VERTEX COLORINGS OF SIMPLICIAL COMPLEXES

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Contents

1. Introduction

Let K be a finite abstract simplicial complex (ASC), V the vertex set of K, and P a finite set of colors. We say that K is (P, s)-colorable if it is possible to paint V with colors from the palette P in such a way that no simplex of K contains more than s vertices of the same color. In other words, a (P, s)-coloring of K is a map $f : V \to P$ so that $|\sigma \cap f^{-1}p| \leq s$ for all simplices σ of K. A $(P, 1)$-coloring is usually called just a coloring. Section 5 and Section 6 contain more detailed discussions of usual colorings and our relaxed colorings.

Figure 1 shows a 2-dimensional complex MB with 5 vertices, triangulating the Möbius band. This complex

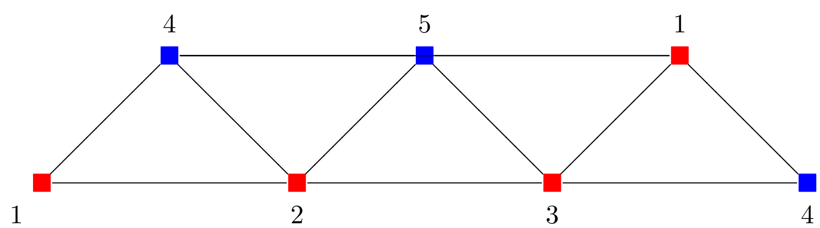

Figure 1. A ({■, ■}, 2)-coloring of the Möbius band

is ({■, ■}, 2)-colorable. One needs a palette P of 5 colors, one for each vertex, for a $(P, 1)$-coloring.

The purpose of this paper is to translate colorability problems from combinatorics to ring and homotopy theory.

Let $\mathrm{SR}(K; R)$ be the Stanley–Reisner face algebra of K with coefficients in the commutative ring R. Also, let $R[V]$ be the graded polynomial R-algebra generated by the vertex set V which we consider to be homogeneous of degree 2. By definition (3.2) there is a surjection $R[V] \twoheadrightarrow \mathrm{SR}(K; R)$. For any subset U of the vertex set V, let $c(U)$ denote the symmetric polynomial $\prod_{u \in U}(1 + u)$ in the polynomial ring $R[V]$ or its image in $\mathrm{SR}(K; R)$ under the projection $R[V] \twoheadrightarrow \mathrm{SR}(K; R)$. Also, for any natural number $s \geq 1$, write

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$c_{\le s}(U)$ for $c(U)$ truncated above degree $2s$ so that $c_{\le s}(U)$ is the the sum of the s first elementary symmetric polynomials in the variables U.

Theorem 1.1. *The map $f\colon V \to P$ is a (P,s)-coloring of K if and only if $c(V) = \prod_{p\in P} c_{\le 2s}(f^{-1}p)$ in $\mathrm{SR}(K;R)$.*

The phrasing of our main theorem uses the Davis–Januszkiewicz space $\mathrm{DJ}(K)$ and the integral Stanley–Reisner face algebra $\mathrm{SR}(K;R)$. By construction, there is an inclusion map $\mathrm{DJ}(K) \xrightarrow{\lambda_K} \mathrm{map}(V,\mathrm{BU}(1))$ inducing the surjection of $H^*(\mathrm{map}(V,\mathrm{BU}(1));R) = R[V]$ onto $H^*(\mathrm{DJ}(K);R) = \mathrm{SR}(K;R)$. (Readers who are not familiar with these functorial constructions can find more information in Section 2 and Section 3.) The element $c(V) = \prod_{v\in V}(1+v)$ is the total Chern class in $\mathrm{SR}(K;R)$ of the stable complex vector bundle $\mathrm{DJ}(K) \xrightarrow{\lambda_K} \mathrm{map}(V,\mathrm{BU}(1)) \xrightarrow{\oplus} \mathrm{BU}$ where $\oplus\colon \mathrm{map}(V,\mathrm{BU}(1)) \to \mathrm{BU}$ is the Whitney sum map.

Theorem 1.2. *The following conditions are equivalent when the ring R is a UFD:*

1. *K admits a (P,s)-coloring.*
2. *The Stanley–Reisner ring $\mathrm{SR}(K;R)$ contains $|P|$ elements c_p of degree $\deg c_p \le 2s$, $p\in P$, so that $\prod_{v\in V}(1+v) = \prod_{p\in P} c_p$.*
3. *There exist $|P|$ s-dimensional complex vector bundles ξ_p, $p\in P$, over $\mathrm{DJ}(K)$ so that λ_K and $\bigoplus_{p\in P}\xi_p$ are stably isomorphic.*
4. *There exists a map $\mathrm{DJ}(K) \to \mathrm{map}(P,\mathrm{BU}(s))$ such that the diagram*

$$\begin{tikzcd} \mathrm{DJ}(K) \arrow[rr] \arrow[dr, "\oplus\circ\lambda_K"'] & & \mathrm{map}(P,\mathrm{BU}(s)) \arrow[dl, "\oplus"] \\ & \mathrm{BU} & \end{tikzcd}$$

commutes up to homotopy.

Of course, only the cardinality of P is relevant here so we shall also say that K is (r,s)-colorable if K is (P,s)-colorable for some palette P of r colors.

The proof of Theorem 1.2 is in Section 6.2. The $s = 1$ version of Theorem 1.2 appeared in [15] by the third author.

1.1. **Notation and the basic definition.** Our convention here is that any nonempty abstract simplicial complex contains the empty set. As we are only interested in finite complexes we shall be working with abstract simplicial complexes in the following sense.

Definition 1.3 (ASC)**.** *An abstract simplicial complex is a finite set of sets closed under formation of subsets.*

We shall use the following notation:

$|V|$**:** The number of elements in the finite set V
$D[V]$**:** The ASC of *all* subsets of the finite set V ($|D[V]| = \Delta^{|V|-1}$)
$\partial D[V]$**:** The finite simplicial complex of all *proper* subsets of V ($|\partial D[V]| = S^{|V|-2}$)
m_+**:** For any integer $m \ge 0$, $m_+ = \{0,1,\dots,m\}$ denotes a set of cardinality $m+1$.
$m(K)$**:** The number of vertices $|V|$ in the ASC K with vertex set V
$n(K)$**:** The number $\max\{|\sigma| \mid \sigma\in K\}$ of vertices in a maximal facet of the ASC K
$\dim K$**:** The dimension $\dim K = n(K) - 1$ of the ASC K
$\mathrm{codim}\, K$**:** The codimension $\mathrm{codim}\, K = m(K) - n(K)$ of the ASC K with vertex set V
$\mathrm{sk}_j(K)$**:** The j-skeleton $\{\sigma\in K \mid \dim\sigma \le j\}$ of the ASC K in K of the simplex $\sigma \in K$
$K_1 * K_2$**:** The join $K_1 * K_2 = \{\sigma_1 \amalg \sigma_2 \mid \sigma_1\in K_1, \sigma_2 \in K_2\} \subset D[V_1 \amalg V_2]$ of ASCs K_1, K_2 with vertex sets V_1, V_2
$P(K)$**:** The face poset of the ASC K [5, Definition 1.5]
i_σ**:** For any simplex σ in the ASC K, $i_\sigma\colon \sigma \to V, D[\sigma]\to K, D[\sigma]\to D[V]$ are inclusion maps of sets or ASCs
S^k**:** The subgroup of homogenous elements of degree k in a $\mathbf{Z}$-graded ring $S = \oplus_{k\in\mathbf{Z}} S^k$
$S^{\le k}$**:** The subgroup $\oplus_{d\le k} S^k$ of a $\mathbf{Z}$-graded ring $S = \oplus_{k\in\mathbf{Z}} S^k$
$\deg(x)$**:** The minimal k so that $x\in S^{\le k}$ for an element x in a $\mathbf{Z}$-graded ring $S = \oplus_{k\in\mathbf{Z}} S^k$

2. Davis–Januszkiewicz spaces

Let V be a finite set and (A, B) a pair of topological spaces. For any subset σ of V, write $\mathrm{map}(V, V-\sigma; A, B)$ for the space of maps $(V, V-\sigma) \to (A, B)$. When $\emptyset \subset \sigma \subset \tau \subset V$, the inclusions $(V, V) \supset (V, V-\sigma) \supset (V, V-\tau) \supset (V, \emptyset)$ induce inclusion maps

$$\mathrm{map}(V, B) = \mathrm{map}(V, V; A, B) \subset \mathrm{map}(V, V-\sigma; A, B) \subset \mathrm{map}(V, V-\tau; A, B) \subset \mathrm{map}(V, \emptyset; A, B) = \mathrm{map}(V, A)$$

of mapping spaces. Thus we have a functor

$$\mathrm{map}(V, V-?; A, B)\colon P(D[V]) \to \mathrm{TOP} \tag{2.1}$$

from the poset of subsets of V to the category of topological spaces. There are natural transformations from the constant functor $\mathrm{map}(V, B)$ to this functor and from this functor to the constant functor $\mathrm{map}(V, A)$.

Let K be an ASC with vertex set V.

Definition 2.2. [5, Construction 6.38] *The Davis–Januszkiewicz space* $\mathrm{DJ}(K; A, B)$ *of* K *with coefficients in* (A, B) *is the colimit over the face poset* $P(K)$

$$\mathrm{DJ}(K; A, B) = \mathrm{colim}(P(K); \mathrm{map}(V, V-?; A, B)) = \bigcup_{\sigma\in K} \mathrm{map}(V, V-\sigma; A, B)$$

of the functor (2.1).

The Davis–Januszkiewicz space $\mathrm{DJ}(K; A, B)$ is born with maps $\mathrm{map}(V, B) \xrightarrow{\varepsilon_K} \mathrm{DJ}(K; A, B) \xrightarrow{\lambda_K} \mathrm{map}(V, A)$ induced by natural transformations.

In the special case where $K = D[V]$ is the full simplex, we see that

$$\mathrm{DJ}(D[V]; A, B) = \mathrm{map}(V, \emptyset; A, B) = \mathrm{map}(V, A)$$

since the face poset $P(D[V])$ has V as terminal object.

We shall just write $\mathrm{DJ}(K; \mathrm{BU}(1), \{1\}) = \mathrm{DJ}(K)$ in case $(A, B) = (K(\mathbf{Z}, 2), \{0\}) = (\mathrm{BU}(1), \mathrm{B}\{1\})$. This space is born with maps $* = \mathrm{map}(V, *) \xrightarrow{\varepsilon_K} \mathrm{DJ}(K) \xrightarrow{\lambda_K} \mathrm{map}(V, \mathrm{BU}(1))$.

Suppose that $f\colon V \to P$ is a map of V into some finite set P and that (C, D) is a pair of topological spaces. Assume also that $\mu = \{\mu_p \mid p \in P\}$ is a set of maps $\mu_p\colon (\mathrm{map}(f^{-1}p, A), \mathrm{map}(f^{-1}p, B)) \to (C, D)$ indexed by P. Define

$$\mathrm{DJ}(f; \mu)\colon\ \mathrm{map}(V, A) \to \mathrm{map}(P, C) \tag{2.3}$$

to be the map given by

$$\mathrm{DJ}(f; \mu)(\chi)(p) = \mu_p(\chi|f^{-1}p), \qquad \chi \in \mathrm{map}(V, A),\ p \in P.$$

For instance, if A is an abelian monoid with operation $+\colon A \times A \to A$ with neutral element 0 and B is submonoid, then we may take $\mu_p\colon (\mathrm{map}(V, A), \mathrm{map}(V, B)) \to (A, B)$ to be the map

$$\mu_p(\chi) = \sum \chi(f^{-1}p) \tag{2.4}$$

for all $\chi \in \mathrm{map}(V, A)$.

Proposition 2.5. $\mathrm{DJ}(-; A, B)$ *is a functor from the category of finite ASCs and injective maps to the category of spaces. If* (A, B) *is a pair of abelian topological monoids then* $\mathrm{DJ}(-; A, B)$ *is a functor from the category of finite ASCs to the category of spaces.*

1. *Any injective simplicial map* $f\colon K \to L$ *induces a map* $\mathrm{DJ}(f; A, B)\colon\ \mathrm{DJ}(K; A, B) \to \mathrm{DJ}(L; A, B)$
2. *Any simplicial map* $f\colon K \to L$ *induces a map* $\mathrm{DJ}(f; A, B)\colon\ \mathrm{DJ}(K; A, B) \to \mathrm{DJ}(L; A, B)$ *provided that* (A, B) *is a pair of abelian topological monoids.*
3. *Any simplicial map* $f\colon K \to L$ *induces a map* $\mathrm{DJ}(f; \mu)\colon\ \mathrm{DJ}(K; A, B) \to \mathrm{DJ}(L; C, D)$ *provided that there are maps* $\mu = \{\mu_p \mid p \in P\}$ *indexed by the vertex set* P *of* L *as above.*
4. $\mathrm{DJ}(K_1 * K_2; A, B) = \mathrm{DJ}(K_1; A, B) \times \mathrm{DJ}(K_2; A, B)$

Proof. (1) Suppose that L is an ASC with vertex set P and that the injective map $f\colon V \to P$ is simplicial map $K \to L$. When $p \in P$, $v \in V$, and $p = f(v)$, take $\mu_p\colon (\mathrm{map}(\{v\}, A), \mathrm{map}(\{v\}, B)) = (A, B) \to (A, B)$ to be the identity map. Let $\mu = (\mu_p)$. Proceed as in the proof of the next item.

(2) Suppose that L is an ASC with vertex set P and that $f\colon V \to P$ is simplicial map $K \to L$. Note that $\mathrm{DJ}(f;\mu)$ (2.3) takes the subspace $\mathrm{map}(V, V-\sigma; A, B)$ into the subspace $\mathrm{map}(V, V-\tau; A, B)$ when $\sigma \in K$, $\tau \in L$ are simplices so that $f(\sigma) \subset \tau$ and $\mu = (\mu_p)$ is as in (2.4). Thus $\mathrm{DJ}(f;\mu)$ restricts to a map

$$\begin{tikzcd} \mathrm{DJ}(K;A,B) \arrow[r, "\mathrm{DJ}(f;\mu)"] \arrow[d, hook] & \mathrm{DJ}(L;A,B) \arrow[d, hook] \\ \mathrm{DJ}(D[V];A,B) \arrow[r, "\mathrm{DJ}(f;\mu)"'] & \mathrm{DJ}(D[P];A,B) \end{tikzcd}$$

between the Davis–Januszkiewicz spaces.
(3) A slight generalization of the above proof.
(4) [5, Construction 6.20]. □

With fixed K, $\mathrm{DJ}(K;A,B)$ is functorial in the pair (A,B). Observe that any homotopy equivalence $(A,B) \to (A',B')$ induces a homotopy equivalence $\mathrm{DJ}(K;A,B) \to \mathrm{DJ}(K;A',B')$ because $\mathrm{map}(V, V-\sigma; A, B) \to \mathrm{map}(V, V-\sigma; A', B')$ is a homotopy equivalence for all $\sigma \in K$ and because in Definition 2.2 we may replace the colimit by the homotopy colimit [17, Lemma 2.7].

3. The Stanley–Reisner face algebra

Let R be a commutative ring and let $\varepsilon\colon P \to Q$ be a homomorphism between commutative graded R-algebras P and Q. (A typical example will be $P = H^*(X;R)$ and $Q = H^*(A;R)$ where (X,A) is a pair of spaces.) Let V be a finite set and σ a subset of V. Write $\otimes\,\mathrm{map}(V, V-\sigma; P, Q)$ for the tensor product $\bigotimes_{v\in V} A_v$ of R-algebras

$$A_v = \begin{cases} P & v \in \sigma \\ Q & v \notin \sigma \end{cases}$$

for v in V. When $\emptyset \subset \sigma \subset \tau \subset V$ there are morphisms of R-algebras

$$Q^{\otimes V} = \otimes\,\mathrm{map}(V,V;P,Q) \leftarrow \otimes\,\mathrm{map}(V,V-\sigma;P,Q) \leftarrow \otimes\,\mathrm{map}(V,V-\tau;P,Q) \leftarrow \otimes\,\mathrm{map}(V,\emptyset;P,Q) = P^{\otimes V}$$

induced by identity and augmentation homomorphisms. For instance, if $V = \{1,2,3\}$ and $\{1\} = \sigma \subset \tau = \{1,3\}$, then there are morphisms

$$Q \otimes_R Q \otimes_R Q \xleftarrow{\varepsilon\otimes\varepsilon\otimes\varepsilon} P \otimes_R Q \otimes_R Q \xleftarrow{1\otimes\varepsilon\otimes\varepsilon} P \otimes_R Q \otimes_R P \xleftarrow{1\otimes\varepsilon\otimes 1} P \otimes_R P \otimes_R P$$

of R-algebras. Thus we have a functor

$$\otimes\,\mathrm{map}(V, V-?; P, Q)\colon P(D[V])^{\mathrm{op}} \to \mathrm{ALG} \tag{3.1}$$

from the opposite of the poset of subsets of V to the category of graded commutative connected R-algebras. There are natural transformations from this functor to the constant functor $Q^{\otimes V}$ and from the constant functor $P^{\otimes V}$ to this functor.

Let K be a finite ASC with vertex set V.

Definition 3.2. *The* Stanley–Reisner algebra of K with coefficients in (P,Q) *is the limit over the opposite face poset* $P(K)^{\mathrm{op}}$

$$\mathrm{SR}(K;P,Q) = \lim(P(K)^{\mathrm{op}}; \otimes\,\mathrm{map}(V, V-?; P, Q))$$

of the functor (3.1).

The Stanley–Reisner algebra $\mathrm{SR}(K;P,Q)$ is born with R-algebra homomorphisms $Q^{\otimes V} \leftarrow \mathrm{SR}(K;P,Q) \leftarrow P^{\otimes V}$ induced by natural transformations.

In the special case where $K = D[V]$ is the full simplex, we see that

$$\mathrm{SR}(D[V];P,Q) = \otimes\,\mathrm{map}(V,\emptyset;P,Q) = P^{\otimes V}$$

since the face poset $P(D[V])$ has V as terminal object.

For any simplicial map $f\colon K \to L$ between two finite ASCs, let $\mathrm{SR}(f)\colon \mathrm{SR}(L;P,Q) \to \mathrm{SR}(K;P,Q)$ be the R-algebra homomorphism induced by the poset map $P(f)\colon P(K) \to P(L)$. This makes $\mathrm{SR}(-;P,Q)$ into a contravariant functor from the category of finite ASCs into the category of graded commutative R-algebras.

Proposition 3.3. [17, Theorem 3.10] *$H^*(\mathrm{DJ}(K;X,*);R) = \mathrm{SR}(K;H^*(X;R),H^*(*;R))$ for any CW-complex X with base point $*$.*

The case where $X = \mathrm{BU}(1) = \mathbf{C}P^\infty$ is the classifying space for complex line bundles will be especially relevant for this paper. We shall abbreviate $\mathrm{DJ}(K;\mathrm{BU}(1),*)$ to $\mathrm{DJ}(K)$ and $\mathrm{SR}(K;H^*(\mathrm{BU}(1);R),H^*(*;R))$ to $\mathrm{SR}(K;R)$. The inclusion $i_K\colon K \to D[V]$ of K into the full simplex on its vertex set induces an inclusion of spaces

$$\mathrm{DJ}(i_K)\colon\ \mathrm{DJ}(K) \to \mathrm{DJ}(D[V]) = \mathrm{BU}(1)^V$$

inducing the algebra morphism

$$\mathrm{SR}(K;R) = H^*(\mathrm{DJ}(K);R) \leftarrow H^*(\mathrm{DJ}(D[V]);R) = H^*(\mathrm{BU}(1);R)^{\otimes V} = R[V]$$

where $R[V]$ is the polynomial ring generated by V in degree 2.

For any subset U of V, write $(U) \subset R[V]$ for the ideal generated by the set $U \subset R[V]$ and $\prod U, \sum U \in R[V]$ for the elements $\prod_{u\in U} u$, $\sum_{u\in U} u$.

Proposition 3.4. [18, (4.7)] *$\mathrm{SR}(K;R)$ is isomorphic to the quotient of $R[V]$ by the ideal $(\prod\tau \mid \tau\in D[V]-K)$.*

Proof. The homomorphism $\mathrm{SR}(i_K)\colon R[V] \to \mathrm{SR}(K;R)$ is quickly seen to be surjective. The kernel is the ideal $\bigcap_{\sigma\in K}(V-\sigma)$ which equals [17, Proposition 3.6] the ideal generated by the set $\{\prod\tau \mid \tau\in D[V]-K\} \subset R[V]$. □

The ideal $(\prod\tau \mid \tau\in D[V]-K)$ of Proposition 3.4 is called the *Stanley–Reisner ideal*.

Define $c(V) \in R[V]$ by

$$c(V) = \prod_{v\in V}(1+v) \tag{3.5}$$

For any map $f\colon V \to P$, the induced map $R[f]\colon R[P] \to R[V]$ takes $c(P)$ in $R[P]$ to

$$R[f]c(P) = \prod_{p\in P}(1+\sum f^{-1}p) \tag{3.6}$$

in $R[V]$.

Lemma 3.7. *$f\colon V \to P$ is injective $\iff$ $R[f]c(P) = c(V)$*

Proof. If $f\colon V\to P$ is injective, then (3.6) shows that $R[f]c(P) = c(V)$ because $f^{-1}(p)$ is nonempty for exactly $|V|$ points of P and for these points $f^{-1}(p)$ is a single point of V.

Conversely, notice that $\deg c(V) = 2|V|$ and $\deg R[f]c(P) = 2|f(V)|$. If $R[f]c(P) = c(V)$ then $|V| = |f(V)|$ and f is injective. □

Define

$$c(\lambda_K) = \mathrm{SR}(i_K)c(V) \tag{3.8}$$

to be the image in $\mathrm{SR}(K;R)$ of $c(V)$ in $R[V]$. (Then $c(V) = c(\lambda_{D[V]})$ in $R[V] = \mathrm{SR}(D[V];R)$.)

Lemma 3.9. *Two elements of $\mathrm{SR}(K;R)$ are identical if and only if they have identical $\mathrm{SR}(i_\sigma)$-images in $\mathrm{SR}(D[\sigma];R) = R[\sigma]$ for all $\sigma\in K$. In particular, for any $\sigma\in K$, $\mathrm{SR}(i_\sigma)c(\lambda_K) = c(\sigma)$ in $\mathrm{SR}(D[\sigma];R) = R[\sigma]$, and $c(\lambda_K)$ is the only element of $\mathrm{SR}(K;R)$ with this property.*

Proof. $\mathrm{SR}(i_\sigma)c(\lambda_K) = \mathrm{SR}(i_\sigma)\,\mathrm{SR}(i_K)c(V) = \mathrm{SR}(i_K\circ i_\sigma)c(V) = \mathrm{SR}(i_\sigma)c(V) = c(\sigma)$. Since $\mathrm{SR}(K;R)$ is a subring of the product of the rings $\mathrm{SR}(D[\sigma];R) = R[\sigma]$, $\sigma\in K$, this property characterizes $c(\lambda_K)$. □

The element $c(\lambda_K)$ of the graded ring $\mathrm{SR}(K;R)$ has degree $\deg c(\lambda_K) = n(K)$.

4. Splittings of vector bundles over Davis–Januszkiewicz spaces

Let $\mathrm{VU}(1) \to \mathrm{BU}(1)$ denote the universal and $\mathrm{BU}(1) \times \mathbf{C} \to \mathrm{BU}(1)$ the trivial complex line bundle.

Lemma 4.1. *There exist a vector bundle map* $\mathrm{VU}(1) \xrightarrow{a} \mathrm{BU}(1) \times \mathbf{C}$ *and a contractible subspace* $\mathrm{BU}(1)^\times$ *of* $\mathrm{BU}(1)$ *such that* $a \mid \mathrm{VU}(1)^\times$ *trivializes the restriction* $\mathrm{VU}(1)^\times \to \mathrm{BU}(1)^\times$ *of the universal line bundle to* $\mathrm{BU}(1)^\times$.

Proof. Let $\mathbf{C}(1)$ and $\mathbf{C}(0)$ be $\mathbf{C}$ with the standard and the trivial $\mathrm{U}(1)$-actions, respectively. The projection $\mathrm{pr}_1 \colon \mathbf{C}(1) \times \mathbf{C}(1) \to \mathbf{C}(1)$ onto the first factor and $a \colon \mathbf{C}(1) \times \mathbf{C}(1) \to \mathbf{C}(0)$ given by $a(x, y) = \overline{x}y$, where $\overline{x}$ is the complex conjugate of x, are $\mathrm{U}(1)$-maps. They determine commutative diagrams

$$\begin{tikzcd} \mathbf{C}(1) \times \mathbf{C}(1) \arrow[rr, "\mathrm{pr}_1 \times a"] \arrow[dr, "\mathrm{pr}_1"'] & & \mathbf{C}(1) \times \mathbf{C}(0) \arrow[dl, "\mathrm{pr}_1"] \\ & \mathbf{C}(1) & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{VU}(1) \arrow[rr, "a"] \arrow[dr] & & \mathrm{BU}(1) \times \mathbf{C} \arrow[dl] \\ & \mathrm{BU}(1) & \end{tikzcd}$$

where the right diagram is obtained from the left diagram of $\mathrm{U}(1)$-spaces and $\mathrm{U}(1)$-maps by applying the homotopy orbit space functor (Definition 4.2). Here we use that the projection $\mathrm{pr}_1 \colon \mathbf{C}(1) \times \mathbf{C}(1) \to \mathbf{C}(1)$ induces the universal line bundle $\mathrm{VU}(1) = (\mathbf{C}(1) \times \mathbf{C}(1))_{h\mathrm{U}(1)} \to \mathbf{C}(1)_{h\mathrm{U}(1)} = \mathrm{BU}(1)$. Let $\mathbf{C}(1)^\times$ be the $\mathrm{U}(1)$-subspace of nonzero elements of $\mathbf{C}(1)$. Observe that the restriction of the vector bundle map $\mathrm{VU}(1) \to \mathrm{BU}(1) \times \mathbf{C}$ trivializes the vector bundle $(\mathbf{C}(1)^\times \times \mathbf{C}(1))_{h\mathrm{U}(1)} \to \mathbf{C}(1)^\times_{h\mathrm{U}(1)}$ over the contractible subspace $\mathrm{BU}(1)^\times = \mathbf{C}(1)^\times_{h\mathrm{U}(1)}$ of $\mathrm{BU}(1) = \mathbf{C}(1)_{h\mathrm{U}(1)}$. □

Definition 4.2. *Let G be a compact Lie group and X a left G-space. The homotopy orbit space X_{hG} is the space $(EG \times X)/G$ of G-orbits for the G-action $(e, x) \cdot g = (eg, g^{-1}x)$, $e \in EG$, $x \in X$, $g \in G$, on $EG \times X$ where EG is a contractible free right G-space.*

By abuse of notation, we shall also let λ_K stand for the $|V|$-dimensional complex vector bundle

$$\mathrm{map}(V, \mathbf{C}) \xrightarrow{\varepsilon_K} \mathrm{DJ}(K; \mathrm{VU}(1), \mathbf{C}) \longrightarrow \mathrm{DJ}(K; \mathrm{BU}(1), *) \tag{4.3}$$

over $\mathrm{DJ}(K) = \mathrm{DJ}(K; \mathrm{BU}(1), *)$ classified by the map $\mathrm{DJ}(K; \mathrm{BU}(1), *) \xrightarrow{\lambda_K} \mathrm{map}(V, \mathrm{BU}(1)) \xrightarrow{\oplus} \mathrm{BU}(|V|)$.

Lemma 4.4. [16, Theorem 1.1] *There exists a fibrewise surjective vector bundle map*

$$\begin{tikzcd} \mathrm{DJ}(K; \mathrm{VU}(1), \mathbf{C}) \arrow[rr] \arrow[dr] & & \mathbf{C}^{\operatorname{codim} K} \times \mathrm{DJ}(K) \arrow[dl] \\ & \mathrm{DJ}(K) & \end{tikzcd}$$

of λ_K to the trivial vector bundle of dimension $\operatorname{codim} K = m(K) - n(K)$ *over* $\mathrm{DJ}(K)$.

Proof. Let $a \colon \mathrm{VU}(1) \to \mathrm{BU}(1) \times \mathbf{C}$ be the vector bundle map from Lemma 4.1. Choose a linear map $A \colon \mathrm{map}(V, \mathbf{C}) \to \mathbf{C}^{\operatorname{codim} K}$ such that the restrictions of A to the subspaces $\mathrm{map}(V, \sigma; \mathbf{C}, 0)$, $\sigma \in K$, are surjective for all $\sigma \in K$. This is possible since $\dim \mathrm{map}(V, \sigma; \mathbf{C}, 0) = |V| - |\sigma| \geq |V| - \max\{|\sigma| \mid \sigma \in K\} = \operatorname{codim} K$. Then the vector bundle map

$$\mathrm{map}(V, \mathrm{VU}(1)) \xrightarrow{\mathrm{map}(V, a)} \mathrm{map}(V, \mathrm{BU}(1)) \times \mathrm{map}(V, \mathbf{C}) \xrightarrow{\mathrm{id} \times A} \mathrm{map}(V, \mathrm{BU}(1)) \times \mathbf{C}^{\operatorname{codim} K}$$

of vector bundles over $\mathrm{map}(V, \mathrm{BU}(1))$ restricts to a surjective vector bundle map

$$\mathrm{DJ}(K; \mathrm{VU}(1), \mathrm{VU}(1)^\times) \to \mathrm{DJ}(K; \mathrm{BU}(1), \mathrm{BU}(1)^\times) \times \mathbf{C}^{\operatorname{codim} K}$$

of vector bundles over $\mathrm{DJ}(K;\mathrm{BU}(1),\mathrm{BU}(1)^{\times})$ because there are commutative diagrams

$$\begin{tikzcd}
\mathrm{map}(V,\sigma,\mathbf{C},0) \arrow[r, equal] \arrow[d, hook] & \mathrm{map}(V,\sigma,\mathbf{C},0) \arrow[dd, hook] \arrow[ddr, "\cong"] & \\
\mathrm{map}(V,\mathbf{C}) \arrow[d, hook] & & \\
\mathrm{map}(V,V-\sigma,\mathrm{VU}(1),\mathrm{VU}(1)^{\times}) \arrow[r, "a"] \arrow[d] & \mathrm{map}(V,\mathrm{BU}(1))\times\mathrm{map}(V,\mathbf{C}) \arrow[r, "A\circ\mathrm{pr}_2"] & \mathbf{C}^{\mathrm{codim}\,K} \\
\mathrm{map}(V,V-\sigma,\mathrm{BU}(1),\mathrm{BU}(1)^{\times}) & &
\end{tikzcd}$$

over $\mathrm{map}(V,V-\sigma,\mathrm{BU}(1),\mathrm{BU}(1)^{\times})\subset\mathrm{DJ}(K;\mathrm{BU}(1),\mathrm{BU}(1)^{\times})$ for every simplex $\sigma\in K$.

The inclusion map $\mathrm{DJ}(K;\mathrm{BU}(1),*)\to\mathrm{DJ}(K;\mathrm{BU}(1),\mathrm{BU}(1)^{\times})$ is a homotopy equivalence covered by the vector bundle map $\mathrm{DJ}(K;\mathrm{VU}(1),\mathbf{C})\to\mathrm{DJ}(K;\mathrm{VU}(1),\mathrm{VU}(1)^{\times})$. $\square$

Let ξ_K be the kernel of the vector bundle morphism from Lemma 4.4 so that the vector bundles λ_K and ξ_K are related by a short exact sequence

$$0\longrightarrow\xi_K\longrightarrow\lambda_K\longrightarrow\mathbf{C}^{\mathrm{codim}\,K}\longrightarrow 0 \tag{4.5}$$

of complex vector bundles over $\mathrm{DJ}(K)$, and $\dim\xi_K=\max\{|\sigma|\mid\sigma\in K\}=n(K)$.

Let $f\colon V\to P$ be any map. For each $p\in P$, define

$$d_f(p)=\max\{|\sigma\cap f^{-1}(p)|\mid\sigma\in K\}$$

to be the maximal number of vertices of color p in any simplex of K. Then

$$n(K)\le\sum_{p\in P}d_f(p)\le m(K) \tag{4.6}$$

and $f\colon V\to P$ is a (P,s)-coloring if and only if $d_f(p)\le s$ for all colors $p\in P$. For each color $p\in P$, consider the subcomplex of K consisting of all monochrome simplices of color p,

$$K_p=K\cap D[f^{-1}p]=\{\sigma\in K\mid f(\sigma)=p\},$$

of dimension $\dim K_p=d_f(p)-1$ and codimension $\mathrm{codim}\,K_p=|f^{-1}(p)|-d_f(p)$. Since $V=\coprod_{p\in P}f^{-1}p$ is the disjoint union of monochrome subsets there is an injective simplicial map $d(f)\colon K\to\bigstar_{p\in P}K_p$ of K into the join of the subcomplexes K_p inducing (Proposition 2.5.(4)) a map

$$\mathrm{DJ}(d(f))\colon\ \mathrm{DJ}(K;A,B)\to\mathrm{DJ}(\bigstar_{p\in P}K_p;A,B)=\prod_{p\in P}\mathrm{DJ}(K_p;A,B) \tag{4.7}$$

of $\mathrm{DJ}(K)$ into the product of the $\mathrm{DJ}(K_p)$, $p\in P$. Write ξ_p and λ_p for the vector bundles ξ_{K_p} and λ_{K_p} over $\mathrm{DJ}(K_p)$ and for their pull backs to $\mathrm{DJ}(K)$. We have $\dim\xi_p=d_f(p)$ and $\dim\lambda_p=|f^{-1}p|$.

Theorem 4.8. *Associated to any map $f\colon V\to P$, there is a short exact sequence*

$$0\longrightarrow\bigoplus_{p\in P}\xi_p\longrightarrow\lambda_K\longrightarrow\mathbf{C}^{m(K)-\sum d_f(p)}\longrightarrow 0$$

of vector bundles over $\mathrm{DJ}(K)$ where $\dim\xi_p=d_f(p)$ and $c(\xi_p)=c(f^{-1}p)$.

Proof. If we pull back the short exact sequence

$$0\longrightarrow\prod_{p\in P}\xi_p\longrightarrow\prod_{p\in P}\lambda_p\longrightarrow\prod_{p\in P}\mathbf{C}^{|f^{-1}p|-d_f(p)}\longrightarrow 0$$

of vector bundles over $\prod_{p\in P}\mathrm{DJ}(K_p)$ along the map (4.7), we obtain a short exact sequence

$$0\longrightarrow\bigoplus_{p\in P}\xi_p\longrightarrow\lambda_K\longrightarrow\mathbf{C}^{m(K)-\sum d_f(p)}\longrightarrow 0$$

of vector bundles of $\mathrm{DJ}(K)$. Here, we use that $\lambda_K = \bigoplus_{p\in P}\lambda_p$ as shown by the pull-back

$$\begin{tikzcd} \mathrm{DJ}(K;\mathrm{VU}(1),\mathbf{C}) \arrow[r, "\mathrm{DJ}(d(f))"] \arrow[d, "\lambda_K"'] & \prod_{p\in P}\mathrm{DJ}(K_p;\mathrm{VU}(1),\mathbf{C}) \arrow[d, "\prod\lambda_p"] \\ \mathrm{DJ}(K;\mathrm{BU}(1),*) \arrow[r, "\mathrm{DJ}(d(f))"'] & \prod_{p\in P}\mathrm{DJ}(K_p;\mathrm{BU}(1),*) \end{tikzcd}$$

of $|V| = \sum |f^{-1}p|$-dimensional vector bundles. □

The isomorphism class of the vector bundle ξ_K is independent of the choice of $\mathrm{map}(V,\mathbf{C}) \xrightarrow{A} \mathbf{C}^{|V|}/\mathbf{C}^{1+\dim K}$ because any two choices are identical up to base changes. One possibility is to take A to be the linear map whose associated $(\operatorname{codim} K \times |V|)$-matrix $A = (i^{j-1})$, $1 \le i \le \operatorname{codim} K$, $1 \le j \le |V|$, is a Vandermonde matrix [4, III, §8, no. 6]. In the situation of Theorem 4.8 we have a commutative diagram

$$\begin{tikzcd} \mathrm{map}(\bigcup_{p\in P} f^{-1}p,\mathbf{C}) \arrow[r, "A"] \arrow[d, equal] & \mathbf{C}^{\sum|f^{-1}p|}/\mathbf{C}^{n(K)} = \mathbf{C}^{\operatorname{codim} K} \arrow[d, two heads] \\ \prod_{p\in P}\mathrm{map}(f^{-1}p,\mathbf{C}) \arrow[r, "\prod_{p\in P}A_p"'] & \prod_{p\in P}\left(\mathbf{C}^{|f^{-1}p|}/\mathbf{C}^{n(K_p)}\right) = \prod_{p\in P}\mathbf{C}^{\operatorname{codim} K_p} \end{tikzcd}$$

where the right vertical epimorphism is induced by an inclusion of $\mathbf{C}^{n(K)}$ into $\prod_{p\in P}\mathbf{C}^{n(K_p)}$ which exists because $n(K) \le \sum_{p\in P} n(K_p)$ or, equivalently, $\operatorname{codim} K \ge \sum_{p\in P}\operatorname{codim} K_p$. The $(\operatorname{codim} K_p \times |f^{-1}p|)$-matrix for A_p is the submatrix of A consisting of the first $\operatorname{codim} K_p$ rows and the columns corresponding to the subset $f^{-1}p$ of V. This is also a Vandermonde matrix so that the linear maps A_p, $p \in P$, satisfy the condition from the proof of Lemma 4.4. Therefore the above diagram induces another commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \xi_K \arrow[r] \arrow[d] & \lambda_K \arrow[r] \arrow[d, equal] & \mathbf{C}^{\operatorname{codim} K} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \bigoplus_{p\in P}\xi_p \arrow[r] & \bigoplus_{p\in P}\lambda_p \arrow[r] & \bigoplus_{p\in P}\mathbf{C}^{\operatorname{codim} K_p} \arrow[r] & 0 \end{tikzcd}$$

of vector bundle morphisms from which we get the short exact sequence

$$0 \longrightarrow \xi_K \longrightarrow \bigoplus_{p\in P}\xi_p \longrightarrow \mathbf{C}^{\operatorname{codim} K - \sum \operatorname{codim} K_p} \longrightarrow 0 \tag{4.9}$$

of vector bundles over $\mathrm{DJ}(K)$.

5. Vertex colorings of simplicial complexes

Let K be a finite ASC with vertex set V and R a commutative ring.

Definition 5.1. *Let P be a finite set, a palette of colors.*

- *A P-coloring of K is a map $f\colon V \to P$ that restricts to injective maps $f|\sigma\colon \sigma \to P$ on all simplices $\sigma \in K$.*
- *K is r-colorable if K admits a coloring from a palette P of r colors.*
- *The chromatic number of K, $\mathrm{chr}(K)$, is the least r so that K is r-colorable.*

The identity map $1_V\colon V \to V$ is a V-coloring of K painting the vertices with distinct colors. The chromatic number of the full simplex $D[V]$ is $\mathrm{chr}(D[V]) = |V|$. If K' is a subcomplex of K then $\mathrm{chr}(K) \ge \mathrm{chr}(K')$. In particular, $|\sigma| = \mathrm{chr}(D[\sigma]) \le \mathrm{chr}(K) \le \mathrm{chr}(D[V]) = |V|$ so that

$$n(K) \le \mathrm{chr}(K) \le m(K)$$

for any ASC K with vertex set V.

Let $\mathrm{Col}(K,P)$ be the set of P-colorings of K. Then

$$\mathrm{Col}(K,P) = \lim(P(K)^{\mathrm{op}}; \mathrm{map}^1(-,P))$$

is the limit of the set-valued functor $\mathrm{map}^1(-,P)\colon P(K)^{\mathrm{op}} \to \mathrm{SET}$, taking $\sigma \in K$ to the set $\mathrm{map}^1(\sigma,P)$ of injective maps $\sigma \to P$. Since $\mathrm{map}^1(\sigma,P) = \mathrm{Col}(D[\sigma],P)$, because the face poset of the full simplex $D[\sigma]$ has σ as a final element, we could also write

$$\mathrm{Col}(K,P) = \lim(P(K)^{\mathrm{op}}; \mathrm{Col}(D[-],P))$$

to emphasize that a global P-coloring of K is a coherent choice of local P-colorings of its simplices.

For any injective simplicial map $f\colon K \to L$ between two finite ASCs, $\mathrm{Col}^1(f,P)\colon \mathrm{Col}(L,P) \to \mathrm{Col}(K,P)$ is the map induced by the injective poset map $P(f)\colon P(K) \to P(L)$. Thus $\mathrm{Col}(-,P)$ is a contravariant set-valued functor on the category of finite ASCs with injective maps.

Definition 5.2. [5, Definition 2.20] *The* flagification *of K is the ASC*

$$\mathrm{fla}(K) = \{\sigma \in D[V] \mid \mathrm{sk}_1(D[\sigma]) \subset K\}$$

and K is a flag complex if K equals its flagification.

We have $\mathrm{sk}_1(K) \subset K \subset \mathrm{fla}(K) \subset D[V]$ and $\mathrm{sk}_1(K) = \mathrm{sk}_1(\mathrm{fla}(K))$; $\mathrm{fla}(K)$ is the largest subcomplex of $D[V]$ with the same 1-skeleton as K.

The *missing faces* of K are the minimal elements of the poset $D[V]-K$ [5, Definition 2.21]; they generate the Stanley–Reisner ideal.

The following two propositions emphasize that coloring issues are 1-dimensional.

Proposition 5.3. [5, Proposition 2.22] *The following conditions are equivalent:*

(1) *K is flag*
(2) $\forall \sigma \in D[V]\colon \mathrm{sk}_1(D[\sigma]) \subset K \implies \sigma \in K$
(3) *The missing faces of K are 1-dimensional*

Proposition 5.4. *The following conditions are equivalent for any map $f\colon V \to P$:*

(1) *f is a P-coloring of K*
(2) *f is a P-coloring of* $\mathrm{sk}_1(K)$
(3) *f is a P-coloring of* $\mathrm{fla}(K)$

Moreover, $\mathrm{Col}(K,P) = \mathrm{Col}(\mathrm{sk}_1(K),P) = \mathrm{Col}(\mathrm{fla}(K),P)$ *and* $\mathrm{chr}(K) = \mathrm{chr}(\mathrm{sk}_1(K)) = \mathrm{chr}(\mathrm{fla}(K))$.

Theorem 5.5. *The map $f\colon V \to P$ is a P-coloring of K if and only if $c(\lambda_K) = \mathrm{SR}(f)c(P)$ in $\mathrm{SR}(K;R)$.*

Proof. We have

$$\begin{aligned} f\colon V \to P \text{ is a coloring } &\iff \forall \sigma \in K\colon f \circ i_\sigma \colon \sigma \to P \text{ is injective} \\ &\iff \forall \sigma \in K\colon c(\sigma) = \mathrm{SR}(f \circ i_\sigma)c(P) \\ &\iff \forall \sigma \in K\colon\ \mathrm{SR}(i_\sigma)\big(c(\lambda_K)\big) = \mathrm{SR}(i_\sigma)\big(\mathrm{SR}(f)c(P)\big) \\ &\iff c(\lambda_K) = \mathrm{SR}(f)c(P) \end{aligned}$$

from Lemma 3.7 and 3.9. (We here regard $f\colon V \to P$ as the simplicial map $K \overset{i_K}{\subset} D[V] \xrightarrow{D[f]} D[P]$.) □

In other words, a partition $V = V_1 \cup \cdots \cup V_r$ of V into r disjoint nonempty subsets is an r-coloring of K if and only if the equation

$$\prod_{v \in V}(1+v) = \prod_{1 \le j \le r}(1 + \sum V_j)$$

holds in the Stanley–Reisner algebra $\mathrm{SR}(K;R)$.

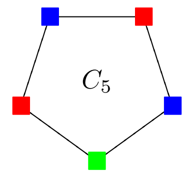
Example 5.6. The cyclic simplicial graph C_5 on the 5 vertices in $V = \{v_1,\ldots,v_5\}$ has Stanley–Reisner ring

$$\mathrm{SR}(C_5;\mathbf{Z}) = \mathbf{Z}[v_1,\ldots,v_5]/(v_1v_3, v_1v_4, v_2v_4, v_2v_5, v_3v_5)$$

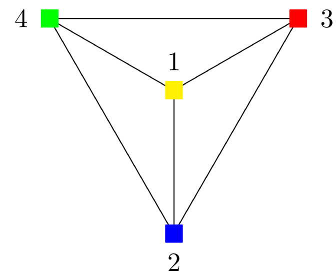
The chromatic number number $\mathrm{chr}(C_5) = 3$ and there are 30 different 3-colorings of C_5. Since $[v_1,v_2,v_3,v_4,v_5] \to [1,2,1,2,3]$ is a coloring, the identity

$$\prod_{1 \le i \le 5}(1+v_i) = (1 + v_1 + v_3)(1 + v_2 + v_4)(1 + v_5)$$

holds in $\mathrm{SR}(C_5;\mathbf{Z})$. C_5 is a flag complex.



Example 5.7. A triangulation $K = \{\{1,2,4\},\{1,3,4\},\{1,2,3\}\} = D[0_+] * \partial D[2_+]$ of the 2-disc with chromatic number 4



and Stanley–Reisner algebra $\mathrm{SR}(K;\mathbf{Z}) = \mathbf{Z}[v_1,v_2,v_3,v_4]/(v_2v_3v_4) = \mathbf{Z}[v_1] \otimes \mathbf{Z}[v_2,v_3,v_4]/(v_2v_3v_4)$.

6. Relaxed vertex colorings of simplicial complexes

We introduce vertex colorings that allow a bounded number of vertices in any simplex to have the same color.

Definition 6.1. *Let P be a finite set (a palette of colors) and s a natural number where $1 \le s \le \dim(K)$.*

- *A (P,s)-coloring of K is a map $f\colon V \to P$ such that every simplex of K contains at most s vertices of the same color: $\forall \sigma \in K \forall p \in P\colon |\sigma \cap f^{-1}(p)| \le s$.*
- *K is (r,s)-colorable if K admits a (P,s)-coloring from a palette P of r colors.*
- *The s-chromatic number of K, $\mathrm{chr}^s(K)$, is the least r so that K is (r,s)-colorable.*

In other words, the vertex coloring $f\colon V \to P$ is a (P,s)-coloring if K has no monochrome s-dimensional simplices. A $(P,1)$-coloring of K is a standard coloring of K, and $\mathrm{chr}(K) = \mathrm{chr}^1(K)$. Clearly,

$$m(K) \ge \mathrm{chr}^1(K) \ge \cdots \ge \mathrm{chr}^s(K) \ge \mathrm{chr}^{s+1}(K) \ge \cdots \ge \mathrm{chr}^{n(K)}(K) = 1$$

as any (P,s)-coloring is also a $(P,s+1)$-coloring. The map $f\colon V \to P$ is a (P,s)-coloring of K if and only if $|f(\sigma)| > 1$ for all s-dimensional simplices $\sigma \in K$, ie if and only if K contains no monochrome s-dimensional simplices. If $f_1\colon V \to P_1$ is a (P_1,s_1)-coloring and $f_2\colon P_1 \to P_2$ is s_2-to-1 then $f_2 \circ f_1$ is an s_1s_2-coloring. Any finite ASC with vertex set V is $(\lceil |V|/s\rceil, s)$- and $(r, \lceil |V|/r\rceil)$- colorable. The s-chromatic number of a full simplex is $\mathrm{chr}^s(D[V]) = \lceil |V|/s\rceil$. (For a real number t, $\lceil t\rceil$ denotes the least integer $\ge t$.)

Let $\mathrm{Col}^s(K,P)$ be the set of (P,s)-colorings of K. Then

$$\mathrm{Col}^s(K,P) = \lim(P(K)^{\mathrm{op}}; \mathrm{map}^s(-,P))$$

is the limit of the set-valued functor $\mathrm{map}^s(-,P)\colon P(K)^{\mathrm{op}} \to \mathrm{SET}$, taking $\sigma \in K$ to the set $\mathrm{map}^s(\sigma,P)$ of s-to-1 maps $\sigma \to P$. In particular, $\mathrm{map}^s(\sigma,P) = \mathrm{Col}^s(D[\sigma],P)$ because the face poset of the full simplex $D[\sigma]$ has σ as a final element, and we could also write

$$\mathrm{Col}^s(K,P) = \lim(P(K)^{\mathrm{op}}; \mathrm{Col}^s(D[-],P))$$

to emphasize that a global (P,s)-coloring of K is a coherent choice of local (P,s)-colorings of its simplices.

For any injective simplicial map $f\colon K \to L$ between two finite ASCs, $\mathrm{Col}^s(f,P)\colon \mathrm{Col}^s(L,P) \to \mathrm{Col}^s(K,P)$ is the map induced by the injective poset map $P(f)\colon P(K) \to P(L)$. Thus $\mathrm{Col}^s(-,P)$ is a contravariant set-valued functor on the category of finite ASCs with injective maps.

In particular, $\mathrm{map}^s(\sigma,P) = \mathrm{Col}^s(D[\sigma],P)$ because the face poset of the full simplex $D[\sigma]$ has σ as a final element, and we could also write

$$\mathrm{Col}^s(K,P) = \lim(P(K)^{\mathrm{op}}; \mathrm{Col}^s(D[-],P))$$

to emphasize that a global (P,s)-coloring of K is a coherent choice of local (P,s)-colorings of its simplices.

If K contains K' as a subcomplex then there is a restriction map $\mathrm{Col}^s(K,P) \to \mathrm{Col}^s(K',P)$ and $\mathrm{chr}^s(K) \geq \mathrm{chr}^s(K')$. In particular, $\lceil |\sigma|/s \rceil = \mathrm{chr}^s(D[\sigma]) \leq \mathrm{chr}^s(K) \leq \mathrm{chr}^s(D[V]) = \lceil |V|/s \rceil$ whenever $\sigma \in K$ so that

$$\left\lceil \frac{n(K)}{s} \right\rceil \leq \mathrm{chr}^s(K) \leq \left\lceil \frac{m(K)}{s} \right\rceil \tag{6.2}$$

for any ASC K.

Since any s-to-1 map is an $(s+1)$-to-1 map there are inclusions

$$\mathrm{map}^1(\sigma,P) \subset \cdots \subset \mathrm{map}^s(\sigma,P) \subset \mathrm{map}^{s+1}(\sigma,P) \subset \cdots \subset \mathrm{map}^{|\sigma|}(\sigma,P) = \mathrm{map}(\sigma,P)$$

that produce inclusions

$$\mathrm{Col}^1(K,P) \subset \cdots \subset \mathrm{Col}^s(K,P) \subset \mathrm{Col}^{s+1}(K,P) \subset \cdots \subset \mathrm{Col}^{\dim(K)+1}(K,P) = \mathrm{map}(V,P)$$

giving $\mathrm{Col}^*(K,P)$ the structure of a filtered set.

For any subset U of the vertex set V, let $c_{\leq s}(U) \in R[V]$ denote the sum of the s first elementary symmetric polynomials in the elements of U. For instance,

$$c_{\leq 0}(U) = 1 = c_{\leq 0}(\emptyset), \quad c_{\leq 1}(U) = 1 + \sum_{u \in U} u, \quad c_{\leq 2}(U) = 1 + \sum_{u\in U} u + \sum_{u_1 \neq u_2} u_1u_2, \ldots, c_{\leq |U|}(U) = \prod_{u \in U}(1+u)$$

Lemma 3.7 says that $f\colon V \to P$ is 1-to-1 if and only if $c(V) = \prod_{p\in P} c_{\leq 1}(f^{-1}p)$ in $R[V]$ and Theorem 5.5 says that $f\colon V \to P$ is a $(P,1)$-coloring of K if and only if $c(\lambda_K) = \prod_{p \in P} c_{\leq 1}(f^{-1}p)$ in $\mathrm{SR}(K;R)$. We can now obtain more general statements.

Lemma 6.3. *$f\colon V \to P$ is s-to-1 $\iff$ $c(V) = \prod_{p\in P} c_{\leq s}(f^{-1}p)$*

Proof. If $f\colon V \to P$ is s-to-1 then $c_{\leq s}(f^{-1}p)$ is the total Chern class of the set $f^{-1}p$ so that $c_{\leq s}(f^{-1}p) = \prod_{f(v)=p}(1+v)$. Therefore, $c(V) = \prod_{v\in V}(1+v) = \prod_{p\in P}\prod_{f(v)=p}(1+v) = \prod_{p\in P} c_{\leq s}(f^{-1}p)$.

If $f^{-1}p$ contains $t > s$ elements for some $p \in P$, then the product of these t elements with the same color p is not a summand in $c_{\leq s}(f^{-1}p)$ but it is a summand in $c(V)$. Thus $c(V)$ and $\prod_{p\in P} c_{\leq s}(f^{-1}p)$ do not have the same homogenous components of degree $2t$. □

Proof of Theorem 1.1. We have

$$\begin{aligned}
f\colon V \to P \text{ is a } (P,s)\text{-coloring} &\iff \forall \sigma \in K\colon f \circ i_\sigma \text{ is } s\text{-to-1} \\
&\iff \forall \sigma \in K\colon c(\sigma) = \sum_{p\in P} c_{\leq s}((f\circ i_\sigma)^{-1}p) \\
&\iff \forall \sigma \in K\colon \mathrm{SR}(i_\sigma)\big(c(V)\big) = \mathrm{SR}(i_\sigma)\big(\sum_{p \in P} c_{\leq s}(f^{-1}p)\big) \\
&\iff c(\lambda_K) = \sum_{p\in P} c_{\leq s}(f^{-1}p)
\end{aligned}$$

from Lemma 6.3 and 3.9. □

In other words, a partition $V = V_1 \cup \cdots \cup V_r$ of V into r disjoint nonempty subsets is an (r,s)-coloring of K if and only if the equation

$$c(V) = \prod_{1 \leq j \leq r} c_{\leq s}(V_j)$$

holds in the Stanley–Reisner algebra $\mathrm{SR}(K;R)$.

Example 6.4. [Surfaces of genus one] As indicated in Figure 2, there is a triangulation P2 of $\mathbf{R}P^2$ with vertex set $V = \{v_1, \ldots, v_6\}$ and f-vector $(1,6,15,10)$. The chromatic numbers $\mathrm{chr}^1(\mathrm{P2}) = 6$ and $\mathrm{chr}^2(\mathrm{P2}) = 3$. P2 has $6! = 720$ $(6,1)$-colorings and $45 \cdot 3! = 270$ $(3,2)$-colorings. Since $[1,2,3,4,5,6] \to [1,1,1,2,2,3]$ is a $(3,2)$-coloring, the identity

$$\prod_{1\leq i \leq 6}(1+v_i) = (1 + v_1 + v_2 + v_3 + v_2v_3 + v_1v_3 + v_1v_2)(1 + v_4 + v_5 + v_4v_5)(1+v_6)$$

holds in the Stanley–Reisner ring

$$\mathrm{SR}(\mathrm{P2};\mathbf{Z}) = \mathbf{Z}[v_1,\ldots,v_6]/(v_1v_2v_3, v_1v_2v_5, v_1v_3v_6, v_1v_4v_5, v_1v_4v_6, v_2v_3v_4, v_2v_4v_6, v_2v_5v_6, v_3v_4v_5, v_3v_5v_6)$$

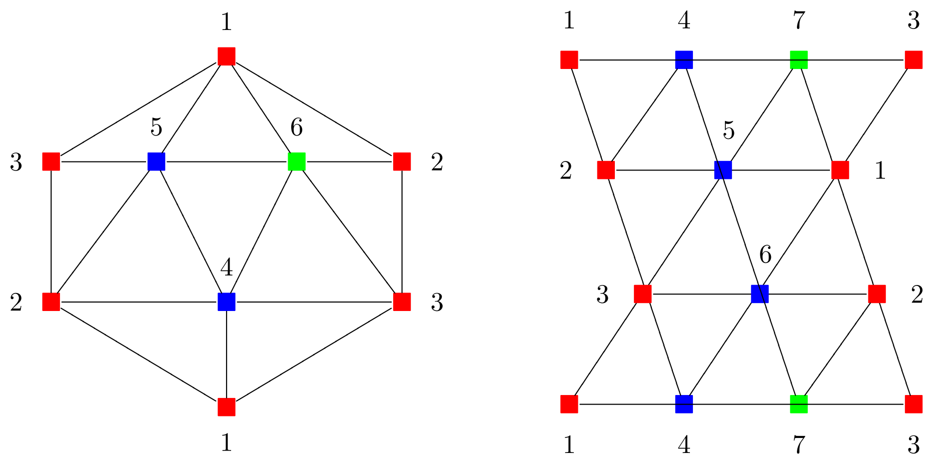


FIGURE 2. ({■, ■, ■}, 2)-colorings of P2 and T2

of P2.

Figure 2 also indicates a $(3,2)$-coloring of T2, a triangulation of the torus with f-vector $(1,7,21,14)$ (Möbius' vertex minimal triangulation). The chromatic numbers in this case are $\mathrm{chr}^1(\mathrm{T2}) = 7$ and $\mathrm{chr}^2(\mathrm{T2}) = 3$. T2 has $7! = 5040$ $(7,1)$-colorings and $84 \cdot 3! = 504$ $(3,2)$-colorings.

Remark 6.5. The number of s-to-1 maps of $V = \{1,\dots,m\}$ *onto* $P = \{1,\dots,r\}$, $m \leq rs$, equals

$$\sum \binom{m}{m_1,\dots,m_r}\binom{r}{r_1,\dots,r_s}$$

where the sum is taken over all r-vectors $(m_1,\dots,m_r)$ such that $s \geq m_1 \geq \cdots \geq m_r$, $m_1 + \cdots + m_r = m$, and the s-vector $(r_1,\dots,r_s)$ consists of the numbers $r_j = |\{p \in P \mid m_p = j\}|$, $1 \leq j \leq s$.

Example 6.6. Let $H = (V,E)$ be a $(s+1)$-uniform hypergraph with vertex set V and hyperedges E [3, Chp 1.3]. We say that H has property B if there exists a red–blue coloring of the vertices with no monochrome hyperedges [14]. Form the pure ASC $K(H)$ with vertex set V and facets E. Then $K(H)$ is $(2,s)$-colorable if and only if H has property B.

6.1. **Coloring flags.** We introduce the notion of an s-flagification procedure that relates to (P,s)-colorings in the same way that flagification (Definition 5.2) relates to P-colorings (Proposition 5.3, Proposition 5.4).

Definition 6.7. *The s-flagification of K is the ASC*

$$\mathrm{fla}_s(K) = \{\sigma \in D[V] \mid \mathrm{sk}_s([D[\sigma]) \subset K\}$$

and K is an s-flag complex if $K = \mathrm{fla}_s(K)$.

An ASC is a 1-flag complex if and only it is a flag complex in the sense of Definition 5.2 and $\mathrm{fla}(K) = \mathrm{fla}_1(K)$. The s-flagification of K is the largest complex on V with the same s-skeleton as K.

Proposition 6.8. *The following conditions are equivalent:*

(1) *K is s-flag*
(2) $\forall \sigma \in D[V]\colon \mathrm{sk}_s(D[\sigma]) \subset K \Longrightarrow \sigma \in K$
(3) *The missing faces of K have dimension at most s*

Proof. If K has a missing face of dimension $s+1$, then that face must be added to form the s-flagification of K, so K is not s-flag. □

For example, the triangulation P2 of the real projective plane from Example 6.4 is 2-flag, and, more generally, the j-skeleton $\mathrm{sk}_j(D[V])$ of a full simplex is $(j+1)$-flag but not j-flag.

Complexes of dimension less than s are s-flag (but the converse does not hold). Any s-flag complex is an $(s+1)$-flag complex so that

$$\mathrm{fla}_1(\mathrm{ASC}) \subset \mathrm{fla}_2(\mathrm{ASC}) \subset \cdots \subset \mathrm{fla}_s(\mathrm{ASC}) \subset \mathrm{fla}_{s+1}(\mathrm{ASC}) \subset \cdots$$

where $\mathrm{fla}_s(\mathrm{ASC})$ stands for the class of s-flag ASCs. The complex $\partial D[N_+]$ is $(N-1)$-dimensional so it is N-flag but it is not $(N-1)$-flag. The complexes MB (Figure 1) and P2 (Figure 2) are 2-flag (but not 1-flag).

The next proposition says that (P,s)-coloring issues are s-dimensional.

Proposition 6.9. *The following conditions are equivalent for any map $f\colon V \to P$:*

1. *f is a (P,s)-coloring of K*
2. *f is a (P,s)-coloring of $\mathrm{sk}_s(K)$*
3. *f is a (P,s)-coloring of $\mathrm{fla}_s(K)$*

Moreover, $\mathrm{Col}^s(K,P) = \mathrm{Col}^s(\mathrm{sk}_s(K),P) = \mathrm{Col}^s(\mathrm{fla}_s(K),P)$ *and* $\mathrm{chr}^s(K) = \mathrm{chr}^s(\mathrm{sk}_s(K)) = \mathrm{chr}^s(\mathrm{fla}_s(K))$.

6.2. **Proof of the main theorem.** We are now ready to prove our main result.

Proof of Theorem 1.2. (1) $\implies$ (3): By Theorem 4.8 and vector bundle theory [8, Theorem 9.6], if K admits a (P,s)-coloring then λ_K is stably isomorphic to a sum of complex vector bundles of dimension at most s.
(3) $\implies$ (2): This is clear.
(2) $\implies$ (1): Since R is a UFD, also the polynomial rings over R are UFDs by Gauss' theorem. Suppose that $\prod_{v\in V}(1+v) = \prod_{p\in P} c_p$ with $c_p \in \mathrm{SR}(K;R)$, $p\in P$, of degree at most $2s$. Recall from Section 3 that for any vertex vertex $v\in V$, $\mathrm{SR}(i_{\{v\}})\colon \mathrm{SR}(K;R) \to \mathrm{SR}(D[\{v\}];R) = R[v]$ denotes the ring homomorphism induced by the functor $\mathrm{SR}(-;R)$ applied to the inclusion $i_{\{v\}}\colon D[\{v\}] \to K$ of that vertex into K. Since the equation $1+v = \prod_{q\in P}\mathrm{SR}(i_{\{v\}})c_q$ holds in the UFD $R[v]$ there is a unique $p\in P$ such that $1+v$ divides $\mathrm{SR}(i_{\{v\}})c_p$. Define $f\colon V\to P$ by

$$\forall v\in V \forall p \in P\colon f(v) = p \iff 1+v \mid \mathrm{SR}(i_{\{v\}})c_p$$

Suppose that σ is a simplex of K. The equation

$$\prod_{v\in\sigma} 1+v = \prod_{p\in P}\mathrm{SR}(i_\sigma)c_p$$

holds in the UFD $R[\sigma]$. For each $v\in\sigma$, the prime element $1+v$ divides exactly one of the factors $\mathrm{SR}(i_\sigma)c_p$, $p\in P$. The only possibility is that $1+v$ divides $\mathrm{SR}(i_\sigma)c_{f(v)}$. It follows that

$$\forall p\in P\colon \prod_{v\in\sigma\cap f^{-1}p}(1+v) \mid \mathrm{SR}(i_\sigma)c_p$$

and for degree reasons we must in fact have that

$$\forall p\in P\colon \prod_{v\in\sigma\cap f^{-1}p}(1+v) = \mathrm{SR}(i_\sigma)c_p$$

up to a unit in R. Therefore $2|\sigma\cap f^{-1}p| = \deg(\mathrm{SR}(i_\sigma)c_p) \le \deg(c_p) \le 2s$ or $|\sigma\cap f^{-1}p| \le s$. This means that $f\colon V\to P$ is a (P,s)-coloring of K.
(1) $\iff$ (4): This is the special case of Theorem 6.14 with $L = D[P]$. Note that the proof of Theorem 6.14 relies only on items (1)–3 from Theorem 1.2. ☐

There is a version of Theorem 1.2 that refers to ξ_K rather than λ_K. Since only complexes K with $\dim K < rs$ admit (r,s)-colorings (6.2) it is no restriction to make this assumption.

Corollary 6.10. *Assume that $n(K)\le rs$. Then K admits an (r,s)-coloring if and only if there exists a map* $\mathrm{DJ}(K)\to \mathrm{BU}(s)^r$ *such that the diagram*

$$\begin{tikzcd} \mathrm{DJ}(K) \arrow[rr] \arrow[dr, "\xi_K\oplus\mathbf{C}^{rs-n(K)}"'] & & \mathrm{BU}(s)^r \arrow[dl, "\oplus"] \\ & \mathrm{BU}(rs) & \end{tikzcd}$$

is homotopy commutative.

Proof. If the diagram has a completion, then K admits an (r,s)-coloring by Theorem 1.2. Conversely, if K admits an (r,s)-coloring, by the short exact sequence (4.9) there are r vector bundles ξ_p, $1\le p\le r$, over $\mathrm{DJ}(K)$ such that

$$\bigoplus_{1\le p\le r}\xi_p \cong \xi_K\oplus\mathbf{C}^{rs-n(K)}$$

where $\dim \xi_p = s$, $1 \le p \le r$. $\square$

6.3. **(L,s)-colorings of simplicial complexes.** Let K be an ASC with vertex set V and L an ASC with vertex set P.

Definition 6.11. *An (L,s)-coloring of K is a simplicial map $f\colon K \to L$ whose vertex map $f\colon V \to P$ is a (P,s)-coloring of K.*

A (P,s)-coloring is the same thing as a $(D[P],s)$-coloring. Any (L,s)-coloring is a (P,s)-coloring. A (P,s)-coloring of K is a $(\partial D[P],s)$-coloring of K if no simplex of K uses the full palette P. There are no $(\partial[\{\blacksquare,\blacksquare,\blacksquare\}],2)$-colorings of P2 from Figure 2 which means that it is impossible to paint the vertices of P2 from a palette of 3 colors so that every facet has exactly 2 colors.

Let

$$\operatorname{map}^s(V,P) = \{f\colon V \to P \mid \forall p \in P\colon |f^{-1}p| \le s\}, \qquad \operatorname{map}^s(V,L) = \{f \in \operatorname{map}^s(V,P) \mid f(V) \in L\}$$

be the set of at most s-to-1 maps from V to P and the subset of those at most s-to-1 maps from V to P whose image is a simplex of L. Then $\operatorname{map}^s(V,L)$ is the set of (L,s)-colorings of the full simplex $D[V]$. More generally, let $\operatorname{Col}^s(K,L)$ stand for the set of (L,s)-colorings of the ASC K. Then

$$\operatorname{Col}^s(K,L) = \lim(P(K)^{\mathrm{op}}; \operatorname{map}^s(-,L)) = \lim(P(K)^{\mathrm{op}}; \operatorname{Col}^s(D[-],L))$$

is the limit of the contravariant functor from $P(K)$ to sets that takes any simplex σ of K to $\operatorname{map}^s(\sigma,L) = \operatorname{Col}^s(D[\sigma],L)$.

For each $p \in P$ let ν_p denote the s-dimensional complex vector bundle

$$\operatorname{map}(P, P-\{p\}; \mathrm{VU}(s), 0) \to \operatorname{map}(P, \mathrm{BU}(s)) \tag{6.12}$$

classified by the evaluation map $\nu_p\colon \operatorname{map}(P,\mathrm{BU}(s)) \to \mathrm{BU}(s)$ at p. Let $\nu_P = \bigoplus_{p\in P} \nu_p$ be the $|P|s$-dimensional Whitney sum

$$\operatorname{map}(P, \mathrm{VU}(s)) \to \operatorname{map}(P, \mathrm{BU}(s)),$$

classified by the map $\nu_P\colon \operatorname{map}(P,\mathrm{BU}(s)) \xrightarrow{\prod \nu_p} \prod_{p\in P} \mathrm{BU}(s) \xrightarrow{\oplus} \mathrm{BU}(|P|s)$, of these bundles.

Lemma 6.13. *There exists a map $f \in \operatorname{map}^s(V,L) = \operatorname{Col}^s(D[V],L)$ if and only if there exists a map*

$$\begin{tikzcd}
\operatorname{map}(V,\mathrm{BU}(1)) \arrow[rr, "F"] \arrow[dr, "\oplus"'] & & \mathrm{DJ}(L;\mathrm{BU}(s),*) \arrow[dl, "\oplus\circ\lambda_L"] \\
& \mathrm{BU} &
\end{tikzcd}$$

over BU. *If this is the case, then $F^*c(\nu_p) = \pm c(f^{-1}p)$ in $H^*(\operatorname{map}(V,\mathrm{BU}(1));\mathbf{Z}) = \mathbf{Z}[V]$ for all $p \in P$.*

Proof. Suppose that $f \in \operatorname{map}^s(V,L)$ is an s-to-1-map. Pick any linear ordering on V and use it to define Lie group homomorphisms $\operatorname{map}(f^{-1}p, \mathrm{U}(1)) \to \mathrm{U}(|f^{-1}p|) \to \mathrm{U}(s)$. Apply the classifying space functor to get maps $\mu_p\colon \operatorname{map}(f^{-1}p, \mathrm{BU}(1)) \to \mathrm{BU}(|f^{-1}p|) \to \mathrm{BU}(s)$, $p \in P$. Set $\mu = \{\mu_p \mid p \in P\}$. The induced map $\mathrm{DJ}(f;\mu)\colon \operatorname{map}(V,\mathrm{BU}(1)) = \mathrm{DJ}(D[V];\mathrm{BU}(1),*) \to \mathrm{DJ}(L;\mathrm{BU}(s),*)$ from Proposition 2.5.(3) will make the diagram homotopy commutative.

Conversely, given a map $F\colon \operatorname{map}(V,\mathrm{BU}(1)) \to \mathrm{DJ}(L;\mathrm{BU}(s),*)$ over BU, we get a map $\operatorname{map}(V,\mathrm{BU}(1)) \to \operatorname{map}(P,\mathrm{BU}(s))$ over BU by composing with the inclusion map $\lambda_L\colon \mathrm{DJ}(L;\mathrm{BU}(s),*) \to \operatorname{map}(P,\mathrm{BU}(s))$ introduced immediately below Definition 2.2. Then

$$\prod_{v\in V}(1+v) = \prod_{p\in P} c_p$$

with $c_p = F^*\lambda_L^* c(\nu_p)$, $p \in P$. According to (the proof of) Theorem 1.2.(1)–(3), there is a map $f \in \operatorname{map}^s(V,P)$ so that

$$\forall p \in P\colon \prod_{v\in f^{-1}p}(1+v) = \pm c_p$$

in the polynomial ring $\mathbf{Z}[V] = H^*(\operatorname{map}(V,\mathrm{BU}(1));\mathbf{Z})$. We claim that $f(V) = \{p \in P \mid c_p \ne 1\}$ is a simplex of L, or, equivalently, that $f \in \operatorname{map}^s(V,L)$.

Assume that $f(V) \notin L$. According to [17, Theorem 3.10] (see Proposition 3.3)

$$H^*(\mathrm{DJ}(L;\mathrm{BU}(s),*);\mathbf{Z}) = \lim(P(L)^{\mathrm{op}}; \otimes \operatorname{map}(P, P-?; H^*(\mathrm{BU}(s);\mathbf{Z}),\mathbf{Z}))$$

The element $\prod_{p\in f(V)}(c(\nu_p)-1)$ is in the kernel of

$$H^*(\mathrm{map}(P,\mathrm{BU}(s));\mathbf{Z}) \xrightarrow{\lambda_L^*} H^*(\mathrm{DJ}(L;\mathrm{BU}(s),*);\mathbf{Z}) \subset \prod_{\tau\in L} \otimes\, \mathrm{map}(P,P-\tau;H^*(\mathrm{BU}(s);\mathbf{Z}),\mathbf{Z}))$$

because $\tau - f(V) \neq \emptyset$ for any simplex τ of L. Thus

$$\prod_{p\in f(V)} \lambda_L^* c(\nu_p) = \sum_{p\in Q\subsetneq f(V)} (-1)^{|f(V)|-|Q|}\lambda_L^* c(\nu_p)$$

in $H^*(\mathrm{DJ}(L;\mathrm{BU}(s),*);\mathbf{Z})$ and, in $H^*(\mathrm{map}(V,\mathrm{BU}(1));\mathbf{Z}) = \mathbf{Z}[V]$,

$$\prod_{v\in V}(1+v) = \prod_{p\in f(V)} c_p = F^*\big(\prod_{p\in f(V)} \lambda_L^* c(\nu_p)\big) = F^*\big(\sum_{p\in Q\subsetneq f(V)} (-1)^{|f(V)|-|Q|}\lambda_L^* c(\nu_p)\big)$$
$$= \sum_{p\in Q\subsetneq f(V)} (-1)^{|f(V)|-|Q|} c_p = \sum_{Q\subsetneq f(V)} \pm(-1)^{|f(V)|-|Q|} \prod_{v\in f^{-1}Q}(1+v)$$

which is a contradiction because these elements of $\mathbf{Z}[V]$ do not have the same homogeneous components in degree $2|V|$. Therefore we must have that $f(V)\in L$. $\square$

Theorem 6.14. *The ASC K is (L,s)-colorable if and only if there exists a map $\mathrm{DJ}(K;\mathrm{BU}(1),*) \to \mathrm{DJ}(L;\mathrm{BU}(s),*)$ such that the diagram*

$$\begin{tikzcd}
\mathrm{DJ}(K;\mathrm{BU}(1),*) \arrow[rr] \arrow[dr, "\oplus\circ\lambda_K"'] & & \mathrm{DJ}(L;\mathrm{BU}(s),*) \arrow[dl, "\oplus\circ\lambda_L"] \\
& \mathrm{BU} &
\end{tikzcd}$$

commutes up to homotopy.

Proof. Suppose that $f\colon V\to P$ is an (L,s)-coloring of K. Pick any linear ordering on V and use it to define Lie group homomorphisms $\mathrm{map}(f^{-1}p,\mathrm{U}(1)) \to \mathrm{U}(|f^{-1}p|) \to \mathrm{U}(s)$. Apply the classifying space functor to get maps $\mu_p\colon \mathrm{map}(f^{-1}p,\mathrm{BU}(1)) \to \mathrm{BU}(|f^{-1}p|) \to \mathrm{BU}(s)$, $p\in P$. Set $\mu = \{\mu_p \mid p\in P\}$. The induced map $\mathrm{DJ}(f;\mu)\colon \mathrm{DJ}(K;\mathrm{BU}(1),*) \to \mathrm{DJ}(L;\mathrm{BU}(s),*)$ from Proposition 2.5.(3) will make the diagram homotopy commutative.

Conversely, given a map $\mathrm{DJ}(K;\mathrm{BU}(1),*) \to \mathrm{DJ}(L;\mathrm{BU}(s),*)$ over BU. Let σ be a simplex of K and $f_\sigma \in \mathrm{map}^s(\sigma,L) = \mathrm{Col}^s(D[\sigma],L)$ the local coloring map that according to Lemma 6.13 corresponds to the map

$$\mathrm{map}(\sigma,\mathrm{BU}(1)) = \mathrm{DJ}(D[\sigma];\mathrm{BU}(1),*) \xrightarrow{\mathrm{DJ}(i_\sigma)} \mathrm{DJ}(K;\mathrm{BU}(1),*) \to \mathrm{DJ}(L;\mathrm{BU}(s),*)$$

over BU. Then $f = (f_\sigma)_{\sigma\in K} \in \mathrm{Col}^s(K,L) = \lim(P(K)^{\mathrm{op}};\mathrm{Col}^s(D[\sigma],L))$ is an (L,s)-coloring of K. $\square$

7. Vertex colorings of polyhedra

Let M be a simplicial manifold, or, more generally, a compact polyhedron; see [5, Definitions 2.31 and 2.33] for definitions of the terminology applied here.

Definition 7.1. $\mathrm{chr}^s(M) = \max\{\mathrm{chr}^s(K) \mid K$ *triangulates* $M\}$ *is the maximum of* $\mathrm{chr}^s(K)$ (6.1) *over all finite triangulations K of the polyhedron M.*

The chromatic number is an invariant of the the homeomorphism type, but not an invariant of the homotopy type as, for instance, triangulations of the plane are very different from triangulations of a point. The assertion $\mathrm{chr}^s(S^d) = r$ means that one needs at most r colors to color *any* simplicial d-sphere in such a way that the s-dimensional simplices are polychrome. For instance,

$$\begin{array}{llll}
\mathrm{chr}^2(S^1) = 1 & \mathrm{chr}^1(S^1) = 3 & & \\
\mathrm{chr}^3(S^2) = 1 & \mathrm{chr}^2(S^2) = 2 & \mathrm{chr}^1(S^2) = 4 & \\
\mathrm{chr}^4(S^3) = 1 & \mathrm{chr}^3(S^3) \geq 2 & \mathrm{chr}^2(S^3) \geq 4 & \mathrm{chr}^1(S^3) = \infty
\end{array}$$

Example 5.7 implies that $\mathrm{chr}^1(\Sigma) \geq 4$ for all compact surfaces Σ (and the exact values are, except for the 2-sphere and the Klein bottle, given by Heawood's inequality and the Map color theorem [19]). The 4-color theorem [3, Chp 11] says that $\mathrm{chr}^1(S^2) = 4$. From a $(4,1)$-coloring of S^2 we get a $(2,2)$-coloring by identifying colors pairwise. Thus $\mathrm{chr}^2(S^2) \leq 2$ and the reverse inequality is (6.2). The assertion about the chromatic numbers of S^3 are justified by the examples below.

Example 7.2. The simplicial 3-sphere $\partial D[3_+] * \partial D[1_+]$ is $(3,2)$-colorable because it has 6 vertices (6.2). $\partial D[3_+] * \partial D[1_+]$ is a 3-flag complex.

Example 7.3. A computer verification shows that Altschuler's 'peculiar' simplicial 3-sphere ALT [1], with f-vector $(1,10,45,70,35)$, is a 2-flag $(4,2)$-colorable ASC with coloring $[1,1,2,2,1,2,3,3,4,4]$. The 2-chromatic number is $\mathrm{chr}^2(\mathrm{ALT}) = 4$.

Example 7.4. A computer verification shows that Mani and Walkup's simplicial 3-spheres C and D [13], with f-vectors $(1,20,126,212,106)$ and $(1,16,106,180,90)$, are 2-flag complexes with 2-chromatic numbers $\mathrm{chr}^2(C) = 3$ and $\mathrm{chr}^2(D) = 4$.

Example 7.5. Klee and Kleinschmidt constructed a simplicial 3-sphere KK with f-vector $(1,16,106,180,90)$ that is not vertex-decomposable and not polytopal [9, Table 1]. This 3-spherical complex is 2-flag, $\mathrm{chr}^2(\mathrm{KK}) = 4$ and $[1,3,1,3,2,2,2,2,4,4,1,1,3,2,2,4]$ is a $(4,2)$-coloring.

These examples show that $\mathrm{chr}^2(S^3) \geq 4$. All these simplicial 3-spheres are $(2,3)$-colorable so that $\mathrm{chr}^3(S^3) \geq 2$. We do not know if $\mathrm{chr}^2(S^3)$ or $\mathrm{chr}^3(S^3)$ are finite numbers.

Example 7.6. The non-constructible simplicial 3-spheres $S^3_{17,74}$ and $S^3_{13,65}$ with f-vectors $(1,17,91,148,74)$ and $(1,13,69,112,56)$ found by Lutz [11] are $(3,2)$-colorable and $\mathrm{chr}^2(S^3_{17,74}) = 3 = \mathrm{chr}^2(S^3_{13,65})$. The non-constructible simplicial 3-sphere with f-vector $f = (1,381,2309,3856,1928)$ containing a knotted triangle [10, 7] is $(4,2)$-colorable. The reduction modulo 4 map $\{1,2,\dots,4m\} \to \{0,1,2,3\}$ is a $(4,2)$-coloring of the centrally symmetric simplicial 3-spheres CS^3_{4m}, $m \geq 2$, from [12, Theorem 6]. The triangulations ${}^{3}_{nn}10^{di}_{1}$ and ${}^{3}_{nn}12^{cy}_{1}$ from [12, Table 2] are $(3,2)$-colorable, ${}^{3}_{nn}14^{cy}_{1}$ is $(4,2)$-colorable, and ${}^{3}_{nn}16^{cy}_{k}$, $1 \leq k \leq 5$, are $(4,2)$-colorable by the map $v \to v \bmod 4$, $1 \leq v \leq 16$.

$[1,1,1,2,2,3,3,1,3,3,4,2,2,4,4,1]$ is a $(4,2)$-coloring of the 2-flag triangulation, BLP, of the Poincare homology 3-sphere with $f = (1,16,106,180,90)$ constructed by Björner and Lutz [2] and $\mathrm{chr}^2(\mathrm{BLP}) = 4$.

Example 7.7. $\partial D[(2n)_+]$ is a triangulation of S^{2n-1}, $n \geq 1$, with n-chromatic number $\mathrm{chr}^n(\partial D[(2n)_+]) = 3$. The map $f\colon \{0,1,\dots,2n\} \to \{1,2,3\}$ given by

$$f(j) = \begin{cases} 1 & 0 \leq j < n \\ 2 & n \leq j < 2n \\ 3 & j = 2n \end{cases}$$

is a $(3,n)$-coloring. Similarly, the ASC $\partial D[(2n+1)_+]$ is a triangulation of S^{2n}, $n \geq 1$, with n-chromatic number

$$\mathrm{chr}^n(\partial D[(2n+1)_+]) = \begin{cases} 4 & n = 1 \\ 3 & n > 1 \end{cases}$$

and the map $f\colon \{0,1,\dots,2n+1\} \to \{1,2,3\}$ given by

$$f(j) = \begin{cases} 1 & 0 \leq j < n \\ 2 & n \leq j < 2n \\ 3 & j = 2n, 2n+1 \end{cases}$$

is a $(3,n)$-coloring for $n > 1$.

We now define the spherical complexes associated to cyclic n-polytopes.

Definition 7.8. $\mathrm{CP}(m,n)$*, $m > n$, is the $(n-1)$-dimensional ASC on the ordered set $V = \{1,\dots,m\}$ with the following facets: An n-subset σ of V is a facet if and only if between any two elements of $V - \sigma$ there is an even number of vertices in σ.*

By Gale's Evenness Theorem [6], the ASC $\mathrm{CP}(m,n)$ triangulates the boundary of the cyclic n-polytope on m vertices. Thus $\mathrm{CP}(m,n)$ is a simplicial $(n-1)$-sphere on m vertices and it is $\lfloor n/2 \rfloor$-neighborly in the sense that $\mathrm{CP}(m,n)$ has the same s-skeleton as the full simplex on its vertex set when $s < \lfloor n/2 \rfloor$ (Proposition 6.9)

[5, Definition 1.15, Example 1.17, Remark after Theorem 6.33]. When $s < \lfloor n/2 \rfloor$, the s-chromatic number of the cyclic polytope $\mathrm{CP}(m,n)$, $\mathrm{chr}^s(\mathrm{CP}(m,n)) = \mathrm{chr}^s(D[V]) = \lceil m/s \rceil$, grows with the number of vertices m. The first interesting chromatic numbers for the cyclic polytopes are $\mathrm{chr}^n(\mathrm{CP}(m,2n))$ and $\mathrm{chr}^n(\mathrm{CP}(m,2n+1))$ and the first first interesting chromatic numbers for the spheres (7.1) are $\mathrm{chr}^n(S^{2n-1})$ and $\mathrm{chr}^n(S^{2n})$.

Proposition 7.9. *For $n \geq 1$,*

$$\mathrm{chr}^n(\mathrm{CP}(m,2n)) = \begin{cases} 2 & m > 2n \text{ even} \\ 3 & m > 2n \text{ odd} \end{cases} \qquad \text{and} \qquad \mathrm{chr}^n(\mathrm{CP}(m,2n+1)) = \begin{cases} 4 & n = 1, \quad m > 3 \\ 3 & n > 1, \quad m > 2n+1 \end{cases}$$

Proof. For general reasons (6.2), $\mathrm{chr}^n(\mathrm{CP}(m,2n)) \geq 2$ and $\mathrm{chr}^n(\mathrm{CP}(m,2n+1)) \geq 3$.

The reduction modulo 2 map $\{1,\ldots,m\} \to \{0,1\}$ is a $(2,n)$ coloring of $\mathrm{CP}(m,2n)$, $n \geq 1$, for even $m > 2n$ and also for odd $m > 2n$ if we modify this map so that it assume the value 2 at vertex m.

The map $f\colon \{1,\ldots,m\} \to \{0,1,2,3\}$ given by

$$f(j) = \begin{cases} 2 & j = 1 \\ j \bmod 2 & 1 < j < m \\ 3 & j = m \end{cases}$$

is a $(4,1)$-coloring of the simplicial 2-sphere $\mathrm{CP}(m,3)$, $m > 3$. There is no $(4,1)$-coloring because there is no $(4,1)$-coloring of $\mathrm{CP}(4,3) = \partial D[3_+]$.

The maps $f\colon \{1,\ldots,m\} \to \{0,1,2\}$ given by

$$f(j) = \begin{cases} 2 & j = 1,2,m \\ j \bmod 2 & n < j < m \end{cases} \qquad f(j) = \begin{cases} 2 & j = 1,m \\ j \bmod 2 & n < j < m \end{cases}$$

when n is odd or n is even, respectively, are $(3,n)$-colorings of $\mathrm{CP}(m,2n+1)$, $n > 1$, $m > 2n+1$. ∎

There is no example contradicting the statement that $\mathrm{chr}^n(S^{2n}) = 4$ for all $n \geq 1$ and $\mathrm{chr}^n(S^{2n-1}) = 4$ for all $n \geq 2$.

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