\,.$$

Proof. It is a consequence of Theorem 4.5, using the above propositions 10.1 and 10.2. $\square$

Now, we will go deeper into the structure of the fusion system $\mathcal{F}=\mathcal{F}_{(S,f)}(B\mathbf{X}_i(q))$. We have seen that the fusion category of elementary abelian p-subgroups is equivalent to that of the p-compact group $\mathbf{X}_i$; in particular, every elementary abelian p-subgroup is toral; that is, $\mathcal{F}$-conjugate to a subgroup of $T_\ell^{(p-1)}$. If we denote $Z=Z(S)$ the center of S, then (10.1) $BC_{\mathbf{X}_i(q)}(Z) = BSU_p(q)^\wedge_p \simeq BSL_p(q)^\wedge_p$, so, the centralizer fusion system $C_{\mathcal{F}}(Z)$ over $C_S(Z)=S$ coincides with the fusion system of $SL_p(q)$ over S. Hence, we can identify S with the Sylow p-subgroup of $SL_p(q)$ and then use the notation of Example 3.5. Recall from 3.5 that any centric radical subgroup of S in $C_{\mathcal{F}}(Z)$ is conjugate to either S, $T_\ell^{(p-1)}$, or an extraspecial group $\Gamma_1(\xi^r)$, $r=0,\dots,p-1$.

Proposition 10.4. *Any centric radical subgroup of S in $\mathcal{F} = \mathcal{F}_{(S,f)}(B\mathbf{X}_i(q))$ is conjugate to one of the groups in the table:*

$$
\begin{array}{ccc}
\hline
Q & \mathrm{Out}_{\mathcal{F}}(Q) & \textit{Conditions} \\
\hline
T_\ell^{(p-1)} & G_i & \\
S & \mathbb{Z}/(p-1)\times\mathbb{Z}/(p-1) & \\
\Gamma_1 & GL_2(p) & \\
\Gamma_1(\xi) & SL_2(p) & \text{if } \ell > 1 \text{ or } p > 3. \\
\hline
\end{array}
\tag{37}
$$

Proof. The proof is divided in four steps, where we first determine a set of representatives for centric radical subgroups of S in $\mathcal{F}$, and then refine it to a minimal set of representatives and compute their automorphisms groups in $\mathcal{F}$.

Step 1: Toral and non toral centric radical subgroups. T_ℓ^{p-1} is centric in $\mathcal{F}$ and $\mathrm{Out}_{\mathcal{F}}(T_\ell^{p-1}) \cong G_i$ is p-reduced, hence T_ℓ^{p-1} is also radical in $\mathcal{F}$. No other subgroup of T_ℓ^{p-1} is centric, so for any other centric and radical subgroup $Q \leq S$ in $\mathcal{F}$, there is a morphism of extensions

$$
\begin{tikzcd}
Q_0 \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & \mathbb{Z}/p \arrow[d, equal] \\
T_\ell^{p-1} \arrow[r] & S \arrow[r] & \mathbb{Z}/p
\end{tikzcd}
\tag{38}
$$

where $Q_0 = T_\ell^{p-1} \cap Q$.

We are assuming that Q is centric, hence the center $Z \cong \mathbb{Z}/p$ of S should be contained in Q_0. But if $Q_0 = Z$, then $Q \cong \mathbb{Z}/p \times \mathbb{Z}/p$ is elementary abelian and then toral in $\mathcal{F}$, hence it would not be centric. Thus $Z \neq Q_0$ and the center of Q is $Z(Q) = Q_0^{\mathbb{Z}/p} = Z$. In particular, every automorphism of Q restricts to an automorphism of Z, so we obtain a homomorphism $\mathrm{Aut}_{\mathcal{F}}(Q) \to \mathrm{Aut}_{\mathcal{F}}(Z)$. The kernel is composed of automorphisms of Q that restrict to the identity in Z; that is, automorphisms of Q in the centralizer fusion system $C_{\mathcal{F}}(Z)$, hence we have an exact sequence

$$1 \to \mathrm{Aut}_{C_{\mathcal{F}}(Z)}(Q) \to \mathrm{Aut}_{\mathcal{F}}(Q) \to \mathrm{Aut}_{\mathcal{F}}(Z) \tag{39}$$

where $\mathrm{Aut}_{\mathcal{F}}(Z) \leq \mathbb{Z}/p-1$ lifts to $\mathrm{Aut}_{\mathcal{F}}(T_\ell^{(p-1)})$ and $\mathrm{Aut}_{\mathcal{F}}(S)$ as unstable Adams operations (the center of G_i). Thus, if Q is radical in $C_{\mathcal{F}}(Z)$, then it is radical in $\mathcal{F}$.

Step 2: Non-abelian centric radical subgroups, all of which abelian characteristic subgroups are cyclic. Assume that all abelian characteristic subgroups of Q are cyclic, then a theorem of Hall implies that Q is the central product of an extraspecial group Γ of exponent p and a cyclic group C, where the elements or order p in C, $\Omega_1(C)$, coincide with the center $Z(\Gamma)$ of Γ (cf. [34, Chap. 5, 4.9, 5.3]).

The faithful irreducible representations of the central product of an extraspecial group Γ or order p^{1+2r} and a cyclic group of order p^ℓ over the algebraic closure of a field of q elements, $(q,p)=1$, have degree p^r, and there are exactly $p^{\ell-1}(p-1)$ inequivalent representations in this degree.

Hence, only the case $r=1$ can appear in $GL_p(q)$. We denote Γ_1 the extraspecial group of order p^3 and exponent p, and Γ_k the central product $\mathbb{Z}/p^k \circ \Gamma_1$. The different irreducible faithful representations of Γ_k in $GL_p(q)$ are obtained by composing with the extension to Γ_k of the automorphisms of $\mathbb{Z}/p^k$, $(\mathbb{Z}/p^k)^*$. Thus, there is at most one subgroup isomorphic to Γ_k in $GL_p(q)$, up to conjugation. A subgroup of $GL_p(q)$ isomorphic to Γ_1 is described in Example 3.4. Since $C_{GL_p(q)}(\Gamma_1) = Z(GL_p(q)) \cong GL_1(q)$, Γ_k is a subgroup of $GL_p(q)$ if and only if $\mathbb{Z}/p^k < GL_1(q)$. Hence Γ_ℓ, $\ell = \nu_p(1-q)$, is the biggest one that can occur in $GL_p(q)$ (see Example 3.4).

Finally, the intersection of Γ_ℓ with $SL_p(q)$, and hence, of any conjugate of Γ_ℓ, is isomorphic to Γ_1, and there are exactly p conjugacy classes of such subgroups $\Gamma_1(\xi^r)$ (see Example 3.5). These are radical in $C_{\mathcal{F}}(Z)$, and so, therefore, they are also radical in $\mathcal{F}$.

Step 3: Non-abelian centric radical subgroups having non-cyclic abelian characteristic subgroups. Assume now that Q contains a non-cyclic abelian characteristic group. If Q is radical in $C_{\mathcal{F}}(Z)$, then it is radical in $\mathcal{F}$. Now, we assume also that Q is not radical in $C_{\mathcal{F}}(Z)$.

We can view $Q \leq S$ as subgroups of $SL_p(q)$ and $GL_p(q)$, for an appropriate prime power q such that S is the Sylow p-subgroup of $SL_p(q)$: $\ell = \nu_p(1-q)$. Write $N = N_{GL_p(q)}(Q)$. The arguments of [4, (4A)] show that (up to conjugacy in $GL_p(q)$)

$$Q \leq N \cap (\mathbb{Z}/p^k \wr \mathbb{Z}/p) \lhd N$$

for some $k \leq \ell$, or, taking the intersection with $SL_p(q)$

$$Q \leq \bar{N} \cap S_k \lhd \bar{N}$$

where $S_k = (\mathbb{Z}/p^k \wr \mathbb{Z}/p) \cap SL_p(q) \leq S$ and $\bar{N} = N \cap SL_p(q) = N_{SL_p(q)}(Q)$, an then

$$\operatorname{Inn} Q \leq (\bar{N} \cap S_k)/Z(Q) \lhd \operatorname{Aut}_{C_{\mathcal{F}}(Z)}(Q)$$

where $\bar{N}/C_{SL_p(q)}(Q) = \operatorname{Aut}_{C_{\mathcal{F}}(Z)}(Q)$. We will see that $(\bar{N} \cap S_k)/Z(Q)$ is still normal in $\operatorname{Aut}_{\mathcal{F}}(Q)$.

Assume that $\varphi \in \operatorname{Aut}_{\mathcal{F}}(Q)$ restricts to Z as the unstable Adams operation ψ^ζ, ζ a $(p-1)$st root of unity. If $\psi^{1/\zeta}(Q) = Q' \leq S$, then $\psi^{1/\zeta} \circ \varphi \colon Q \to Q'$ is a morphism of $\mathcal{F}$, that restricted to Z is trivial, hence a morphism of $C_{\mathcal{F}}(Z)$. Since, we have assumed that Q is not radical in $C_{\mathcal{F}}(Z)$, $\psi^{1/\zeta} \circ \varphi$ should be obtained as composition of restrictions of automorphisms of centric radical subgroups of $C_{\mathcal{F}}(Z)$, by Alperin fusion theorem [13, A.10]. This is the fusion system of $SL_p(q)$, and the Sylow p-subgroup S itself is the only centric radical that contains Q, hence, there is $\chi \in \operatorname{Aut}_{C_{\mathcal{F}}(Z)}(S)$ with $\chi|_Q = \psi^{1/\zeta} \circ \varphi$, hence $\varphi = \psi^\zeta \circ \chi|_Q$ extends to an automorphism $\psi^\zeta \circ \chi$ of $\operatorname{Aut}_{\mathcal{F}}(S)$. Notice that $\psi^\zeta(S_k) = S_k$ and also $\chi(S_k) = S_k$, hence, if $g \in S_k$ normalizes Q, we have $\varphi \circ c_g \circ \varphi^{-1} = c_{\varphi(g)}$, with $\varphi(g) \in \bar{N} \cap S_k$. This proves that we have

$$\operatorname{Inn} Q \leq (\bar{N} \cap S_k)/Z(Q) \lhd \operatorname{Aut}_{\mathcal{F}}(Q)$$

and since Q is radical in $\mathcal{F}$, $Q = S_k$.

We claim that only the case $S_k = S$ is radical. First we compute the normalizer of $\mathbb{Z}/p^k \wr \mathbb{Z}/p$ in $GL_p(q)$. The subgroup $(\mathbb{Z}/p^k)^p$ is a characteristic subgroup of $\mathbb{Z}/p^k \wr \mathbb{Z}/p$, for it is the only abelian subgroup of index p, hence, $N_{GL_p(q)}(\mathbb{Z}/p^k \wr \mathbb{Z}/p) \leq N_{GL_p(q)}((\mathbb{Z}/p^k)^p)$. It is not difficult to compute $N_{GL_p(q)}((\mathbb{Z}/p^k)^p) = GL_1(q) \wr \Sigma_p$, the group of invertible matrices with only one non-trivial entry in each line and column. By direct computation one can obtain that $N_{GL_p(q)}(\mathbb{Z}/p^k \wr \mathbb{Z}/p) = GL_1(q) \cdot (\mathbb{Z}/p^k \wr N_{\Sigma_p}(\mathbb{Z}/p))$, where $GL_1(q)$ is identified with the subgroup of all diagonal matrices of $GL_p(q)$; that is, the center of $GL_p(q)$.

Call $N_k = N_{GL_p(q)}(\mathbb{Z}/p^k \wr \mathbb{Z}/p) \cap SL_p(q)$. We have $N_k \cong B_k \rtimes N_{\Sigma_p}(\mathbb{Z}/p)$, with

$$B_k = \left\{ (z \cdot x_1, \ldots, z \cdot x_p) \in GL_1(q)^p \,\middle|\, x_i \in Z/p^k,\ z^p x_1 \ldots x_p = 1 \right\}$$

and $N_{SL_p(q)}(S_k) = N_k$. Notice that, when $k < \ell$, S_k has index p in the Sylow p-subgroup $B_k \rtimes \mathbb{Z}/p$, and this is normal in N_k, hence only $S = S_\ell$ is radical in $SL_p(q)$.

The centralizer of S_k in $SL_p(q)$ is $C_{SL_p(q)}(S_k) = Z \cong \mathbb{Z}/p$ and then $\mathrm{Aut}_{C_\mathcal{F}(Z)}(S_k) \cong \mathrm{Aut}_{SL_p(q)}(S_k) \cong N_k/Z$. $(B_k/Z) \rtimes \mathbb{Z}/p$ is normal in N_k/Z, and, since the Adams operations ψ^ζ, ζ a $(p-1)$st root of unity, act internally in B_k, $(B_k/Z) \rtimes \mathbb{Z}/p$ is also a normal of $\mathrm{Aut}_\mathcal{F}(S_k)$:

$$\mathrm{Inn}\, S_k = S_k/\mathbb{Z}/p \lhd (B_k/\mathbb{Z}/p) \rtimes \mathbb{Z}/p \lhd \mathrm{Aut}_\mathcal{F}(S_k)$$

thus, S_k is radical in the fusion system $\mathcal{F}$ if and only if $k = \ell$; that is, only the case $S_k = S$ is radical. In this case we have obtained $\mathrm{Aut}_\mathcal{F}(S) \cong N_\ell/Z \rtimes \mathbb{Z}/(p-1)$, where $\mathbb{Z}/(p-1)$ on the right is generated by the Adams operations of exponent a primitive $(p-1)$st root of unity, and $\mathrm{Out}_\mathcal{F}(S) \cong \mathbb{Z}/(p-1) \times \mathbb{Z}/(p-1)$, given by the Adams operations and $N_{\Sigma_p}(\mathbb{Z}/p)/\mathbb{Z}/p$.

Step 4: Minimal set of representatives and automorphism groups. It remains to check which of those are $\mathcal{F}$-conjugate to one of the others in the list and also to compute their $\mathcal{F}$-automorphisms.

For $Q = S$ the restriction $\mathrm{Aut}_\mathcal{F}(Q) \to \mathrm{Aut}_\mathcal{F}(Z)$ is split because unstable Adams operations extend to S. Moreover, since they are realized by the center of G_i, the $\mathcal{F}$-automorphisms of S are given by conjugation in the normalizer $N_{\ell,i}$ of the maximal finite torus $T_\ell^{(p-1)}$. We have seen already that the same is true for $Q = T_\ell^{(p-1)}$.

Finally, we analyse the case $Q = \Gamma_1(\xi^r)$, $r = 0, \dots, p-1$. Assume that $\varphi \in \mathrm{Aut}_\mathcal{F}(Q)$ and that the restriction to the center Z is the unstable Adams operation ψ^z. This extends to an $\mathcal{F}$-automorphism of S. Write $Q' = \psi^z(Q)$. Then $\chi = \psi^z \circ \varphi^{(-1)} \colon Q \to Q'$ is a homomorphism of $\mathcal{F}$ that restricts to the identity in Z, hence it belongs to the centralizer fusion system $C_\mathcal{F}(Z)$. In other words, every automorphism $\varphi \in \mathrm{Aut}_\mathcal{F}(Q)$ is the composite of an isomorphism $\chi \colon Q \to Q'$ of $C_\mathcal{F}(Z)$ and a unstable Adams operation ψ^z.

It is then enough to compute the effect of unstable Adams operations on the family of subgroups $\Gamma_1(\xi^r)$. It turns out that unstable Adams operations restrict to automorphisms of $\Gamma_1 = \Gamma_1(\xi^0)$ so that $\mathrm{Out}_\mathcal{F}(\Gamma_1) = GL_p(q)$, while, for $p > 3$ or $\ell > 1$, they conjugate $\Gamma_1(\xi^r)$ for $r = 1, \dots, p-1$ to each other and $\mathrm{Out}_\mathcal{F}(\Gamma_1(\xi) = SL_p(q)$. □

Corollary 10.5. *The fusion system of $B\mathbf{X}_i(q)$ is*

$$\mathcal{F}_{(S,f)}(B\mathbf{X}_i(q)) = \langle\, \mathcal{F}_{N_{\ell,i}}(S)\, ; \mathcal{F}_{\Gamma_1}(GL_2(p))\, , \mathcal{F}_{\Gamma_1(\xi)}(SL_2(p))\,\rangle\,,$$

for $p > 3$ or $\ell > 1$, and $\mathcal{F}_{(S,f)}(B\mathbf{X}_{12}(q)) = \langle\, \mathcal{F}_{N_{1,i}}(S)\, ; \mathcal{F}_{\Gamma_1}(GL_2(p))\,\rangle$, for $p = 3$ and $\ell = 1$, where $N_{\ell,i} = N_{\mathbf{X}_i(q)}(T_\ell^{(p-1)}) \cong T_\ell^{(p-1)} \rtimes G_i$.

Proof. It is a consequence of Proposition 10.4 and Alperin's fusion theorem for saturated fusion systems (see section 3). □

We end this section with a case by case study in order to determine which spaces $B\mathbf{X}_i(q)$ are p-completed classifying spaces of finite groups and which cases correspond to exotic examples of p-local finite groups.

As we shall see, S contains no proper strongly closed subgroups in $\mathcal{F} = \mathcal{F}_{(S,f)}(B\mathbf{X}_i(q))$ and so, according to [13, 9.2], if $\mathcal{F}$ is the p-completed classifying space of a finite group, this group is almost simple.

In fact, a strongly closed subgroup of S in $\mathcal{F}$ is a normal subgroup P of S such that no element of P is $\mathcal{F}$-conjugate to any element in $S \setminus P$. Now, if P is non trivial it contains at least an element of order p, and this is $\mathcal{F}$-conjugate to an element of order p in $T_\ell^{(p-1)}$.

Now, the maximal elementary abelian p-subgroup t of $T_\ell^{(p-1)}$ turns out to be an irreducible G_i-module, hence $t \subset P$ and since the cycle of order p generating $S/T_\ell^{(p-1)}$ is conjugate to an element of t, the extension of t by this cycle is in P. Thus we have a diagram of extensions

$$\begin{tikzcd} P_T \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & \mathbb{Z}/p \arrow[d, equal] \\ T_\ell^{(p-1)} \arrow[r] & S \arrow[r] & \mathbb{Z}/p \end{tikzcd}$$

where $t \leq P_T = P \cap T$. Now $S/P \cong T_\ell^{(p-1)}/P_T$ is abelian. The abelianization of S is seen to be $Z/p \times \mathbb{Z}/p$, and then we obtain that $T_\ell^{(p-1)}/P_T$ is either trivial or has order p. It follows that all elements of order up to $p^{\ell-1}$ of $T_\ell^{(p-1)}$ belong to P_T. Taking the quotient by this subgroup we obtain an inclusion of G_i-modules $\overline{P_T} \leq \overline{T_\ell^{(p-1)}}$, but again, this last is an irreducible G_i-module, hence $\overline{P_T} = \overline{T_\ell^{(p-1)}}$, and then $P = S$.

Example 10.6. $B\mathbf{X}_{29}(q)$ at $p = 5$ and $B\mathbf{X}_{34}(q)$ at $p = 7$ are classifying spaces of exotic p-local finite groups. We have seen that the Sylow subgroup does not contain any proper strongly closed subgroup in $\mathcal{F}_{(S,f)}(B\mathbf{X}_i(q))$, hence if this is the p-completed classifying space of a finite group G, then G is almost simple [13, Lemma 9.2]. A complete list of almost simple groups with a Sylow subgroup isomorphic to S is provided by [13, Proposition 9.5]. No group in the list contains G_{29} or G_{34} as automorphisms of $T_\ell^{(p-1)}$ induced by conjugation in the group. Hence $\mathbf{X}_{29}(q)$ at $p = 5$ and $\mathbf{X}_{34}(q)$ at $p = 7$ are exotic.

Example 10.7. $BX_{12}(q)$ at $p = 3$ is the 3-completed classifying space of a twisted Chevalley group of type F_4. More precisely, $BX_{12}(q) = B({}^2F_4(2^{3^{\ell-1}}))^\wedge_3$ where $\ell = \nu_3(q^2-1)$.

The 3-completed classifying space of the twisted Chevalley group ${}^2F_4(2^{2n+1})$ can be described at $p = 3$ as $B({}^2F_4(2^{2n+1})) \simeq BF_4^\alpha$, for $\alpha = \varphi \circ \psi^{2^n}$, where φ is the Friedlander's exceptional isogeny of F_4 [31]. φ has the effect of reflecting the Dynkin diagram of F_4 by sending the short roots to the long roots and the long roots to 2 times short roots. Furthermore, $\varphi^2 \simeq \psi^2$, and then we can choose ζ a square root of -2 in $\mathbb{Z}_3$ such that $\beta = \varphi \circ \psi^{1/\zeta}$ is a self equivalence of BF_4 at $p = 3$ of order two and $2^n\zeta \equiv 1 \bmod 3$. We can write $\alpha = \beta \circ \psi^{2^n\zeta}$, and then, by Proposition 6.2, $BF_4^\alpha \simeq (BF_4)^{h\beta}(2^n\zeta)$. In [15] it is shown that $(BF_4)^{h\beta} \simeq BX_{12}$, hence $BX_{12}(2^n\zeta) \simeq B({}^2F_4(2^{2n+1}))^\wedge_3$. Since ψ^{-1} belongs to the Weyl group of X_{12}, $BX_{12}(q) \simeq BX_{12}(-q)$, and then, according to Theorem E, the homotopy type of $BX_{12}(\pm q)$ does only depend on $\ell = \nu_3(q^2-1)$, thus, if we choose n with $\ell = \nu_3(q^2-1) = \nu_3(1-2^n\zeta) = \nu_3(1+2^{2n+1})$, then we have

$$BX_{12}(q) \simeq BX_{12}(2^n\zeta) \simeq B({}^2F_4(2^{2n+1}))^\wedge_3 \,.$$

In particular, $BX_{12}(q) \simeq B({}^2F_4(2^{3^{\ell-1}}))^\wedge_3$. The local structure of ${}^2F_4(2^{2n+1})$, also called Ree groups of characteristic two, was studied by Malle [42].

Example 10.8. For any 5-adic unit, $q \in \mathbb{Z}_5^*$, $BX_{31}(q)$ at $p = 5$ is the 5-completed classifying space of a Chevalley group of type E_8, namely, $BX_{31}(q) \simeq BE_8(2^{2m+1})^\wedge_5$ if $\nu_5(q^4-1) = \nu_5(1+2^{4m+2})$.

Let $i = \sqrt{-1}$ be a primitive 4th root of unity. Since ψ^i belongs to the Weyl group of X_{31}, we can assume that $q \equiv 1 \bmod 5$ for otherwise we can multiply q by an appropriate power of i and still have $BX_{31}(q) \simeq BX_{31}(i^r q)$. Moreover, according to Theorem E, the homotopy type of $BX_{31}(q)$ will only depend on $\ell = \nu_5(q^4-1)$.

We fix a prime power q_0 with $q_0 \equiv \pm 2 \bmod 5$ and $\ell = \nu_5(\pm i q_0 - 1) = \nu_5({q_0}^4 - 1) = \nu_5({q_0}^2+1)$, where we choose $+i$ or $-i$ in order that the equality makes sense.

We can write $q_0 = i \cdot (-i \cdot q_0)$, where now $-i \cdot q_0 \equiv \pm 1 \bmod p$. Since ψ^{-1} belongs to the Weyl group of E_8, we can apply Proposition 6.2 and get $BE_8(q_0) \simeq (BE_8)^{h\psi^i}(-iq_0)$. Now we have seen in Example A.12(2), that $(BE_8)^{h\psi^i} \simeq BX_{31}$, so, therefore

$$BE_8(q_0) \simeq BX_{31}(-iq_0) \simeq BX_{31}(q_0)\,,$$

and this last is homotopy equivalent to $BX_{31}(q)$ by our choice of q_0 with $\nu_5({q_0}^4-1) = \nu_5(q^4-1)$.

Similar considerations can be made, more generally, at any prime p such that $p \equiv 1 \bmod 4$; that is, any prime at which X_{31} can be defined, and then obtain that $BE_8(q_0) \simeq BX_{31}(q_0)$ for a prime power q_0 with ${q_0}^2+1 \equiv 0 \bmod p$.

The local structure of $E_8(q)$ was described in [40].

Remark 10.9. One can easily obtain natural maps $B\mathbf{X}_i(q^{p^n}) \to B\mathbf{X}_i(q^{p^{n+1}})$, that at the level of maximal finite tori induce inclusions $T_{\ell+n}^{(p-1)} \le T_{\ell+n+1}^{(p-1)}$, and then obtain that the p-compact group $\mathbf{X}_i$ can be reconstructed by means of a telescope construction

$$B\mathbf{X}_i \simeq \operatorname*{hocolim}_n B\mathbf{X}_i(q^{p^n})\,.$$

In particular, we may obtain the p-compact groups $B\mathbf{X}_{12}$ (at $p=3$) and $B\mathbf{X}_{31}$ (at $p=5$) as telescopes

$$\begin{aligned} B\mathbf{X}_{12} &= \operatorname{hocolim} B\mathbf{X}_{12}(4^{3^n}) = \operatorname{hocolim} B({}^2F_4(2^{3^n})) \\ B\mathbf{X}_{31} &= \operatorname{hocolim} B\mathbf{X}_{31}(16^{5^n}) = \operatorname{hocolim} BE_8(2^{5^n}) \end{aligned}$$

of p-completed classifying spaces of finite Chevalley groups.

11. Chevalley p-local finite groups from generalized Grassmannians

We discuss here the Chevalley p-local finite groups of type $\mathbf{X}(m,r,n)$. Let p be an odd prime, $m \ge 1$, $r \ge 1$, and $n \ge 1$ with $r|m|(p-1)$. The simply connected polynomial irreducible p-compact group $\mathbf{X}(m,r,n)$ has Weyl group $G(m,r,n)$ (see Section 2) and its cohomology is the invariant ring

$$H^*(B\mathbf{X}(m,r,n);\mathbb{F}_p) = H^*(BT(\mathbf{X}(m,r,n));\mathbb{F}_p)^{G(m,r,n)} \cong P[x_1,\dots,x_{n-1},e]$$

with $\deg(x_i) = 2mi$ and $\deg(e) = \frac{2mn}{r}$. See [59, 57, 52] for the construction of these spaces. We are here interested in the associated spaces $B\mathbf{X}(m,r,n)(q)$ defined by the pull-back diagram (22) with $\alpha = \psi^q$ where q is a p-adic unit.

Remark 11.1. Many cases already appear in the literature ([29, 33, 59]). We can extract the following equivalences, up to p-completion, for a prime power q, prime to p:

1. $BSU(n+1)(q) \simeq BSL_{n+1}(q)$.
2. $BU(n)(q) \simeq B\mathbf{X}(1,1,n)(q) \simeq BGL_n(q)$.
3. $B\mathbf{X}(m,1,n)(q) \simeq BGL_{mn}(q)$.
4. $B\mathbf{X}(2,2,n)(q) \simeq BSO(2n)(q) \simeq BSO^+_{2n}(q)$.

By Remark 6.6, we have that, also for any p-adic unit q, $BSU(n+1)(q)$, $B\mathbf{X}(m,1,n)(q)$ and $B\mathbf{X}(2,2,n)(q)$ are homotopy equivalent to classifying spaces of finite groups, up to p-completion.

These also include the cases $B\mathbf{X}(m,2,n)(q)$, that can be reduced to $B\mathbf{X}(2,2,n)(q')$ using propositions A.10 and 6.2, so they are also equivalent, up to p-completion, to classifying spaces of orthogonal groups over finite fields.

The above observations will be used as the starting point of the induction arguments that we will develop in the rest of this section in order to study the structure of $B\mathbf{X}(m,r,n)(q)$, for $q \equiv 1 \bmod p$, $q \neq 1$, and general values of m, r, and n.

Fix $q \equiv 1 \mod p$, $q \neq 1$. The p-compact groups $\mathbf{X}(m,r,n)$ are polynomial, hence propositions 7.5 and 7.6 apply. The maximal elementary abelian p-subgroup of $\mathbf{X}(m,r,n)$, (t_X,ν), factors as a p-subgroup, (t_X,g), of $\mathbf{X}(m,r,n)(q)$, and the maximal finite torus of $\mathbf{X}(m,r,n)(q)$ is

$$BT_\ell^n \simeq BC_{\mathbf{X}(m,r,n)(q)}(t_X,g)$$

where $\ell = \nu_p(q-1)$. The Weyl group is $W_{\mathbf{X}(m,r,n)(q)}(T_\ell^n) \cong G(m,r,n)$, and the extension $N_{\mathbf{X}(m,r,n)(q)}(T_\ell^n) \cong T_\ell^n \rtimes G(m,r,n)$ sits in the maximal torus normalizer of $\mathbf{X}(m,r,n)$, making the following diagram homotopy commutative:

$$\begin{tikzcd} BN_{\mathbf{X}(m,r,n)(q)}(T_\ell^n) \arrow[r] \arrow[d] & BN_{\mathbf{X}(m,r,n)}(T^n) \arrow[d] \\ B\mathbf{X}(m,r,n)(q) \arrow[r, "\iota"] & B\mathbf{X}(m,r,n)\,. \end{tikzcd}$$

Corollary 7.7 implies that the functor

$$\iota_\sharp\colon \mathcal{F}_p^e(\mathbf{X}(m,r,n)(q)) \to \mathcal{F}_p^e(\mathbf{X}(m,r,n)) \tag{40}$$

is an equivalence of categories. The next result is a description of the centralizers of elementary abelian p-subgroups.

Proposition 11.2. [52, 7.11] *Let p be an odd prime, $m \geq 1$, $r \geq 1$, $n \geq 1$ with $r|m|(p-1)$, and $q \equiv 1 \bmod p$, $q \neq 1$. Then,*

(1) *any elementary abelian p-subgroup $h\colon BE \to B\mathbf{X}(m,r,n)(q)$, factors through the maximal finite torus, and*
(2) *for any subgroup $E \leq t_x \leq T_\ell^n$, the centralizer of $(E, g|_{BE})$ in $\mathbf{X}(m,r,n)(q)$,*
$$BC_{\mathbf{X}(m,r,n)(q)}(E,g|_{BE}) \simeq B\mathbf{X}(m,r,n_0)(q) \times BU(n_1)(q) \times \cdots \times BU(n_s)(q)\,,$$
$n = n_0 + n_1 + \cdots + n_s$, is determined by the point-wise stabilizer of $E \leq T_\ell^n$ in the Weyl group $G(m,r,n)$, $G(m,r,n)(E) \cong G(m,r,n_0) \times \Sigma_{n_1} \times \cdots \times \Sigma_{n_s}$.

Proof. All elementary abelian p-subgroups of $\mathbf{X}(m,r,n)$ are toral, hence the same is true for $\mathbf{X}(m,r,n)(q)$ by the equivalence (40). If $E \leq t_X$, by Corollary 7.4, the restriction of ψ^q to the centralizer of $(E, g|_{BE})$, is ψ^q again, $\psi^q|_{C_{\mathbf{X}(m,r,n)(q)}(E,g|_{BE})} = \psi^q$, and

$$BC_{\mathbf{X}(m,r,n)(q)}(E,g|_{BE}) \simeq BC_{\mathbf{X}(m,r,n)}(E,\nu|_{BE})(q)\,.$$

The centralizers $C_{\mathbf{X}(m,r,n)}(E,\nu|_{BE})$ are known to be connected p-compact groups of maximal rank, with Weyl group $G(m,r,n_0) \times \Sigma_{n_1} \times \cdots \times \Sigma_{n_s}$, the point-wise stabilizer of E in T^n by the action of the Weyl group $G(m,r,n)$:

$$BC_{\mathbf{X}(m,r,n)}(E,\nu|_{BE}) \simeq B\mathbf{X}(m,r,n_0) \times BU(n_1) \times \cdots \times BU(n_s)\,,$$

thus,

$$BC_{\mathbf{X}(m,r,n)}(E,\nu|_{BE})(q) \simeq B\mathbf{X}(m,r,n_0)(q) \times BU(n_1)(q) \times \cdots \times BU(n_s)(q)$$

contains the same maximal finite torus T_ℓ^n as $\mathbf{X}(m,r,n)(q)$, $\ell = \nu_p(q-1)$, $n = n_0+n_1+\cdots+n_s$ and the Weyl group is $G(m,r,n_0) \times \Sigma_{n_1} \times \cdots \times \Sigma_{n_s}$ (see Propositions 7.5 and 7.6). $\square$

Proposition 11.3. *Let p be an odd prime, $m \geq 1$, $r \geq 1$, $n \geq 1$ with $r|m|(p-1)$, and $q \equiv 1 \bmod p$, $q \neq 1$. The natural map*

$$\operatorname*{hocolim}_{\mathcal{F}_p^e(\mathbf{X}(m,r,n)(q))^{op}} BC_{\mathbf{X}(m,r,n)(q)} \longrightarrow B\mathbf{X}(m,r,n)(q)$$

is a mod p homology equivalence.

Proof. According to Theorem F and Example 9.5

$$H^*(B\mathbf{X}(m,r,n)(q);\mathbb{F}_p) \cong H^*(BT_\ell^n;\mathbb{F}_p)^{G(m,r,n)} \cong P[x_1,\dots,x_{n-1},e] \otimes E[y_1,\dots,y_{n-1},u]$$

with $\deg(x_i) = 2mi$, $\deg(e) = \frac{2mn}{r}$, $\deg(y_i) = 2mi-1$, and $\deg(u) = \frac{2mn}{r} - 1$.

Since this is true for all values of m, r, n, we obtain from Proposition 11.2 that also, for every elementary abelian p-subgroup $E \leq t_X$,

$$H^*(BC_{\mathbf{X}(m,r,n)(q)}(E,g|_{BE});\mathbb{F}_p) \cong H^*(BT_\ell^n;\mathbb{F}_p)^{G(m,r,n)(E)}$$

where $G(m,r,n)(E)$ is the point-wise stabilizer of E in T_ℓ^n, by the action of the Weyl group $G(m,r,n)$. So, then, the result follows from Proposition 7.9. $\square$

Fix a Sylow p-subgroup of $N_{\mathbf{X}(m,r,n)(q)}(T_\ell^n)$, $S_{n,\ell} \cong \mathbb{Z}/p^\ell \wr S_n$, where S_n is the Sylow p-subgroup of the symmetric group Σ_n. Call f the composition $BS_{n,\ell} \to BN_{\mathbf{X}(m,r,n)(q)}(T_\ell^n) \to B\mathbf{X}(m,r,n)(q)$, Thus $(S_{n,\ell},f)$ is a p-subgroup of $B\mathbf{X}(m,r,n)(q)$.

We will denote by

$$\mathcal{F}(m,r,n,q) = \mathcal{F}_{(S_{n,\ell},f)}(B\mathbf{X}(m,r,n)(q))$$

the fusion system of $B\mathbf{X}(m,r,n)(q)$ over $(S_{n,\ell},f)$ and by

$$\mathcal{L}(m,r,n,q) = \mathcal{L}_{(S_{n,\ell},f)}(B\mathbf{X}(m,r,n)(q))\,,$$

the associated centric linking system. Recall that the underlying category of $\mathcal{F}(m,r,n,q)$ is equivalent to $\mathcal{F}_p(B\mathbf{X}(m,n,r)(q))$.

Theorem 11.4. *If q is a p-adic unit such that $q \equiv 1 \bmod p$, $q \neq 1$, and $\ell = \nu_p(1-q)$, then, $(S_{n,\ell},f)$ is a Sylow p-subgroup for $B\mathbf{X}(m,r,n)(q)$ and*

$$(S_{n,\ell},\mathcal{F}(m,r,n,q),\mathcal{L}(m,r,n,q))$$

is a p-local finite group with classifying space

$$|\mathcal{L}(m,r,n,q)|_p^\wedge \simeq B\mathbf{X}(m,r,n)(q)\,.$$

Proof. We proceed by induction on n, the p-rank of $\mathbf{X}(m,r,n)(q)$. For $n < p$, $\mathbf{X}(m,r,n)$ is a Clark–Ewing p-compact group, and then, $\mathbf{X}(m,r,n)(q)$ is the p-completed classifying space of a finite group (see 9.8). Also, for $BX(1,1,n) \simeq BU(n)_p^\wedge$, Remark 11.1 characterizes $BX(1,1,n)(q)$ as p-completed classifying spaces of finite groups. In all that cases, the conclusion of the theorem is clearly satisfied (see Section 3).

Assume that n is large and that the theorem holds for every $n_0 < n$. That is, for every $n_0 < n$, the space $BX(m,r,n_0)(q)$ is the classifying space of the p-local finite group

$(S_{n_0,\ell}, \mathcal{F}(m,r,n_0,q), \mathcal{L}(m,r,n_0,q))$. The result about $B\mathbf{X}(m,r,n)(q)$ will follow from Theorem 4.5. We will show that the space $B\mathbf{X}(m,r,n)(q)$ and its p-subgroup $(S_{n,\ell}, f)$ meet the conditions of 4.5. Condition (1) of 4.5 is satisfied by Proposition 7.1.

Condition (2a) of Theorem 4.5 amounts to show that if $E \leq t_X$, then the centralizer $BC_{\mathbf{X}(m,r,n)(q)}(E, g|_{BE})$ is the classifying space of a p-local finite group. This follows by the induction hypothesis. In fact, by 11.2, there is a homotopy equivalence $BC_{\mathbf{X}(m,r,n)(q)}(E, g|_{BE}) \simeq B\mathbf{X}(m,r,n_0)(q) \times BU(n_1)(q) \times \cdots \times BU(n_s)(q)$, for $n = n_0 + n_1 + \ldots n_s$, a non-trivial decomposition of n into positive summands, and by the induction hypothesis and [13, 1.4] this is the classifying space of the p-local finite group defined as the product

$$\begin{aligned}(S_{n_0,\ell}, &\mathcal{F}(m,r,n_0,q), \mathcal{L}(m,r,n_0,q)) \\ &\times (S_{n_1,\ell}, \mathcal{F}(1,1,n_1,q), \mathcal{L}(1,1,n_1,q)) \times \cdots \times (S_{n_s,\ell}, \mathcal{F}(1,1,n_s,q), \mathcal{L}(1,1,n_s,q)).\end{aligned}$$

Condition (2b) of 4.5 establishes that Sylow p-subgroups of centralizers of elementary abelian subgroups of $B\mathbf{X}(m,r,n)(q)$ factor through $(S_{n,\ell}, f)$. This is proved by reducing the question to unitary groups, obtained as centralizers of the center of $S_{n,\ell}$.

Let $Z \cong \mathbb{Z}/p$ denote the diagonal elements of order p in $T_\ell^n \cong (\mathbb{Z}/p^\ell)^n \leq S_{n,\ell}$. Then, the point-wise stabilizer of Z in T_ℓ^n by the action of $G(m,r,n)$ is Σ_n and therefore, according to Proposition 11.2, $BC_{B\mathbf{X}(m,r,n)(q)}(Z, g|_{BZ}) \simeq BU(n)(q)$.

By naturality of the construction of the normalizer of the maximal finite torus, we obtain a diagram

$$\begin{tikzcd} BN_{U(n)(q)}(T_\ell^n) \arrow[r] \arrow[d] & BN_{\mathbf{X}(m,r,n)(q)}(T_\ell^n) \arrow[d] \\ BU(n)(q) \arrow[r, "Bj_n"] & B\mathbf{X}(m,r,n)(q) \end{tikzcd}$$

hence a factorization of $(S_{n,\ell}, f)$:

$$\begin{tikzcd} & BS_{n,\ell} \arrow[dl, "f'"'] \arrow[dr, "f"] & \\ BU(n)(q) \arrow[rr, "Bj_n"] & & B\mathbf{X}(m,r,n)(q). \end{tikzcd} \tag{41}$$

Choose any other subgroup $E \leq t_X \leq S_{n,\ell}$. Assume that the point-wise stabilizer of E in T_ℓ^n by the action of $G(m,r,n)$ is $G(m,r,n)(E) \cong G(m,r,n_0) \times \Sigma_{n_1} \times \cdots \times \Sigma_{n_s}$. Define $E' = Z \cdot E \leq t_X$, then, the point-wise stabilizer of E' will be $G(m,r,n)(E') \cong \Sigma_{n_0} \times \Sigma_{n_1} \times \cdots \times \Sigma_{n_s}$. The inclusions $E \leq E' \geq Z$ induce a commutative diagram of centralizers

$$\begin{tikzcd} BC_{\mathbf{X}(m,r,n)(q)}(E', g|_{BE'}) \arrow[r, "Bj_n^\sharp"] \arrow[d] & BC_{\mathbf{X}(m,r,n)(q)}(E, g|_{BE}) \arrow[d] \\ BC_{\mathbf{X}(m,r,n)(q)}(Z, g|_{BZ}) \arrow[r, "Bj_n"] & B\mathbf{X}(m,r,n)(q). \end{tikzcd} \tag{42}$$

Now,

$$BC_{\mathbf{X}(m,r,n)(q)}(E, g|_{BE}) \simeq BX(m,r,n_0)(q) \times BU(n_1)(q) \times \cdots \times BU(n_s)(q)$$

with Sylow p-subgroup $S_{n_0,\ell} \times \cdots \times S_{n_s,\ell}$ while

$$BC_{\mathbf{X}(m,r,n)(q)}(E', g|_{BE'}) \simeq BU(n_0)(q) \times BU(n_1)(q) \times \cdots \times BU(n_s)(q)$$

and from the above discussion we have a factorization

$$\begin{tikzcd} B(S_{n_0,\ell} \times \cdots \times S_{n_s,\ell}) \arrow[r] \arrow[dr] & BU(n_0)(q) \times BU(n_1)(q) \times \cdots \times BU(n_s)(q) \arrow[d, "Bj_n^\sharp \simeq Bj_{n_0} \times 1 \times \cdots \times 1"] \\ & BX(m,r,n_0)(q) \times BU(n_1)(q) \times \cdots \times BU(n_s)(q)\,. \end{tikzcd} \tag{43}$$

Diagrams (41), (42), and (43) provide a homotopy commutative diagram

$$\begin{tikzcd} B(S_{n_0,\ell} \times \cdots \times S_{n_s,\ell}) \arrow[r] \arrow[d, dotted, "B\rho"] \arrow[rr, bend left] & BC_{\mathbf{X}(m,r,n)(q)}(E', g|_{BE'}) \arrow[r, "Bj_n^\sharp"] \arrow[d] & BC_{\mathbf{X}(m,r,n)(q)}(E, g|_{BE}) \arrow[d] \\ BS_{n,\ell} \arrow[r] \arrow[rr, bend right] & BC_{\mathbf{X}(m,r,n)(q)}(Z, g|_{BZ}) \arrow[r, "Bj_n"] & B\mathbf{X}(m,r,n)(q) \end{tikzcd}$$

where the existence of the homomorphism $\rho\colon S_{n_0,\ell} \times \cdots \times S_{n_s,\ell} \to S_{n,\ell}$ making homotopy commutative the left square is obtained because $S_{n,\ell}$ is a Sylow p-subgroup of $U(n)(q)$.

We have proved that $B\mathbf{X}(m,r,n)(q)$ and $(S_{n,\ell}, f)$ satisfy the conditions (1) and (2) of Theorem 4.5, and therefore, that $(S_{n,\ell}, f)$ is a Sylow p-subgroup of $B\mathbf{X}(m,r,n)(q)$ and $(S_{n,\ell}, \mathcal{F}(m,r,n,q), \mathcal{L}(m,r,n,q))$ is a p-local finite group.

Finally, $B\mathbf{X}(m,r,n)(q)$ is the classifying space $|\mathcal{L}(m,r,n,q)|^\wedge_p$ according to Proposition 11.3 and Theorem 4.5. $\square$

Proposition 11.5. *For $q \equiv 1 \bmod p$, $q \neq 1$, $\mathbf{X}(m,r,n)(q)$ is a exotic p-local finite group if $r > 2$, $n \geq p$.*

Notice that in the above hypothesis $r|(p-1)$, thus $r > 2$ can only occur with $p \geq 5$, so that we are implicitely assuming also that $p \geq 5$.

Proof. We wil first reduce the question to the rank p-case. Then we classify the centric radical subgroups in the fusion system of $B\mathbf{X}(m,r,p)(q)$ and show that they coincide with the p-local finite groups of [13, Example 9.4].

There is an elementary abelian p-subgroup $E \leq t_X$, in $\mathbf{X}(m,r,n)(q)$, of rank $n-p$ such that

$$C_{\mathbf{X}(m,r,n)(q)}(E, g|_{BE}) \cong \mathbf{X}(m,r,p)(q) \times U(1)^\wedge_p(q)^{n-p}$$

(see Proposition 11.2). If we assume that there is a finite group G such that $B\mathbf{X}(m,r,n)(q) \simeq BG^\wedge_p$, then the map $Bg|_{BE}\colon BE \to B\mathbf{X}(m,r,n)(q) \simeq BG^\wedge_p$ is induced by a homomorphism $\varphi\colon E \to G$, and

$$BC_G(\varphi(E))^\wedge_p \simeq B\mathbf{X}(m,r,p)(q) \times BU(1)^\wedge_p(q)^{n-p}\,.$$

Since $BU(1)^\wedge_p(q) \simeq B\mathbb{Z}/p^\ell$, the projection $BC_G(\varphi(E))^\wedge_p \to BU(1)^\wedge_p(q)^{n-p}$ is the p-completion of the map induced by a homomorphism $\rho\colon C_G(\varphi(E)) \to (\mathbb{Z}/p^\ell)^{n-p}$. It has a section, also induced by a homomorphims $\sigma\colon (\mathbb{Z}/p^\ell)^{n-p} \to C_G(\varphi(E))$, hence ρ is an epimorphism. Therefore, we have a short exact sequence $\operatorname{Ker}\rho \to C_G(\varphi(E)) \to (\mathbb{Z}/p^\ell)^{n-p}$ and an induced fibration $B(\operatorname{Ker}\rho)^\wedge_p \to BC_G(\varphi(E))^\wedge_p \to B(\mathbb{Z}/p^\ell)^{n-p}$, from which we obtain an equivalence $B(\operatorname{Ker}\rho)^\wedge_p \simeq B\mathbf{X}(m,r,p)(q)$. This reduces the question to showing that $\mathbf{X}(m,r,p)(q)$ is an exotic p-local finite group.

We will show now that $\mathbf{X}(m,r,p)(q)$ coincide with the p-local finite groups constructed in [13, Example 9.4] in purely algebraic terms. For this aim we will need to describe the centric and radical p-subgroups of $\mathbf{X}(m,r,p)(q)$.

Recall that $T_\ell^p \cong (\mathbb{Z}/p^\ell)^p$ is the maximal finite torus of $\mathbf{X}(m,r,p)(q)$ with Weyl group $G(m,r,p)$ and they form a split extension

$$T_\ell^p \to N_{\mathbf{X}(m,r,p)(q)}(T_\ell^p) \to G(m,r,p)$$

that contains $S_{p,\ell} = T_\ell^p \rtimes \mathbb{Z}/p \leq N_{\mathbf{X}(m,r,p)(q)}(T_\ell^p)$, a Sylow p-subgroup of $\mathbf{X}(m,r,p)(q)$. For simplicity we will denote $\mathcal{F} = \mathcal{F}(m,r,p,q)$, the fusion system of $B\mathbf{X}(m,r,p)(q)$ over $(S_{p,\ell}, f)$.

The center of the Sylow p-subgroup is $Z(S_{p,\ell}) \cong \mathbb{Z}/p^\ell$ embeded diagonally in T_ℓ^p, and, if we write $Z(t_X)$ for the elements of order p in $Z(S_{p,\ell})$, then we obtain $BC_{\mathbf{X}(m,r,p)(q)}(Z(S_{p,\ell}) \simeq BC_{\mathbf{X}(m,r,p)(q)}(Z(t_X)) \simeq BU(p)^\wedge_p(q)$ (see Proposition 11.2). We also know (see Remark 11.1) that $BU(p)^\wedge_p(q) \simeq BGL_p(q_0)^\wedge_p$ for a prime power q_0 with $\ell = \nu_p(1-q) = \nu_p(1-q_0)$, hence we conclude that the centralizer fusion system $C_\mathcal{F}(Z(S_{p,\ell}))$ coincides with the fusion system of $GL_p(q_0)$, that has been described in Example 3.4.

The Sylow p-subgroup $S_{p,\ell}$ is clearly centric and radical. T_ℓ^p is centric and $\mathrm{Out}_\mathcal{F}(T_\ell^p) = G(m,r,p)$ hence it is also radical ($p \geq 5$). Proper subgroups of T_ℓ^p are not centric, so we will look at subgroups $Q \leq S_{p,\ell}$ not contained in T_ℓ^p. such a subgroup fits in an extension

$$\begin{tikzcd}
Q_0 \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & \mathbb{Z}/p \arrow[d, equal] \\
T_\ell^p \arrow[r] & S_{p,\ell} \arrow[r] & \mathbb{Z}/p
\end{tikzcd}$$

where $Q_0 = Q \cap T_\ell^n$, and since Q is centric, $Z(S_{p,\ell}) \leq Q_0$. It turns out that this is actually a characteristic subgroup of Q, Hence there is an exact sequence of groups:

$$1 \to \mathrm{Aut}_{C_\mathcal{F}(Z(S_{p,\ell}))}(Q) \to \mathrm{Aut}_\mathcal{F}(Q) \to \mathrm{Aut}_\mathcal{F}(Z(S_{p,\ell}))$$

where $\mathrm{Aut}_\mathcal{F}(Z(S_{p,\ell})) \cong \mathbb{Z}/(m/r)$ is given by the action of the Adams operations of exponents a (m/r)th root of unity.

Assume that Q is abelian. Then $Q_0 = Z(S_{p,\ell})$ and Q is either $\mathbb{Z}/p \times Z(S_{p,\ell})$ or cyclic $Z/p^{\ell+1}$. In the first case, Q is $\mathcal{F}$-conjugate to a subgroup of T_ℓ^n, hence it is not centric while in the second case, it is conjugate to the group $U_{\ell+1}$ described in Example 3.4. Adams operations do not act internally in $U_{\ell+1}$, hence $\mathrm{Out}_\mathcal{F}(U_{\ell+1}) \cong \mathrm{Out}_{C_\mathcal{F}(Z(S_{p,\ell}))}(U_{\ell+1}) \cong \mathbb{Z}/p$ and then $U_{\ell+1}$ is not radical in $\mathcal{F}$.

Assume that Q is non-abelian. The same arguments as in 10.5 show that Q is either $S_{p,\ell}$ or Γ_ℓ, and both are radical in $C_\mathcal{F}(Z(S_{p,\ell}))$. Thus we obtain that they complete the list of conjugacy classes of centric radical subgroups of $S_{p,\ell}$ in $\mathcal{F}$.

In order to complete the picture it remains to compute the $\mathcal{F}$-automorphisms of Γ_ℓ. We have $\mathrm{Out}_{C_\mathcal{F}(Z(S_{p,\ell}))}(\Gamma_\ell) \cong SL_2(p)$. Now, the Adams operations act internally in Γ_ℓ and we get $\mathrm{Out}_\mathcal{F}(\Gamma_\ell) \cong SL_2(p).(m/r)$.

By Alperin's fusion theorem, a fusion system over S is generated by the automorphisms of its fully normalized centric radical subgroups in S. Since in our case al the automorphisms of T_ℓ^p are induced by conjugation in $N_{\mathbf{X}(m,r,p)(q)}(T_\ell^p)$, we can write

$$\mathcal{F}(m,r,p,q) = \langle \mathcal{F}_{N_{\mathbf{X}(m,r,p))(q)}(T_\ell^p)}(S_{p,\ell}); \mathcal{F}_{\Gamma_\ell}(SL_2(p).(m/r)) \rangle$$

(see Section 3) but this is precisely the definition of the fusion systems in [13, Example 9.4]. $\square$

The cases $B\mathbf{X}(m,r,n)(q)$ with $r=1,2$ or $n<p$, are homotopy equivalent to p-completed classifying spaces of finite groups according to Theorem 9.8 and Remark 11.1.

Appendix A. Recognition of homotopy fixed point p-compact groups

The objective of this appendix is to obtain a recognition principle for the homotopy fixed point p-compact group BX^{hG} where p is an odd prime, X a connected p-compact group, G a finite group of order prime to p, and $\rho\colon G \to \mathrm{Out}(X)$ and outer action.

Let $N \to X$ be the maximal torus normalizer for the p-compact group X. The short exact sequence of topological monoids

$$BZ(N) = \mathrm{aut}(BN)_1 \to \mathrm{aut}(BN) \to \mathrm{Out}(N)$$

induces a fibration sequence

$$B^2Z(N) \to B\mathrm{aut}(BN) \to B\mathrm{Out}(N)$$

which shows that equivalence classes of fibrations over BG with fibre BN is in one-to-one correspondence with

$$[BG, B\mathrm{Out}(N)] = \mathrm{Hom}(G, \mathrm{Out}(N))\,.$$

Also, we know from Theorem B that equivalence classes of fibrations over BG with fibre BX is in one-to-one correspondence with

$$[BG, B\mathrm{Out}(X)] = \mathrm{Hom}(G, \mathrm{Out}(X))\,.$$

However, $\mathrm{Out}(X) \cong \mathrm{Out}(N)$ [52, 7] and therefore there is a bijective correspondence between fibrations with fibre BX over BG and fibrations with fibre BN over BG. We shall now make this correspondence more explicit.

Define the group-like topological monoid $\mathrm{aut}(Bj)$ to be the submonoid of $\mathrm{aut}(BN) \times \mathrm{aut}(BX)$ consisting of all pairs $(a,b) \in \mathrm{aut}(BN) \times \mathrm{aut}(BX)$ such that the diagram

$$\begin{tikzcd} BN \arrow[r, "a"] \arrow[d, "Bj"'] & BN \arrow[d, "Bj"] \\ BX \arrow[r, "b"'] & BX \end{tikzcd}$$

commutes.

Lemma A.1. *Assume that p is odd. The forgetful homomorphisms*

$$\mathrm{aut}(BN) \longleftarrow \mathrm{aut}(Bj) \longrightarrow \mathrm{aut}(BX)$$

are homotopy equivalences.

Proof. The group homomorphisms $\pi_0\,\mathrm{aut}(BN) \leftarrow \pi_0\,\mathrm{aut}(Bj) \to \pi_0\,\mathrm{aut}(BX)$ are injective because X has N-determined automorphisms [52, 7]. The group homomorphism to the left is surjective because X is N-determined and the one to the right is surjective because any self-homotopy equivalence of BX lifts to a self-homotopy equivalence of BN [50, §3]. The identity components fit into a map of fibrations [27, 11.10]

$$\begin{tikzcd} \mathrm{aut}_{BX}(Bj)_1 \arrow[r] \arrow[d, equal] & \mathrm{aut}(Bj)_1 \arrow[r] \arrow[d] & \mathrm{aut}(BX)_1 \arrow[d, "\simeq"] \\ \mathrm{aut}_{BX}(Bj)_1 \arrow[r] & \mathrm{aut}(BN)_1 \arrow[r] & \mathrm{Map}(BN, BX)_{Bj} \end{tikzcd}$$

where the right vertical map, defined by composition with Bj, is a homotopy equivalence [27, 7.5, 1.3] [25, 9.1] [50, 3.4]. The fibre, consisting of the space of maps $BN \to BN$ over BX and vertically homotopic to the identity map of BN, is (one component) of the space $(X/N)^{hN}$ which is contractible [48, 5.1]. $\square$

Thus we have bijections

$$[B, B\mathrm{aut}(BN)] = [B, B\mathrm{aut}(Bj)] = [B, B\mathrm{aut}(BX)]$$

for any space B and this means BN-fibrations and BX-fibrations over B are in bijective correspondence.

Proposition A.2. *Let X be a connected p-compact group with maximal torus normalizer $N \to X$. If G is a finite group of order prime to p, then any outer action $\rho\colon G \to \mathrm{Out}(X)$, lifts to a unique G-action on BX and unique G-action on BN. Moreover, these actions make the map $BN \to BX$ G-equivariant; that is, the diagram*

$$\begin{tikzcd} BN \arrow[r] \arrow[d] & BN_{hG} \arrow[r] \arrow[d] & BG \arrow[d, equal] \\ BX \arrow[r] & BX_{hG} \arrow[r] & BG \end{tikzcd}$$

is homotopy commutative.

Proof. Let us say that our input is an outer action

$$\rho\colon G \to \mathrm{Out}(X) = W\backslash N_{GL(L)}(W) = \mathrm{Out}(N) \tag{44}$$

of the finite group G on X and N. By Theorem B, ρ lifts to a unique action of G on BX, and by Lemma A.1 the same is true for BN. In particular, ρ determines a unique map, up to homotopy,

$$\widetilde{B\rho}\colon BG \longrightarrow B\mathrm{aut}(Bj)$$

inducing ρ on fundamental groups.

Over $B\mathrm{aut}(Bj)$ there are two related fibrations

$$\begin{tikzcd} BN_{h\,\mathrm{aut}(Bj)} \arrow[rr] \arrow[dr] & & BX_{h\,\mathrm{aut}(Bj)} \arrow[dl] \\ & B\mathrm{aut}(Bj) & \end{tikzcd}$$

with fibre BN and BX, respectively. Pull back these two related fibrations along the map $\widetilde{B\rho}$ to obtain the commutative diagram of the Proposition. $\square$

Next, we need to lift the action of G on BN and BX to an action on the loop spaces N and X (see Definition 5.3), such that the inclusion $N \to X$ is still equivariant.

Lemma 5.1 applies to show that the fibration $BX \to BX_{hG} \to BG$ admits a section, unique up to vertical homotopy, when X is connected; that is, there is a unique lifting of the action on BX to an action on X. However, BN is not simply connected as $\pi_1(BN) \cong W$ and then Lemma 5.1 ensures neither the existence nor the uniqueness of a lifting of the action of G on BN to an action of G on N. Instead, it leads to the next description of the possible actions.

Proposition A.3. *If a finite group G of order prime to p acts on BN with outer action $\rho\colon G \to W\backslash N_{GL(L)}(W) \cong \mathrm{Out}(N)$, then there are natural one-to-one correspondences between the sets:*

(1) $\pi_0(BN^{hG})$,

(2) *W-conjugacy classes of lifts in the diagram*

$$\begin{tikzcd} & N_{GL(L)}(W) \arrow[d] \\ G \arrow[ur, dotted] \arrow[r, "\rho"'] & W\backslash N_{GL(L)}(W)\,. \end{tikzcd}$$

If these sets are non-empty, then they are also in one-to-one correspondence with $H^1(G;W)$.

Proof. An action of G on BN is by definition a fibration

$$BN \to BN_{hG} \to BG\,, \tag{45}$$

and according to A.2 this action of G on BN is uniquely determined by ρ.

Next, we map $\pi_0(BN^{hG})$ directly to the set (2). Let $\varphi\colon BG \to B\,\mathrm{aut}(BN)$ be a classifying map for the fibration (45). Thus, φ extends to a map of fibrations

$$\begin{tikzcd} BN \arrow[r] \arrow[d, equal] & BN_{hG} \arrow[r] \arrow[d] & BG \arrow[d, "B\rho"] \\ BN \arrow[r] & B\mathrm{aut}_*(BN) \arrow[r] & B\mathrm{aut}(BN) \end{tikzcd}$$

into the universal BN-fibration. Here, $\mathrm{aut}_*(BN)$ is the topological monoid of based self-homotopy equivalences of BN. On the level of fundamental groups we get an induced morphism

$$\begin{tikzcd} W \arrow[r] \arrow[d, equal] & \pi_0(N_{hG}) \arrow[r] \arrow[d] & G \arrow[d, "\rho"] \\ W \arrow[r] & N_{GL(L)}(W) \arrow[r] & W\backslash N_{GL(L)}(W) \end{tikzcd} \tag{46}$$

of group extensions. Here we use the short exact sequence from [49, 5.2] in combination with the vanishing results from [6, 3.3].

We have seen (Lemma 5.1) that the existence of an action of G on N lifting the action on BN is equivalent to the existence of a section of the exact sequence on the top row of (46), and the diagram shows that this is equivalent to the existence of a lifting of ρ to a homomorphism $\sigma\colon G \to N_{GL(L)}(W)$. This gives the bijection between $\pi_0(BN^{hG})$ and the set (2).

Finally, if these sets are nonempty, then obstruction theory as in Lemma 5.1 shows that they are in one-to-one correspondence with the set $H^1(G;W) = H^1(G;\pi_1(BN))$. □

Proposition A.4. *Let X be a connected p-compact group with Weyl group W and maximal torus normalizer $N \to X$. If G is a finite group of order prime to p and*

$$\rho\colon G \to \mathrm{Out}(X) \cong W\backslash N_{GL(L)}(W)$$

is an outer action, then ρ lifts to a unique action of G on X, and each lift

$$\sigma\colon G \to N_{GL(L)}(W)$$

determines a unique action of G on N such that the inclusion $N \to X$ is G-equivariant.

Proof. The first part was proved in Proposition A.2. According to Proposition A.3, the actions of G on N that lift the given outer action are in one-to-one correspondence with lifts of ρ to $N_{GL(L)}(W)$. If we view one of these actions as a sectioned fibration

$$\begin{tikzcd} BN \arrow[r] & BN_{hG} \arrow[r, shift left] & BG \arrow[l, shift left] \end{tikzcd}$$

it clearly induces an action on X that makes $N \to X$ equivariant:

$$\begin{tikzcd} BN_{hG} \arrow[rr] \arrow[dr] & & BX_{hG} \arrow[dl] \\ & BG . \arrow[ul, dotted, bend left] \arrow[ur, dotted, bend right] & \end{tikzcd}$$

The proposition follows because there is only one action of G on X inducing ρ. $\square$

Proposition A.5. *Let p be an odd prime and G a finite group of order prime to p. Assume that G acts on the connected p-compact group X and that*

$$\overline{\rho}\colon G \to N_{GL(L)}(W)$$

is a lift of the given outer action. If Y is a connected p-compact group that satisfies

(1) $W^{\overline{\rho}G}$ contains a subgroup $\overline{W}$, complementary to the kernel of $W^{\overline{\rho}G} \to GL(L^{\overline{\rho}G})$, such that $(\overline{W}, L(X)^{\overline{\rho}G})$ is a reflection group similar to $(W(Y), L(Y))$, and

(2) $QH^*(BY;\mathbb{Q}_p) \cong QH^*(BX;\mathbb{Q}_p)_G$,

then $BY = BX^{hG}$.

Proof. By the classification theorem for p-compact groups at odd primes [52, 7], it suffices [51, 1.2] to find an map $BN(Y) \to BX^{hG}$ that induces an isomorphism on $H^*(-;\mathbb{Q}_p)$ and restricts to monomorphism on the p-normalizer $N_p(Y)$, is a p-monomorphism. The homomorphism $\overline{\rho}$ corresponds (A.4) to compatible G-actions $BG \to BN(X)_{hG} \to BX_{hG}$ on $N(X)$ and X. Taking homotopy fixed points we obtain a commutative diagram of loop space morphisms

$$\begin{tikzcd} N(X)^{hG} \arrow[r] \arrow[d] & X^{hG} \arrow[d] \\ N(X) \arrow[r] & X \end{tikzcd}$$

which shows that $N(X)^{hG} \to X^{hG}$ is a p-monomorphism. Since the discrete approximation to $N(X)$, $N(X)^{hG}$, and $N(Y)$ are semi-direct products [6], there is a p-monomorphism $N(Y) \to N(X)^{hG}$ for $W(Y)$ is a subgroup $W^{\overline{\rho}G} = \pi_0 N(X)^{hG}$ by the first condition. By the second condition, $H^*(BY;\mathbb{Q}_p) = H^*(BN(Y);\mathbb{Q}_p)$ and $H^*(BX^{hG};\mathbb{Q}_p)$ are abstractly isomorphic graded vector spaces. Therefore, Y and X^{hG} have the same rank [25, 5.9] so that $T(Y) \to N(X)^{hG} \to X^{hG}$ is a maximal torus and $H^*(BX^{hG};\mathbb{Q}_p) \to H^*(BN(Y);\mathbb{Q}_p)$ is injective [25, 9.7], hence bijective. $\square$

A special case arises when G acts through unstable Adams operations so that the action $\pi_0\rho\colon G \to \mathrm{Out}(N) \to \mathrm{Out}(W)$ is trivial. Then the image of G in $\mathrm{Out}(N) = W\backslash N_{GL(L)}(W)$

is contained in the subgroup $Z(W)\backslash C_{GL(L)}(W)$ [52, 3.16] and we have a morphism

$$\begin{tikzcd} W \arrow[r] \arrow[d, equal] & \pi_0(N_{hG}) \arrow[r] \arrow[d] & G \arrow[d, "B\rho"] \\ W \arrow[r] & W.C_{GL(L)}(W) \arrow[r] & Z(W)\backslash C_{GL(L)}(W) \end{tikzcd}$$

of group extensions. The possible extensions occurring in the upper line, realizing the trivial action $G \to \mathrm{Out}(W)$, are classified by $H^2(G; Z(W))$; they are all isomorphic to

$$W \to Z(W)\backslash(D \times W) \to G$$

for some central extension $Z(W) \to D \to G$ [41, IV.§8]. If $Z(W)$ is trivial, $\pi_0(N_{hG}) = G \times W$ and $H^1(G; W) = \mathrm{Rep}(G, W)$.

Assume that $G = C_r$ is a cyclic group of order r, and the outer action of G on X, $\rho\colon C_r \to \mathrm{Out}(X)$, is given by an Adams operation $\rho(\lambda) = \psi^\lambda$, where $\lambda \in \mathbb{Z}_p^\times$ is a p-adic unit of order $r|(p-1)$. We can lift $\psi^\lambda \in Z(W)\backslash C_{GL(L)}(W)$ to an element $\zeta \in C_{GL(L)}(W)$, such that $\zeta^r \in Z(W)$. If there is a choice of ζ with $\zeta^r = 1$, then $\bar\rho\lambda = \zeta$ provides a lifting of ρ.

Assume, otherwise, that ζ^r has order s in $Z(W)$. Since p is odd, $Z(W)$ has order prime to p, hence s is prime to p. Now, even if there is no lift of the action of C_r on X to an action on N, we can reduce the problem by extending the action of C_r to an action of C_{sr} on X determined by $\rho'(\lambda) = \psi^\lambda \in Z(W)\backslash C_{GL(L)}(W) \subset \mathrm{Out}(X)$, that now admits the lift $\bar\rho'(\lambda) = \zeta$. Notice that $C_s = \langle \lambda^r \rangle$ acts trivially on X, so that $BX^{hC_s} \simeq BX$, and then $BX^{hC_{sr}} \cong BX^{hC_r}$, so we can still determine BX^{hC_r} by analyzing the equivariant action of C_{sr} on N and X.

Notice also, that if W is irreducible, then $C_{GL(L)}(W)$ consists of diagonal matrices and therefore ζ is an Adams operation.

Corollary A.6. *Let $\lambda \in \mathbb{Z}_p^\times$ be a p-adic unit of order $r|(p-1)$. Consider the outer action $\rho\colon C_r = \langle\lambda\rangle \to W\backslash N_{GL(L)}(W)$ through unstable Adams operations given by $\rho(\lambda) = \psi^\lambda$. Then, if ρ admits a lift $\bar\rho\colon C_r \to N_{GL(L)}(W)$, then all possible lifts are parameterized by $H^1(C_r; W) = \mathrm{Rep}(C_r, W)$, the set of conjugacy classes of order r elements w of W, and*

$$(W^{\overline{\rho}C_r}, L^{\overline{\rho}C_r}) = (C_W(w), L^{\langle\lambda w\rangle})$$

for the lift $\overline{\rho}(\lambda) = \lambda w$ corresponding to w.

Proof. The lifts

$$\begin{tikzcd} & W.C_{GL(L)}(W) \arrow[d] \\ C_r = \langle\lambda\rangle \arrow[ur, dotted, "\overline{\rho}"] \arrow[r, "\rho"'] & Z(W)\backslash C_{GL(L)}(W) \end{tikzcd}$$

are given by $\overline{\rho}(\lambda) = w\psi^\lambda$ where $w \in W$ is any element of order r. $\square$

We next apply the recognition principle (A.6) in some concrete cases.

A.7. **The three infinite families.** We identify the fixed point p-compact groups for the actions of finite cyclic groups of order prime to p through unstable Adams operations on the p-compact groups of the three infinite families of irreducible p-compact groups, namely the projective or special unitary groups, the generalized Grassmannians, and the Sullivan spheres (as defined in Section 2).

Proposition A.8 (Sullivan spheres). *Let p be an odd prime. Suppose that m and $r > 1$ divide $p-1$. Consider the outer action through unstable Adams operations $\psi\colon C_r \to \mathrm{Out}(S^{2m-1}) \cong \mathbb{Z}_p^\times/C_m$ of the cyclic group $C_r \leq \mathbb{Z}_p^\times$ on the Sullivan sphere S^{2m-1}. Then the homotopy fixed point group is*

$$(S^{2m-1})^{hC_r} = \begin{cases} S^{2m-1} & r \mid m \\ * & \textit{otherwise,} \end{cases}$$

Proof. Let λ be a primitive rth root of unity, so that $C_r = \langle \lambda \rangle \leq \mathbb{Z}_p^*$. According to Theorem B, $(S^{2m-1})^{h\langle\lambda\rangle}$ is a connected polynomial p-compact group. If r does not divide m, $H^{2m}(\psi^\lambda) = \lambda^m$ is nontrivial, so that the vector space of covariants $QH^*(BS^{2m-1};\mathbb{Q}_p)_{\langle\lambda\rangle}$ vanishes in positive degrees, and the fixed point p-compact group is trivial. If r does divide m, ψ^λ acts trivially on S^{2m-1}, because the kernel of ψ is C_m which contains C_r, and the fixed point p-compact group is again S^{2m-1}. □

Proposition A.9 (Special unitary groups). *Let p be an odd prime. Suppose that $m > 1$ divides $p-1$, and let $C_m = \langle\lambda\rangle \subset \mathbb{Z}_p^\times$ be the cyclic group generated by a primitive mth root of unity acting through unstable Adams operations. Then*

$$X(mn+s)^{hC_m} = U(mn+s)^{hC_m} = \begin{cases} X(m,1,n) & n > 0 \\ * & n = 0 \end{cases}$$

for any p-compact group $X(mn+s)$ locally isomorphic to $SU(mn+s)$, $0 \leq s < m$.

Proof. In the rational cohomology algebras $H^*(BU(mn+s);\mathbb{Q}_p) = \mathbb{Q}_p[c_1,\dots,c_{mn+s}]$ and $H^*(BX(mn+s);\mathbb{Q}_p) = \mathbb{Q}_p[c_2,\dots,c_{mn+s}]$ we have

$$c_i \text{ is preserved by } H^{2i}(\psi^\lambda) \iff m|i$$

and therefore

$$\begin{aligned} QH^*(BU(mn+s);\mathbb{Q}_p)_{C_m} = \mathbb{Q}_p\{c_m,\dots,c_{mn}\} &= QH^*(BX(m,1,n);\mathbb{Q}_p) \\ &= QH^*(BX(mn+s);\mathbb{Q}_p)_{C_m}\,. \end{aligned}$$

The Weyl group $W = \Sigma_{mn+s}$ is the symmetric group in its natural representation on $L = \mathbb{Z}_p^{mn+s}$. Let $e_1,\dots,e_{mn+s}$ be the canonical basis vectors of L. The permutation

$$w = (1\cdots m)(m+1\cdots 2m)\cdots(m(n-1)+1\cdots mn) \in \Sigma_{mn+s}$$

has order m and

$$\begin{aligned} &(C_{\Sigma_{mn+s}}(w), L^{\langle\lambda w\rangle}) \\ &\quad= (C_m \wr \Sigma_n \times \Sigma_s, \mathbb{Z}_p\{\lambda e_1 + \lambda^2 e_2 + \cdots + \lambda^m e_m, \dots, \lambda e_{m(n-1)+1} + \cdots + \lambda^m e_{mn}\}) \end{aligned}$$

contains the reflection group $G(m,1,n) = C_m \wr \Sigma_n$ as a a subgroup complementary to the kernel, Σ_s, of the action of $(C_{\Sigma_{mn+s}}(w)$ on $L^{\langle\lambda w\rangle}$. This means (A.6) that the fixed point p-compact group $U(mn+s)^{hC_m} = X(m,1,n)$.

From the two short exact sequences of $\mathbb{Z}_p\Sigma_{mn+s}$-modules [52, §10]

$$0 \to \mathbb{Z}_p \xrightarrow{\Delta} L \to LPU(mn+s) \to 0, \quad 0 \to LX(mn+s) \to LPU(mn+s) \to \check{\pi} \to 0$$

where Δ is the diagonal and $\check{\pi}$ a subgroup of $\pi_1(PU(mn+s)) = \mathbb{Z}_p/\mathbb{Z}_p(mn+s)$ (with trivial Σ_{mn+s}-action), we get that

$$L^{\langle\lambda w\rangle} = LPU(mn+s)^{\langle\lambda w\rangle} = LX(mn+s)^{\langle\lambda w\rangle}$$

as $\mathbb{Z}_p C_{\Sigma_{mn+s}}(w)$-modules. $\square$

The proof of (A.10) will make use of the following elementary facts:

- For arbitrary natural numbers m and n we write m_n for $m/\gcd(m,n)$. Then $m_n n = \mathrm{lcm}(m,n)$ and $m_n n_m = \mathrm{lcm}(m,n)/\gcd(m,n)$.
- $C_{\mathrm{lcm}(q,m)} = \langle \lambda, \mu | \lambda^q = 1, \mu^m = 1, \lambda\mu = \mu\lambda, \lambda^{q_m} = \mu^{m_q} \rangle$.
- Let $A(a,t) \in GL(\mathbb{Z}_p, t)$ denote the linear automorphism
$$A(a,t)(x_1,\dots,x_t) = (ax_t, x_1, \dots, x_{t-1})$$
where $a \in \mathbb{Z}_p^\times$ is a unit. The ith power $A(a,t)^i$ has characteristic polynomial $(x^{t_i} - a^{i_t})^{t/t_i}$ and $A(a,t)^t = aE$.
- If $\lambda \in \mathbb{Z}_p^\times$ has order q, then $A(\lambda^{-q_m}, q_m)$ also has order q for $A(\lambda^{-q_m}, q_m)^{q_m} = \lambda^{-q_m} E$ has order $\gcd(q,m)$. The λ^{-1} eigenspace of $A(\lambda^{-q_m}, q_m)$ has rank one and $A(\lambda^{-q_m}, q_m)^{-1}$ acts on it as multiplication by λ.
- In the exact sequence $1 \to A^{\langle g \rangle} \to C_{A \rtimes G}(a,g) \to C_G(g)$ the image in $C_G(g)$ consists of those $h \in C_G(g)$ that fix $a \in A/(1-g)A$.

Proposition A.10 (Generalized Grassmannians)**.** *Let $\mathbf{X}(m,r,n)$, $m \geq 2$, $r \geq 1$, $n \geq 2$, $r \mid m \mid p-1$, be the irreducible polynomial p-compact group corresponding to the imprimitive reflection group $G(m,r,n)$. Suppose that the natural number ℓ divides $p-1$ and let the cyclic group $C_\ell \subset C_{\mathrm{lcm}(\ell,m)} \subset \mathbb{Z}_p^\times$ act on $\mathbf{X}(m,r,n)$ through unstable Adams operations. The homotopy fixed point group for this action is*
$$\mathbf{X}(m,r,n)^{hC_\ell} = \begin{cases} \mathbf{X}(\mathrm{lcm}(\ell,m), r, n/\ell_m) & r\ell \mid mn \\ \mathbf{X}(\mathrm{lcm}(\ell,m), 1, n/\ell_m - 1) & r\ell \nmid mn\,, \ell \mid mn, \\ \mathbf{X}(\mathrm{lcm}(\ell,m), 1, [n/\ell_m]) & \ell \nmid mn \end{cases}$$
where $\ell_m = \ell/\gcd(\ell,m)$ and $[n/\ell_m]$ is the biggest integer $\leq n/\ell_m$. (By convention, $G(m,r,1)$ is cyclic of order m/r and $G(m,r,0)$ is the trivial group.)

Proof. Let $\lambda \in \mathbb{Z}_p^\times$ be a primitive ℓth root of unity. In the rational cohomology algebra $H^*(B\mathbf{X}(m,r,n);\mathbb{Q}_p) \cong \mathbb{Q}_p[x_1,\dots,x_{n-1},e]$ the degrees $|x_i| = 2im$ and $|e| = 2\frac{m}{r}n$ so that
$$x_i \text{ is preserved by } H^{2im}(\psi^\lambda) = \lambda^{im} \Leftrightarrow \ell \mid im \Leftrightarrow \ell_m \mid i$$
$$e \text{ is preserved by } H^{2\frac{m}{r}n}(\psi^\lambda) = \lambda^{\frac{m}{r}n} \Leftrightarrow \ell \mid nm/r \Leftrightarrow \ell_{m/r} \mid n$$
and thus $QH^*(B\mathbf{X}(m,r,n);\mathbb{Q}_p)_{C_\ell}$ is isomorphic to the indecomposables of the rational cohomology algebra of the p-compact group on the right hand side of the equation.

We have $r\ell \mid mn \Leftrightarrow \ell_{m/r} \mid n$, $\ell \mid mn \Leftrightarrow \ell_m \mid n$, and $\ell_m \mid \ell_{m/r} \mid \ell | p-1$.

$\underline{\ell_{m/r} \mid n}$: The element
$$w = \mathrm{diag}\,\big(\underbrace{A(\lambda^{-\ell_m}, \ell_m), \dots, A(\lambda^{-\ell_m}, \ell_m)}_{n/\ell_m}\big) \in G(m,r,n)$$
has order ℓ. Since $((\lambda^{-\ell_m})^{n/\ell_m})^{m/r} = \lambda^{-mn/r} = 1$ because $\ell | (mn/r)$ by assumption, w does indeed belong to the index r subgroup $G(m,r,n)$ of $G(m,1,n) = C_m \wr \Sigma_n$. Let $\{e_1,\dots,e_n\}$ be the canonical basis for the free $\mathbb{Z}_p$-module $L = \mathbb{Z}_p^n$ on which $G(m,r,n)$ acts. The free $\mathbb{Z}_p$-module
$$L^{\langle \lambda w \rangle} = \left\langle e_1 + \lambda e_2 + \cdots + \lambda^{\ell_m - 1} e_{\ell_m}, \dots, e_{(n-\ell_m)+1} + \lambda e_{(n-\ell_m)+2} + \cdots + \lambda^{\ell_m-1} e_n \right\rangle,$$

has rank n/ℓ_m. We shall now compute the centralizer of w. Let ζ be a generator of the cyclic group $C_{\mathrm{lcm}(\ell,m)} \subset \mathbb{Z}_p^\times$ so that $C_m = \langle \mu \rangle$ and $C_\ell = \langle \lambda \rangle$ with $\mu = \zeta^{\ell_m}$ and $\lambda = \zeta^{m_\ell}$. The homomorphisms $A(\ell, 1, n/\ell_m) \longrightarrow C_{G(m,1,n)}(w) \longleftarrow A(m, 1, n/\ell_m)$ defined by

$$\lambda_i \to \mathrm{diag}\left(\underbrace{E, \dots, E}_{i-1}, A(\lambda^{-\ell_m}, \ell_m)^{-1}, E, \dots, E\right), \quad \mathrm{diag}\left(\underbrace{E, \dots, E}_{i-1}, \mu E, E, \dots, E\right) \leftarrow \mu_i$$

combine to a homomorphism defined on $A(\mathrm{lcm}(\ell, m), 1, n/\ell_m)$ since they agree on their common domain $A(\gcd(\ell, m), 1, n/\ell_m) = \langle \mu^{m_\ell} \rangle^{n/\ell_m} = \left\langle \lambda^{\ell_m} \right\rangle^{n/\ell_m}$, Observe that $(\lambda^{a_1}, \dots, \lambda^{a_{n/\ell_m}}) \in A(\ell, 1, n/\ell_m)$ lies in the subgroup $A(\mathrm{lcm}(\ell, m), r, n/\ell_m)$ if and only if its image lies in $G(m, r, n)$ and that $(\mu^{a_1}, \dots, \mu^{a_{n/\ell_m}}) \in A(m, 1, n/\ell_m)$ lies in the subgroup $A(m, r, n/\ell_m)$ if and only if its image lies in $G(m, r, n)$. Together with the diagonal $\Delta\colon \Sigma_{n/\ell_m} \to \Sigma_n$ given by $\Delta(\sigma)((i-1)\ell_m + j) = (\sigma(i) - 1)\ell_m + j$, $1 \leq \ell i \leq n/\ell_m$, $1 \leq j \leq \ell_m$, we obtain a group isomorphism

$$G(\mathrm{lcm}(\ell, m), 1, n/\ell_m) \xrightarrow{\cong} C_{G(m,1,n)}(w)$$

that restricts to a group isomorphism $G(\mathrm{lcm}(\ell, m), r, n/\ell_m) \cong C_{G(m,r,n)}(w)$ between index r subgroups. This isomorphism identifies the pair $(C_{G(m,r,n)}(w), L^{\langle \lambda w \rangle})$ and the imprimitive reflection group $(G(\mathrm{lcm}(\ell, m), r, n/\ell_m), \mathbb{Z}_p^{n/\ell_m})$.

$\underline{\ell_{m/r} \nmid n,\ \ell_m \mid n}$: It will suffice to consider the case of $G(m, m, n)$ where $\ell \nmid n$ and $\ell_m \mid n$. The element

$$w = \mathrm{diag}\left(\underbrace{A(\lambda^{-\ell_m}, \ell_m), \dots, A(\lambda^{-\ell_m}, \ell_m)}_{n/\ell_m - 1}, A(\lambda^{-\ell_m}, \ell_m)^{1-n/\ell_m}\right) \in G(m, m, n)$$

has order ℓ. Note that λ^{-1} is not an eigenvalue for $A(\lambda^{-\ell_m}, \ell_m)^{1-n/\ell_m}$ because

$$\begin{aligned} A(\lambda^{-\ell_m}, \ell_m)^{1-n/\ell_m} \text{ has eigenvalue } \lambda^{-1} &\Leftrightarrow A(\lambda^{-\ell_m}, \ell_m)^{n/\ell_m - 1} \text{ has eigenvalue } \lambda \\ &\Leftrightarrow \lambda^{(\ell_m)_{n/\ell_m - 1}} = \lambda^{-\ell_m (n/\ell_m - 1)\ell_m} \Leftrightarrow \ell \mid (\ell_m)_{n/\ell_m - 1} + \ell_m (n/\ell_m - 1)_{\ell_m} \\ &\Leftrightarrow \ell \mid n/\gcd(\ell_m, n/\ell_m - 1) \Rightarrow \ell \mid n \Rightarrow \ell_m \mid n \end{aligned}$$

which is not the case. Therefore the λ^{-1}-eigenspace

$$L^{\langle \lambda w \rangle} = \left\langle e_1 + \lambda e_2 + \dots + \lambda^{\ell_m - 1} e_{\ell_m}, \dots, e_{(n-2\ell_m)+1} + \lambda e_{(n-2\ell_m)+2} + \dots + \lambda^{\ell_m - 1} e_{n-\ell_m} \right\rangle$$

has rank $n/\ell_m - 1$. The two monomorphisms $A(\ell, 1, n/\ell_m - 1) \to C_{G(m,m,n)}(w)$ and $C_{G(m,m,n)}(w) \leftarrow A(m, 1, n/\ell_m - 1)$ given by

$$\lambda_i \to \mathrm{diag}\left(\underbrace{E, \dots, E}_{i-1}, A(\lambda^{-\ell_m}, \ell_m)^{-1}, E, \dots, E, A(\lambda^{-\ell_m}, \ell_m)\right)$$

$$\mathrm{diag}\left(\underbrace{E, \dots, E}_{i-1}, \mu E, E, \dots, E, \mu^{-1} E\right) \leftarrow \mu_i$$

agree on their common domain $A(\gcd(\ell, m), 1, n/\ell_m - 1)$ and together with the monomorphism $\Sigma_{n/\ell_m - 1} \xhookrightarrow{\Delta} \Sigma_{n-\ell_m} \hookrightarrow \Sigma_m$ they define a homomorphism on the group $A(\mathrm{lcm}(\ell, m), 1, n/\ell_m - 1) \rtimes \Sigma_{n/\ell_m - 1}$ such that the composition

$$A(\mathrm{lcm}(\ell, m), 1, n/\ell_m - 1) \rtimes \Sigma_{n/\ell_m - 1} \hookrightarrow C_{G(m,m,n)}(w) \twoheadrightarrow \mathrm{Im}\left(C_{G(m,m,n)}(w) \to GL(L^{\langle \lambda w \rangle})\right)$$

is an isomorphism with image similar to $G(\operatorname{lcm}(\ell,m),1,n/\ell_m-1)$.
$\underline{\ell_m\nmid n}$: It will suffice to consider the case of $G(m,m,n)$. The element

$$w=\operatorname{diag}\big(\underbrace{A(\lambda^{-\ell_m},\ell_m),\dots,A(\lambda^{-\ell_m},\ell_m)}_{[n/\ell_m]},\underbrace{\lambda^{\ell_m[n/\ell_m]},1,\dots,1}_{n-\ell_m[n/\ell_m]}\big)\in G(m,m,n)$$

has order ℓ. Note that λ^{-1} is not an eigenvalue for $\lambda^{\ell_m[n/\ell_m]}$ because

$$\lambda^{\ell_m[n/\ell_m]}=\lambda^{-1}\Leftrightarrow \ell\mid \ell_m[n/\ell_m]+1\Leftrightarrow \ell_m\gcd(\ell,m)\mid \ell_m[n/\ell_m]+1\Rightarrow \ell_m\mid 1$$

which is not the case as $\ell_m>1$. Therefore the λ^{-1} eigenspace $L^{\langle\lambda w\rangle}$ has rank $[n/\ell_m]$. The two monomorphisms $A(\ell,1,[n/\ell_m])\longrightarrow C_{G(m,m,n)}(w)\longleftarrow A(m,1,[n/\ell_m])$ given by

$$\lambda_i\to\operatorname{diag}\big(\underbrace{E,\dots,E}_{(i-1)\ell_m},A(\lambda^{-\ell_m},\ell_m)^{-1},E,\dots,E,\underbrace{\lambda^{-\ell_m},1,\dots,1}_{n-\ell_m[n/\ell_m]}\big)$$

$$\operatorname{diag}\big(\underbrace{E,\dots,E}_{(i-1)\ell_m},\mu E,E,\dots,E,\underbrace{\mu^{-\ell_m},1,\dots,1}_{n-\ell_m[n/\ell_m]}\big)\leftarrow\mu_i$$

agree on their common domain $A(\gcd(\ell,m),1,[n/\ell_m])$ and together with the inclusion of permutation groups $\Sigma_{[n/\ell_m]}\overset{\Delta}{\hookrightarrow}\Sigma_{\ell_m[n/\ell_m]}\hookrightarrow\Sigma_m$, they define a homomorphism on the group $A(\operatorname{lcm}(\ell,m),1,[n/\ell_m])\rtimes\Sigma_{[n/\ell_m]}$ such that the composition

$$A(\operatorname{lcm}(\ell,m),1,[n/\ell_m])\rtimes\Sigma_{[n/\ell_m]}\hookrightarrow C_{G(m,m,n)}(w)\twoheadrightarrow\operatorname{Im}\big(C_{G(m,m,n)}(w)\to GL(L^{\langle\lambda w\rangle})\big)$$

is an isomorphism with image similar to the reflection group $G(\operatorname{lcm}(\ell,m),1,[n/\ell_m])$. ☐

The outer automorphism group of $\mathbf{X}(m,r,n)$ is isomorphic to $A(m,r,n)\backslash\mathbb{Z}_p^\times A(m,1,n)$ except in the cases $(m,r,n)\in\{(2,1,2),(4,2,2),(3,3,3),(2,2,4)\}$ [57, §6] [52, 7.14]. The (exotic) homotopy action

$$\rho\colon C_m=\langle\mu\rangle\to\operatorname{Out}(\mathbf{X}(m,r,n))\cong A(m,r,n)\backslash\mathbb{Z}_p^\times A(m,1,n)$$

that takes the generator μ of C_m to $A(m,r,n)(\mu,1,\dots,1)$ is distinct from the actions through unstable Adams operations of (A.10) when $\gcd(r,n)>1$ [52, 7.14].

Proposition A.11 (Exotic actions on generalized Grassmannians)**.** *Assume that $m\geq 2$, $r\geq 1$, $n\geq 2$, and $(m,r,n)\notin\{(2,1,2),(4,2,2),(3,3,3),(2,2,4)\}$. Then the homotopy fixed point p-compact group*

$$\mathbf{X}(m,r,n)^{hC_m}=\mathbf{X}(m,1,n-1)$$

for the above exotic homotopy action on $\mathbf{X}(m,r,n)$.

Proof. The second assumption of (A.5) is clearly satisfied as the action preserves the generators $x_1,\dots,x_{n-1}$ but does not preserve the generator e. To verify the first assumption, take $\overline{\rho}\colon C_m\to N_{GL(L)}(G(m,r,n))=\mathbb{Z}_p^\times G(m,1,n)$ to be the obvious choice $\overline{\rho}(\mu)=(\mu,1,\dots,1)$. Then

$$G(m,r,n)^{\overline{\rho}C_m}=A(m,r,n)\rtimes\Sigma_{n-1},\quad L^{\overline{\rho}C_m}=\mathbb{Z}_p^{n-1}$$

and the composition

$$A(m,1,n-1)\rtimes\Sigma_{n-1}\hookrightarrow G(m,r,n)^{\overline{\rho}C_m}\twoheadrightarrow\operatorname{Im}\big(G(m,r,n)^{\overline{\rho}C_m}\to GL(L^{\overline{\rho}C_m})\big)$$

where the first morphism is $(\mu_2,\dots,\mu_m)\to((\mu_2\cdots\mu_n)^{-1},\mu_2,\dots,\mu_n)$, $\Sigma_{n-1}\hookrightarrow\Sigma_n$, identifies the group to the right as the reflection group $G(m,1,n-1)$. ☐

The results of A.9 and A.11 were obtained by Castellana [17] using different methods.

A.12. **The sporadic p-compact groups.** As in Section 2 we write G_i, $4 \leq i \leq 37$, for the sporadic irreducible and simply connected $\mathbb{Z}_p$-reflection group with number i in the Clark–Ewing classification table, and $\mathbf{X}_i$ for the corresponding simply connected p-compact group. When $\mathbf{X}_i$ is defined at the odd prime p and r divides $p-1$, $\mathbf{X}_i^{hC_r}$ denotes the fixed point p-compact group for the homotopy action $\psi\colon C_r \to \mathrm{Out}(\mathbf{X}_i)$ through unstable Adams operations on the p-compact group $\mathbf{X}_i$. We identify the fixed point p-compact groups for actions through unstable Adams operations on the 34 sporadic irreducible p-compact groups. We may summarize our results in the following diagrams

$$\begin{tikzcd}
 & S^{59} & \mathbf{X}_{32} \arrow[l, "5"'] \arrow[r, "5"] \arrow[dr, "4"] & S^{59} & S^{35} & \\
\mathbf{X}_{16} \arrow[ur, "3"] \arrow[dr, "4"'] & \mathbf{X}_{37} \arrow[l, "5"'] \arrow[ur, "3"] \arrow[dr, "4"] & & \mathbf{X}_{10} \arrow[d, "8"] & \mathbf{X}_{34} \arrow[l, "4"'] \arrow[u, "9"'] \arrow[ul, "5"'] \arrow[r, "7"] \arrow[d, "5"] & S^{83} \\
 & S^{39} & \mathbf{X}_{31} \arrow[l, "5"] \arrow[ur, "3"] \arrow[d, "8"] & S^{47} & S^{59} & \\
 & & \mathbf{X}_{9} \arrow[ur, "3"'] & & &
\end{tikzcd} \tag{13}$$

$$\begin{tikzcd}
S^{35} & \mathbf{X}_{36} \arrow[l, "18"'] \arrow[dl, "6"'] \arrow[dr, "4"] & & S^{9} & \mathbf{X}_{35} \arrow[l, "5"'] \arrow[r, "9"] \arrow[dl, "2"'] \arrow[dr, "3"] & S^{17} \\
\mathbf{X}_{26} \arrow[u, "18"] \arrow[dr, "12"'] & & \mathbf{X}_{8} \arrow[dl, "12"] \arrow[d, "8"] & \mathbf{X}_{28} \arrow[l, "4"'] \arrow[dr, "3"'] & & \mathbf{X}_{25} \arrow[dl, "2"] \arrow[d, "12"] \\
 & S^{23} & S^{15} & & \mathbf{X}_{5} \arrow[r, "12"'] & S^{23}
\end{tikzcd}$$

where, for instance, $\mathbf{X}_{32} \xrightarrow{4} \mathbf{X}_{10}$ means that $\mathbf{X}_{32}^{hC_4} = \mathbf{X}_{10}$ (when $p \equiv 1 \bmod 12$) and $\mathbf{X}_{32} \xrightarrow{5} S^{59}$ means that $\mathbf{X}_{32}^{hC_5} = S^{59}$ (when $p \equiv 1 \bmod 30$). The relevant primes are mentioned in the more detailed explanations below but not displayed in the above diagrams. We use (A.6) to identify the homotopy fixed point groups. With a computer algebra program it is quite easy to find eigenspaces for the elements of these reflection groups. We used the program MAGMA.

(1) ($G_{37} = W(E_8), C_3, G_{32}, p \equiv 1 \bmod 3$) There is an element $w \in G_{37}$ of order 3 and a primitive 3rd root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{37}}(w), L_{37}^{\langle \lambda w \rangle}) = (G_{32}, L_{32})$$

meaning that that $E_8^{hC_3} = \mathbf{X}_{37}^{hC_3} = \mathbf{X}_{32}$.

(2) ($G_{37} = W(E_8), C_4, G_{31}, p \equiv 1 \bmod 4$) There is an element $w \in G_{37}$ of order 4 such that

$$(C_{G_{37}}(w), L_{37}^{\langle iw \rangle}) = (G_{31}, L_{31})$$

meaning that $E_8^{hC_4} = \mathbf{X}_{37}^{hC_4} = \mathbf{X}_{31}$.

(3) ($G_{37} = W(E_8), C_5, G_{16}, p \equiv 1 \bmod 15$) There is an element $w \in G_{37}$ of order 5 and a primitive 5th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{37}}(w), L_{37}^{\langle\lambda w\rangle}) = (G_{16}, L_{16})$$

meaning that $E_8^{hC_5} = \mathbf{X}_{37}^{hC_5} = \mathbf{X}_{16}$.

(4) ($G_{34}, C_4, G_{10}, p \equiv 1 \bmod 12$). There exists an element $w \in G_{34}$ of order 4, a (index 4) subgroup G of $C_{G_{34}}(w)$, and a primitive 4th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(G, L_{34}^{\langle\lambda w\rangle}) = (G_{10}, L_{10})$$

meaning that $\mathbf{X}_{34}^{hC_4} = \mathbf{X}_{10}$.

(5) ($G_{32}, C_4, G_{10}, p \equiv 1 \bmod 12$) There is an element $w \in G_{32}$ of order 4 and a primitive 4th root of unity $i \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{32}}(w), L_{32}^{\langle iw\rangle}) = (G_{10}, L_{10})$$

which means that $\mathbf{X}_{32}^{hC_4} = \mathbf{X}_{10}$.

(6) ($G_{32}, C_{30}, C_5, p \equiv 1 \bmod 30$) There is an element $w \in G_{32}$ of order 5 and a primitive 5th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{32}}(w), L_{32}^{\langle\lambda w\rangle}) = (C_{30}, \mathbb{Z}_p)$$

which means that $\mathbf{X}_{32}^{hC_5} = S^{59}$.

(7) ($G_{31}, C_3, G_{10}, p \equiv 1 \bmod 12$). There exists an element $w \in G_{31}$ of order 3 and a primitive 3rd root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{31}}(w), L_{31}^{\langle\lambda w\rangle}) = (G_{10}, L_{10})\,.$$

This means that $\mathbf{X}_{31}^{hC_3} = \mathbf{X}_{10}$. (The group that the computer finds is G_{10} and not G_{15} (of the same rank and the same degrees) because the elements of order 8 square to central elements [61, p. 281].)

(8) ($G_{31}, C_8, G_9, p \equiv 1 \bmod 24$). There exists an element $w \in G_{31}$ of order 8 and a primitive 8th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that the reflection group

$$(C_{G_{31}}(w), L_{31}^{\langle\lambda w\rangle}) = (G_9, L_9)$$

which means that $\mathbf{X}_{31}^{hC_8} = \mathbf{X}_9$.

(9) ($G_{10}, C_8, C_{24}, p \equiv 1 \bmod 24$) There is an element $w \in G_{10}$ of order 8 and a primitive 8th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{10}}(w), L_{10}^{\langle\lambda w\rangle}) = (C_{24}, \mathbb{Z}_p)$$

which means that $\mathbf{X}_{10}^{hC_8} = S^{47}$.

(10) ($G_9, C_3, C_{24}, p \equiv 1 \bmod 24$) There is an element $w \in G_9$ of order 3 and a primitive 3rd root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_9}(w), L_9^{\langle\lambda w\rangle}) = (C_{24}, \mathbb{Z}_p)$$

which means that $\mathbf{X}_9^{hC_3} = S^{47}$.

(11) ($G_{34}, C_9, C_{18}, p \equiv 1 \bmod 18$) There is an element $w \in G_{34}$ of order 9 and a primitive 9th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{34}}(w), L_{34}^{\langle\lambda w\rangle}) = (C_{18}, \mathbb{Z}_p)$$

which means that $\mathbf{X}_{34}^{hC_9} = S^{37}$.

(12) $(G_{36} = W(E_7), C_6, G_{26}, p \equiv 1 \bmod 6)$ There is an element $w \in G_{36}$ of order 6 and a primitive 6th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{36}}(w), L_{36}^{\langle \lambda w\rangle}) = (G_{26}, L_{26})$$

which means that $E_7^{hC_6} = \mathbf{X}_{36}^{hC_6} = \mathbf{X}_{26}$.

(13) $(G_{36} = W(E_7), C_4, G_8, p \equiv 1 \bmod 8)$. There is an element $w \in G_{36}$ of order 4, a subgroup $\overline{W} < C_{G_{36}}(w)$ of index 8, faithfully represented in $L_{36}^{\langle iw\rangle}$, and a primitive 4th root of unity $i \in \mathbb{Z}_p^\times$ such that

$$(\overline{W}, L_{36}^{\langle iw\rangle}) = (G_8, L_8)$$

which means that $E_7^{hC_4} = \mathbf{X}_{36}^{hC_4} = G_8$. (The reflection group $\overline{W}$ contains elements of order 8 with central square so it is not similar to G_{13} [61, p. 281].)

(14) $(G_{36} = W(E_7), C_{14}, C_{14}, p \equiv 1 \bmod 14)$ There is an element $w \in G_{36}$ of order 14 and a primitive 14th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{36}}(w), L_{36}^{\langle \lambda w\rangle}) = (C_{14}, \mathbb{Z}_p)$$

which means that $E_7^{hC_{14}} = \mathbf{X}_{36}^{hC_{14}} = S^{27}$.

(15) $(G_{36} = W(E_7), C_{18}, C_{18}, p \equiv 1 \bmod 18)$ There is an element $w \in G_{36}$ of order 18 and a primitive 18th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{36}}(w), L_{36}^{\langle \lambda w\rangle}) = (C_{18}, \mathbb{Z}_p)$$

which means that $E_7^{hC_{18}} = \mathbf{X}_{36}^{hC_{18}} = S^{35}$.

(16) $(G_{26}, C_{18}, C_{18}, p \equiv 1 \bmod 18)$ There is an element $w \in G_{26}$ of order 18 and a primitive 18th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{26}}(w), L_{26}^{\langle \lambda w\rangle}) = (C_{18}, \mathbb{Z}_p)$$

which means that $\mathbf{X}_{26}^{hC_{18}} = S^{35}$.

(17) $(G_8, C_{12}, C_{12}, p \equiv 1 \bmod 12)$ There is an element $w \in G_8$ of order 12 and a primitive 12th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_8}(w), L_8^{\langle \lambda w\rangle}) = (C_{12}, \mathbb{Z}_p)$$

which means that $\mathbf{X}_8^{hC_{12}} = S^{23}$.

(18) $(G_8, C_8, C_8, p \equiv 1 \bmod 8)$ There is an element $w \in G_8$ of order 8 and a primitive 8th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_8}(w), L_8^{\langle \lambda w\rangle}) = (C_8, \mathbb{Z}_p)$$

which means that $\mathbf{X}_8^{hC_8} = S^{15}$.

(19) $(G_{35} = W(E_6), C_2, G_{28} = W(F_4), p \equiv 1 \bmod 2)$ There is an element $w \in G_{35}$ of order 2 such that

$$(C_{G_{35}}(w), L_{35}^{\langle -w\rangle}) = (G_{28}, L_{28})$$

which means that $E_6^{hC_2} = \mathbf{X}_{35}^{hC_2} = F_4$.

(20) ($G_{35} = W(E_6), C_3, G_{25}, p \equiv 1 \bmod 3$) There is an element $w \in G_{35}$ of order 3 and a primitive 3rd root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{35}}(w), L_{35}^{\langle \lambda w \rangle}) = (G_{25}, L_{25})$$

which means that $E_6^{hC_3} = \mathbf{X}_{35}^{hC_3} = G_{25}$.

(21) ($G_{35} = W(E_6), C_5, G_{25}, p \equiv 1 \bmod 5$) There is an element $w \in G_{35}$ of order 5 and a primitive 5th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{35}}(w), L_{35}^{\langle \lambda w \rangle}) = (C_5, \mathbb{Z}_p)$$

which means that $E_6^{hC_5} = \mathbf{X}_{35}^{hC_5} = S^9$

(22) ($G_{35} = W(E_6), C_4, G_8, p \equiv 1 \bmod 4$) There is an element $w \in G_{35}$ of order 4 and a primitive 4th root of unity $i \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{35}}(w), L_{35}^{\langle iw \rangle}) = (G_8, L_8)$$

which means that $E_6^{hC_4} = \mathbf{X}_{35}^{hC_4} = G_8$.

(23) ($G_{25}, C_2, G_5, p \equiv 1 \bmod 6$) There is an element $w \in G_{25}$ of order 2 such that

$$(C_{G_{25}}(w), L_{25}^{\langle -w \rangle}) = (G_5, L_5)$$

which means that $\mathbf{X}_{25}^{hC_2} = \mathbf{X}_5$.

(24) ($G_{28} = W(F_4), C_3, G_5, p \equiv 1 \bmod 6$) There is an element $w \in G_{28}$ of order 3 and a primitive 3rd root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{28}}(w), L_{28}^{\langle \lambda w \rangle}) = (G_5, L_5)$$

which means that ${F_4}^{hC_3} = \mathbf{X}_{28}^{hC_3} = \mathbf{X}_5$.

(25) ($G_{28} = W(F_4), C_4, G_4, p \equiv 1 \bmod 4$) There is an element $w \in G_{28}$ of order 4 and a primitive 4th root of unity $i \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{28}}(w), L_{28}^{\langle iw \rangle}) = (G_8, L_8)$$

which means that ${F_4}^{hC_4} = \mathbf{X}_{28}^{hC_4} = \mathbf{X}_8$.

(26) ($G_{25}, C_{12}, C_{12}, p \equiv 1 \bmod 12$) There is an element $w \in G_{25}$ of order 12 and a primitive 12th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{25}}(w), L_{25}^{\langle \lambda w \rangle}) = (C_{12}, \mathbb{Z}_p)$$

which means that $\mathbf{X}_{25}^{hC_{12}} = S^{23}$.

(27) ($G_{25}, C_{12}, C_{12}, p \equiv 1 \bmod 12$) There is an element $w \in G_{25}$ of order 12 and a primitive 12th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_5}(w), L_5^{\langle \lambda w \rangle}) = (C_{12}, \mathbb{Z}_p)$$

which means that $\mathbf{X}_{25}^{hC_{12}} = S^{23}$.

Appendix B. Derived functors of inverse limit functor

In this appendix we discuss higher limits over some finite categories of a special type.

Given a finite group G and subgroups $H_1, H_2, \ldots, H_k \leq G$, we define a finite category $\mathbb{I}(k)$ with objects $\{0, 1, 2, \ldots, k\}$, where G is the group of automorphisms of 0 and for each $i > 0$, $H_i \backslash G = \mathrm{Hom}_{\mathbb{I}(k)}(i, 0)$ as G-sets and $\mathrm{Aut}_{\mathbb{I}(k)}(i) = N_G(H_i)/H_i$, and other morphism sets are empty. Those categories appear in the context of the Aguadé p-compact groups and other compact Lie groups, as categories of elementary abelian subgroups. The next result is essentially contained in [1, 52].

Proposition B.1. *Let* $\mathbf{M}$ *be a given diagram of* $\mathbb{Z}_p$*-modules index by the category* $\mathbb{I}(k)$*. Assume that*

(a) *Restriction gives an isomorphism* $H^j(G; A) \cong H^j(H_1; A)$*, for any* $\mathbb{Z}_{(p)}G$*-module* A *and* $j \geq 1$*.*

(b) $p \nmid |N_G(H_i)|$ *and* $M_i = M_0^{H_i}$*, for every* $i \geq 2$*.*

Then there is an exact sequence

$$0 \to \varprojlim_{\mathbb{I}(k)}{}^0 \mathbf{M} \to M_1^{N_G(H_1)/H_1} \oplus M_0^G \to M_0^{N_G(H_1)} \to \varprojlim_{\mathbb{I}(k)}{}^1 \mathbf{M} \to 0\,,$$

and $\varprojlim^j{}_{\mathbb{I}(k)} \mathbf{M} = 0$ *if* $j \geq 2$*.*

Proof. We consider a star-shaped category $\mathbb{I}(k)$ with $k+1$ objects $\{0, 1, 2, \ldots, k\}$. There is an exact sequence of the form [52]

$$\begin{aligned}
0 \to & \lim{}^0 \mathbf{M} \to M_0^G \times \prod_{i>0} M_i^{N_G(H_i)/H_i} \to \prod_{i>0} M_0^{N_G(H_i)} \\
\to & \lim{}^1 \mathbf{M} \to H^1(G; M_0) \times \prod_{i>0} H^1(N_G(H_i)/H_i; M_i) \to \prod_{i>0} H^1(N_G(H_i); M_0) \\
\to & \lim{}^2 \mathbf{M} \to H^2(G; M_0) \times \prod_{i>0} H^2(N_G(H_i)/H_i; M_i) \to \prod_{i>0} H^2(N_G(H_i); M_0) \\
\to & \lim{}^3 \mathbf{M} \to \cdots
\end{aligned}$$

Under condition (b) this exact sequence reduces to the exact sequence

$$\begin{aligned}
0 \to & \lim{}^0 \mathbf{M} \to M_0^G \times M_1^{N_G(H_1)/H_1} \to M_0^{N_G(H_1)} \\
\to & \lim{}^1 \mathbf{M} \to H^1(G; M_0) \times H^1(N_G(H_1)/H_1; M_1) \to H^1(N_G(H_1); M_0) \\
\to & \lim{}^2 \mathbf{M} \to H^2(G; M_0) \times H^2(N_G(H_1)/H_1; M_1) \to H^2(N_G(H_1); M_0) \\
\to & \lim{}^3 \mathbf{M} \to \cdots
\end{aligned}$$

Condition (a) implies that H_1 and G have the same Sylow p-subgroup. Hence p does not divide $|N_G(H_1)/H_1|$ and so therefore $H^*(N_G(H_1); A) \cong H^*(H_1; A)^{N_G(H_1)/H_1}$. Now, in the diagram given by restrictions $H^j(G; A) \to H^j(N_G(H_1); A) \to H^j(H_1; A)$, $j \geq 1$, the composition is an isomorphism and the second arrow is a monomorphism, hence both arrows are isomorphisms:

$$H^j(G; A) \cong H^j(N_G(H_1); A) \cong H^j(H_1; A)\,, \quad j \geq 1\,,$$

and the Proposition follows. $\square$

References

[1] J. Aguadé, *Constructing modular classifying spaces*, Israel J. Math. **66** (1989), 23–40

[2] J. Aguadé, *Constructing modular classifying spaces, II*, preprint

[3] J. Aguadé and L. Smith, *Modular cohomology algebras*, Amer. J. Math. **107** (1985), 507–530

[4] J.L. Alperin and P. Fong, *Weights for symmetric and general linear groups*, J. Algebra **31** (1990), 2–22

[5] M. Aubry, *Homotopy theory and models. Based on lectures held at a DMV seminar in Blaubeuren by H. J. Baues, S. Halperin and J.-M. Lemaire.* DMV Seminar, 24. Birkhuser Verlag, Basel, 1995. x+117 pp.

[6] K.K.S. Andersen, *The normalizer splitting conjecture for p-compact groups*, Algebraic topology (Kazimierz Dolny, 1997), Fund. Math. **161** (1999), 1–16

[7] K.K.S. Andersen, J. Grodal, J.M. Møller, and A. Viruel, *The classification of p-compact groups for p odd*, preprint

[8] D. Benson, *Cohomology of sporadic groups, finite loop spaces, and the Dickson invariants*, Geometry and cohomology in group theory (Durham, 1994), 10–23, London Math. Soc. Lecture Note Ser., 252, Cambridge Univ. Press, Cambridge, 1998

[9] A.K. Bousfield, *The localization of spaces with respect to homology*, Topology **14** (1975), 133–150

[10] A.K. Bousfield and D.M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics, vol. 304, Springer-Verlag, 1972

[11] C. Broto, *Caràcters lineals i fibracions esfèriques* (Catalan). Thesis. Universitat Autònoma de Barcelona, 1988

[12] C. Broto, R. Levi, and B. Oliver, *Homotopy equivalences of p-completed classifying spaces of finite groups*, Invent. Math. **151** (2003), 611–664

[13] C. Broto, R. Levi, and B. Oliver, *The homotopy theory of fusion systems,* J. Amer. Math. Soc. **16** (2003), 779–856

[14] C. Broto, R. Levi, and B. Oliver, *The theory of p-local groups: a survey*, to appear in Contemporary Math. Proceedings of Northwestern University International Conference on Algebraic Topology, March 24-28, 2002

[15] C. Broto and J.M. Møller, *Embeddings of DI_2 in F_4*, Trans. Amer. Math. Soc. **353** (2001), 4461–4479

[16] C. Broto, L. Smith, and R. Stong, *Thom modules and pseudoreflection groups*, J. Pure Appl. Algebra **60** (1989), 1–20

[17] N. Castellana, *Quillen grasmmaniann̈s as non-modular homotopy fixed points*, Math. Z. **248** (2004), 477–493

[18] A. Clark and J. Ewing, *The realization of polynomial algebras as cohomology rings*, Pacific J. Math. **50** (1974), 425–434

[19] E. Dror Farjoun, *Cellular spaces, null spaces and homotopy localization*, Lecture Notes in Mathematics, 1622. Springer-Verlag, Berlin, 1996

[20] W.G. Dwyer, *Sharp homology decompositions for classifying spaces of finite groups*, Group representations: cohomology, group actions and topology (Seattle, WA, 1996), Amer. Math. Soc., Providence, RI, 1998, pp. 197–220

[21] W.G. Dwyer, *The centralizer decomposition of BG*. Algebraic topology: new trends in localization and periodicity (Sant Feliu de Guxols, 1994), 167–184, Progr. Math., 136, Birkhäuser, Basel, 1996

[22] W.G. Dwyer, *Lie groups and p-compact groups*, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998). Doc. Math. 1998, Extra Vol. II, 433–442 (electronic)

[23] W.G. Dwyer, H.R. Miller, and C.W. Wilkerson, *Homotopical uniqueness of classifying spaces*, Topology **31** (1992), 29–45

[24] W.G. Dwyer and C.W. Wilkerson, *A new finite loop space at the prime two,* J. Amer. Math. Soc. **6** (1993), 37–64

[25] W.G. Dwyer and C.W. Wilkerson, *Homotopy fixed-point methods for Lie groups and finite loop spaces*, Ann. of Math. **139** (1994), 395–442

[26] W.G. Dwyer and C.W. Wilkerson, *Product splittings for p-compact groups*, Fund. Math. **147** (1995), 279–300

[27] W.G. Dwyer and C.W. Wilkerson, *The center of a p-compact group*, The Čech centennial (Boston, MA, 1993), 119–157, Contemp. Math., 181, Amer. Math. Soc., Providence, RI, 1995

[28] W.G. Dwyer and C.W. Wilkerson, *Kähler differentials, the T-functor, and a theorem of Steinberg,* Trans. Amer. Math. Soc. **350** (1998), 4919–4930

[29] Z. Fiedorowicz and S. Priddy, *Homology of classical groups over finite fields and their associated infinite loop spaces*, Lecture Notes in Mathematics, 674. Springer, Berlin, 1978

[30] Z. Fiedorowicz and S. Priddy, *Homology of classical groups over a finite field.* Algebraic K-theory (Proc. Conf., Northwestern Univ., Evanston, Ill., 1976), pp. 269–277. Lecture Notes in Mathematics, 551, Springer, Berlin, 1976

[31] E.M. Friedlander, *Exceptional isogenies and the classifying spaces of simple Lie groups*, Ann. Math. **101** (1975), 510–520

[32] E.M. Friedlander, *Unstable K-theories of the algebraic closure of a finite field,* Comment. Math. Helv. **50** (1975), 145–154

[33] E.M. Friedlander, *Étale homotopy of simplicial schemes,* Annals of Mathematics Studies, 104. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1982

[34] D. Gorenstein, *Finite Groups*. Second edition. Chelsea Publishing Co., New York, 1980

[35] B. Harris, *Suspensions and characteristic maps for symmetric spaces*, Ann. of Math. **76** (1962), 295–305

[36] J.R. Harper, *The* mod 3 *homotopy type of* F_4, Localization in group theory and homotopy theory, and related topics (Sympos., Battelle Seattle Res. Center, Seattle, Wash., 1974), pp. 58–67. Lecture Notes in Mathematics 418, Springer, Berlin, 1974

[37] S. Jackowski, J. McClure, and B. Oliver, *Self-homotopy equivalences of classifying spaces of compact connected Lie groups*, Fund. Math. **147** (1995), 99–126

[38] S.N. Kleinerman, *The cohomology of Chevalley groups of exceptional Lie type,* Mem. Amer. Math. Soc. **39** (1982), no. 268

[39] R. Levi, and B. Oliver, *Construction of 2-local finite groups of a type studied by Solomon and Benson*, Geometry and Topology, **6** (2002) Paper no. 28, pages 917–990

[40] M.W. Liebeck, J. Saxl, and G.M. Seitz, *Subgroups of maximal rank in finite exceptional groups of Lie type*, Proc. London Math. Soc. **65** (1992), 297–325

[41] S. MacLane, *Homology*, Reprint of the 1975 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995

[42] G. Malle, *The maximal subgroups of* ${}^2F_4(q^2)$, J. Algebra **139** (1991), 52–69

[43] M. Mimura, M. Tezuka, and S. Tsukuda, *On the cohomology of finite Chevalley groups*, in preparation

[44] M. Mimura, G. Nishida, and H. Toda, Mod*p decomposition of compact Lie groups*, Publ. Res. Inst. Math. Sci. **13** (1977/78), 627–680

[45] M. Mimura, H. Toda, *Topology of Lie groups. I, II.* Translated from the 1978 Japanese edition by the authors. Translations of Mathematical Monographs, 91. American Mathematical Society, Providence, RI, 1991

[46] J.M. Møller, *Spaces of sections of Eilenberg-Mac Lane fibrations*, Pacific J. Math. **130** (1987), 171–186

[47] J.M. Møller, *Homotopy Lie groups*, Bull. Amer. Math. Soc. (N.S.) **32** (1995), 413–428

[48] J.M. Møller, *Rational isomorphism of p-compact groups*, Topology **35** (1996), 201–225

[49] J.M. Møller, *Extensions of p-compact groups*, Algebraic topology: new trends in localization and periodicity (Sant Feliu de Guíxols, 1994), Birkhäuser, Basel, 1996, pp. 307–327

[50] J.M. Møller, *Deterministic p-compact groups*, Stable and unstable homotopy (Toronto, ON, 1996), 255–278, Fields Inst. Commun., 19, Amer. Math. Soc., Providence, RI, 1998

[51] J.M. Møller, *Normalizers of maximal tori*, Math. Z. **231** (1999), 51–74

[52] J.M. Møller, *N-determined p-compact groups*, Fund. Math. **173** (2002), 201–300

[53] J.M. Møller and D. Notbohm, *Connected finite loop spaces with maximal tori,* Trans. Amer. Math. Soc. **350** (1998), 3483–3504

[54] D. Notbohm, *Unstable splittings of classifying spaces of p-compact groups,* Q. J. Math. **51** (2000), 237–266

[55] H. Mui, *Modular invariant theory and cohomology algebras of symmetric groups*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **22** (1975), 319–369

[56] D. Notbohm, *Classifying spaces of compact Lie groups and finite loop spaces*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 1049–1094

[57] D. Notbohm, *Topological realization of a family of pseudoreflection groups,* Fund. Math. **155** (1998), 1–31

[58] L. Puig, *Frobenius categories,* J. Algebra **303** (2006), 309–357
[59] D. Quillen, *On the cohomology and K-theory of the general linear groups over a finite field*, Ann. of Math. **96** (1972), 552–586
[60] J.-P. Serre, *Cohomologie galoisienne*, Lecture Notes in Mathematics, 5. Springer-Verlag, Berlin, 1994
[61] G.C. Shephard and J.A. Todd, *Finite unitary reflection groups*, Can. J. Math **6** (1954), 274–304
[62] L. Smith, *On the realization and classification of symmetric algebras as cohomology rings,* Proc. Amer. Math. Soc. 87 (1983), 144–148
[63] R. Solomon, *Finite groups with Sylow 2-subgroups of type* .3, J. Algebra **28** (1974), 182–198
[64] J. Stasheff, *The* mod$p$ *decomposition of Lie groups.* Localization in group theory and homotopy theory, and related topics (Sympos., Battelle Seattle Res. Center, Seattle, Wash., 1974), pp. 142–149. Lecture Notes in Mathematics, 418, Springer, Berlin, 1974
[65] D.P. Sullivan, *Geometric topology: localization, periodicity and Galois symmetry. The 1970 MIT notes.* Edited and with a preface by Andrew Ranicki. K-Monographs in Mathematics, 8. Springer, Dordrecht, 2005. pp. xiv+283
[66] C.W. Wilkerson, *Self-maps of classifying spaces.* Localization in group theory and homotopy theory, and related topics (Sympos., Battelle Seattle Res. Center, Seattle, Wash., 1974), pp. 150–157. Lecture Notes in Mathematics, 418, Springer, Berlin, 1974
[67] C.W. Wilkerson, *Rings of invariants and inseparable forms of algebras over the Steenrod algebra.* Recent progress in homotopy theory (Baltimore, MD, 2000), pp. 381–396, Contemp. Math., 293, Amer. Math. Soc., Providence, RI, 2002
[68] C. Xu, *Computing invariant polynomials of p-adic reflection groups.* Mathematics of Computation 1943–1993: a half-century of computational mathematics (Vancouver, BC, 1993), 599–602, Proc. Sympos. Appl. Math., 48
[69] A. Zabrodsky,*On the realization of invariant subgroups of* $\pi_*(X)$, Trans. Amer. Math. Soc. **285** (1984), 467–496

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