N-determined 2-compact groups

Jesper M. Møller

Matematisk Institut, Universitetsparken 5, DK–2100 København
E-mail address: moller@math.ku.dk
URL: http://www.math.ku.dk/~moller

2000 *Mathematics Subject Classification.* 55R35, 55P15

Key words and phrases. Classification of p-compact groups at the prime $p = 2$, p-compact group, compact Lie group, Quillen category, homology decomposition, nontoral elementary abelian 2-group, preferred lift

ABSTRACT. The main purpose of this paper is to formulate a general scheme for the possible classification of 2-compact groups in terms of maximal torus normalizer pairs. As an application it is shown that the connected 2-compact groups associated to the simple compact Lie groups of the A-, B-, C, and D-families, as well as G_2 and F_4, are determined up to isomorphism by their maximal torus normalizers. Also the exotic 2-compact group DI(4) is uniquely determined by its maximal torus normalizer.

Contents

CHAPTER 1

Introduction

sec:intro

A p-compact group, where p is a prime number, is a p-complete space BX whose loop space $X = \Omega BX$ has finite mod p singular cohomology. If G is a Lie group and $\pi_0(G)$ is a finite p-group then the p-completed classifying space of G is a p-compact group. The Sullivan spheres $(BS^{2n-1})^{\wedge}_p$, $n|(p-1)$, or, more generally, the Clark–Ewing spaces [**9**] are also examples of p-compact groups. These homotopy Lie groups were defined and explored by W.G. Dwyer and C.W. Wilkerson in a series of papers [**17**, **18**, **19**, **16**]. (The reader may also consult one of the survey articles [**11**, **33**, **50**, **41**].) They show that any p-compact group BX has a maximal torus $BT \to BX$ and a Weyl group such that the normalizer $BN \to BX$ of the maximal torus is an extension

$$BT \to BN \to BW$$

of the maximal torus by the Weyl group. (Stricly speaking, BN is in general not a p-compact group as its fundamental group may not be a finite p-group; instead, BN is an example of an extended p-compact torus [**18**, 3.12].) It is a conjecture, suggested by the analogous situation for connected compact Lie groups [**10**] (and some nonconnected ones [**25**, **26**]), that BN determines BX. This classification conjecture has been verified for odd primes [**44**, **47**, **2**]. For $p = 2$, however, only scattered results are known. It is the main purpose of this paper to establish an environment facilitating a structured approach to the classification problem for 2-compact groups. First, we have to be clear about what it means for two 2-compact groups to have the same maximal torus normalizer.

The first obstacle for a classification of 2-compact groups in terms of their maximal torus normalizers is that the maximal torus normalizer does not retain information about the component group of the 2-compact group. For instance, the nonconnected 2-compact group $BO(2n)$ and the connected 2-compact group $BSO(2n+1)$ have identical maximal torus normalizers. Thus we have to use a stronger invariant. One way to store information about component groups is to replace maximal torus normalizers by maximal torus normalizer *pairs*.

Let BN be an extended 2-compact torus with a normal maximal rank subgroup $BN_0 \to BN$; this simply means that there is a fibration sequence of the form $BN_0 \to BN \to B\pi$ where $\pi = N/N_0$ is a finite group. Consider a 2-compact group BX with identity component $BX_0 \to BX$ and component group $\pi_0(X) = X/X_0$. We shall say that the pair (BN, BN_0) is a *maximal torus normalizer pair* for the 2-compact group BX (2.2) if there exists a map of fibrations

$$\begin{array}{ccc} BX_0 & \xleftarrow{Bj_0} & BN_0 \\ \downarrow & & \downarrow \\ BX & \xleftarrow{Bj} & BN \\ \downarrow & & \downarrow \\ B\pi_0(X) & \xleftarrow{\cong} & B\pi \end{array}$$

where the horizontal maps are maximal torus normalizers of 2-compact groups (so that the map between the base spaces is a homotopy equivalence). In particular, the maximal torus normalizer pair determines the component group in that $\pi_0(X) \cong N/N_0$.

According to this definition, two 2-compact groups, BX and BX' have the same maximal torus normalizer pair if there exists a commutative diagram

$$\begin{array}{ccccc} BX_0 & \xleftarrow{Bj_0} & BN_0 & \xrightarrow{Bj'_0} & BX'_0 \\ \downarrow & & \downarrow & & \downarrow \\ BX & \xleftarrow{Bj} & BN & \xrightarrow{Bj'} & BX' \\ \downarrow & & \downarrow & & \downarrow \\ B\pi_0(X) & \xleftarrow{\cong} & B\pi & \xrightarrow{\cong} & B\pi_0(X') \end{array}$$

where the horizontal maps are maximal torus normalizers of 2-compact groups.

We shall say that a 2-compact group BX with maximal torus normalizer pair $(BN, BN_0) \to (BX_0, BX)$ is *totally N-determined* (2.11) if

(1) automorphisms of BX are determined by their restrictions to BN, and
(2) for any other connected 2-compact group BX' with the same maximal torus normalizer pair as BX there exist an isomorphism $Bf\colon BX \to BX'$ and an automorphism $B\alpha\colon BN \to BN$, inducing the identity map on homotopy groups, making the diagram

$$\begin{array}{ccc} BN & \xrightarrow[\cong]{B\alpha} & BN \\ \downarrow & & \downarrow \\ BX & \xrightarrow[Bf]{\cong} & BX' \end{array}$$

commutative

We shall say that BX is *uniquely N-determined* if in addition the automorphisms of BX are determined by their effect on the (two nontrivial) homotopy groups of BN. (See Lemma 2.13 for a justification of the terminology.) The role of $B\alpha$ is to compensate for the automorphisms of BN that do not extend to automorphisms of BX; such automorphisms do exist when $p = 2$ whereas they do not occur at odd primes.

1.1. Conjecture. All (connected) 2-compact groups are (uniquely) totally N-determined.

The proposed plan for proving the conjecture has two stages. The first stage, which is completed in this paper, consists of a reduction of the problem to case of connected, simple and centerless 2-compact groups. The next stage, only partially solved in this paper, is an inductive case-by-case checking of the simple 2-compact groups. This approach follows the inductive principle for 2-compact groups [**18**, 9.1].

The classification conjecture for 2-compact groups can be reduced to the case of connected, centerless, and simple 2-compact groups because N-determinism is to a large extent hereditary (2.§2): The product of two N-determined 2-compact groups is N-determined, and any connected 2-compact group whose adjoint form is N-determined is itself N-determined. Furthermore, any connected and centerless 2-compact group can be decomposed into a homotopy colimit of a system of 2-compact group of smaller dimension [**15**] and, under certain hypotheses (2.48, 2.51), N-determinism is hereditary also under homotopy colimits. Thus there is a theoretical possibility of proving the classification conjecture by induction over the dimension.

In this paper we prove the classification conjecture for the 2-compact groups associated to the classical matrix Lie groups of the infinite A-, B-, C- and D-families, for the exceptional compact Lie groups G_2 and F_4, and for the exotic 2-compact group $\mathrm{DI}(4)$ from [**16**].

Any connected Lie group G has an associated 2-compact group obtained as the 2-completion of the classifying space of G. We shall denote the 2-compact group associated to G by G also. As 2-compact groups, $\mathrm{SL}(2n+1, \mathbf{R})$ and $\mathrm{SO}(2n+1)$, for instance, are synonyms because their classifying spaces are homotopy equivalent.

The main results are the theorems below (whose proofs are in Chapter 7 and in Chapter 8).

thm:afam

THEOREM 1.2. *The simple 2-compact group* $\mathrm{PGL}(n+1,\mathbf{C})$, $n\geq 1$, *is uniquely N-determined, and its automorphism group is*

$$\mathrm{Aut}(\mathrm{PGL}(n+1,\mathbf{C}))=\begin{cases}\mathbf{Z}^\times\backslash\mathbf{Z}_2^\times & n=1\\ \mathbf{Z}_2^\times & n>1\end{cases}$$

mainthm

THEOREM 1.3. *The simple 2-compact groups* $\mathrm{PSL}(2n,\mathbf{R})$, $n\geq 4$, $\mathrm{SL}(2n+1,\mathbf{R})$, $n\geq 2$, *and* $\mathrm{PGL}(n,\mathbf{H})$, $n\geq 3$, *are uniquely N-determined for all* $n\geq 1$. *Their automorphism groups are*

$$\mathrm{Aut}(\mathrm{PSL}(2n,\mathbf{R}))=\begin{cases}\mathbf{Z}^\times\backslash\mathbf{Z}_2^\times\times\Sigma_3 & n=4\\ \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times\times\langle c_1\rangle & n>4 \text{ even}\\ \mathbf{Z}_2^\times & n>4\text{ odd}\end{cases}$$

where $\langle c_1\rangle$ *is a group of order two,* $\mathrm{Aut}(\mathrm{SL}(2n+1,\mathbf{R}))=\mathbf{Z}^\times\backslash\mathbf{Z}_2^\times$ *for* $n\geq 2$, *and* $\mathrm{Aut}(\mathrm{PGL}(n,\mathbf{H}))=\mathbf{Z}^\times\backslash\mathbf{Z}_2^\times$ *for* $n\geq 3$.

thm:g2

THEOREM 1.4. [60, 1.3] *The simple 2-compact group* G_2 *is uniquely N-determined and its automorphism group is* $\mathrm{Aut}(\mathrm{G}_2)=\mathbf{Z}^\times\backslash\mathbf{Z}_2^\times\times C_2$.

thm:di4

THEOREM 1.5. *The simple 2-compact group* $\mathrm{DI}(4)$ *is uniquely N-determined and its automorphism group is* $\mathrm{Aut}(\mathrm{DI}(4))=\mathbf{Z}^\times\backslash\mathbf{Z}_2^\times$.

thm:f4

THEOREM 1.6. *The simple 2-compact group* F_4 *is uniquely N-determined and its automorphism group is* $\mathrm{Aut}(\mathrm{F}_4)=\mathbf{Z}^\times\backslash\mathbf{Z}_2^\times$.

The methods are not limited to simple nor even to connected 2-compact groups. Here are two examples of the type of consequences that can obtained for more general 2-compact groups.

cor:afam

1.7. COROLLARY. [38, 1.9] *The 2-compact group* $\mathrm{GL}(n,\mathbf{C})$ *is uniquely N-determined and its automorphism group is*

$$\mathrm{Aut}(\mathrm{GL}(n,\mathbf{C}))=\begin{cases}\mathbf{Z}^\times\backslash\mathrm{Aut}_{\mathbf{Z}_2\Sigma_2}(\mathbf{Z}_2^2) & n=2\\ \mathrm{Aut}_{\mathbf{Z}_2\Sigma_n}(\mathbf{Z}_2^n) & n>2\end{cases}$$

cor:glnR

1.8. COROLLARY. *The 2-compact group* $\mathrm{GL}(n,\mathbf{R})$ *is totally N-determined for all* $n\geq 2$ *and its automorphism group is*

$$\mathrm{Aut}(\mathrm{GL}(n,\mathbf{R}))=\begin{cases}\mathbf{Z}^\times\backslash\mathbf{Z}_2^\times & n\geq 3\text{ odd}\\ \mathbf{Z}_2^\times & n=2\\ \mathbf{Z}_2^\times\times\langle\delta\rangle & n\equiv 2 \bmod 4,\ n>2\\ \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times\times\langle c_1\rangle\times\langle\delta\rangle & n\equiv 0\bmod 4\end{cases}$$

where $\langle\delta\rangle$ *and* $\langle c_1\rangle$ *are subgroups of order two.*

Related uniqueness results can be found in the papers [49, 51, 52, 60, 59, 58] by Dietrich Notbohm, Antonio Viruel and Aleš Vavpetič, respectively.

It has been conjectured [11, 5.1, 5.2] that any connected 2-compact group splits as a product of the form $B\mathrm{DI}(4)^s\times BG$ for some connected Lie group G and some $s\geq 0$. This conjecture is true on the level of maximal torus normalizers [14, 1.12] (and hence also true rationally [52, 1.5]). If it can be shown that also the three members of the E-family are uniquely N-determined, the conjecture will follow.

CHAPTER 2

N-determined 2-compact groups

cha:ndet This chapter contains the fundamental definitions and the first general results. Whereas p-compact groups are determined by their maximal torus normalizers [47, 2] when $p > 2$, a finer invariant is needed for 2-compact groups as there are examples (2.3) of distinct 2-compact groups with identical maximal torus normalizers.

1. Maximal torus normalizer pairs

sec:Z2refl Let $N_0 \to N$ be a maximal rank normal monomorphism between two extended 2-compact tori, meaning simply that there exists a short exact sequence [17, 3.2] of loop spaces $N_0 \to N \to \pi$ for some finite group π. For a 2-compact group, X, let (X, X_0) be the pair consistsing of X and its identity component X_0. Then there is a short exact sequence $X_0 \to X \to \pi_0(X)$ of loop spaces where $\pi_0(X) = X/X_0$ is a finite 2-group, the component group of X.

defn:mtnp 2.2. Definition. *If there exists a morphism of loop space short exact sequences* [18, 2.1]

$$\begin{array}{ccccc} N_0 & \longrightarrow & N & \longrightarrow & \pi \\ {\scriptstyle j_0}\downarrow & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \cong} \\ X_0 & \longrightarrow & X & \longrightarrow & \pi_0(X) \end{array}$$

where $j_0\colon N_0 \to X_0$ and $j\colon N \to X$ are maximal torus normalizers [17, 9.8]*, and $\pi \to \pi_0(X)$ an isomorphism of finite 2-groups, then we say that (N, N_0) is a maximal torus normalizer pair for (X, X_0).*

A maximal torus normalizer pair for X determines the maximal torus $T(X)$, isomorphic to the identity component of N, the Weyl groups, $W(X) = \pi_0(N)$ and $W(X_0) = \pi_0(N_0)$, of X and X_0, the component group $\pi_0(X) = N/N_0 = W(X)/W(X_0)$ [37, 3.8], and [18, 7.5] the center $Z(X_0) \to X_0$ of X_0 [37, 18].

onson 2.3. Example. (1) Since $\mathrm{GL}(2,\mathbf{R}) \wr \Sigma_n = N(\mathrm{SL}(2n+1,\mathbf{R})) \subset \mathrm{GL}(2n,\mathbf{R}) \subsetneq \mathrm{SL}(2n+1,\mathbf{R})$, $\mathrm{GL}(2n,\mathbf{R})$ and $\mathrm{SL}(2n+1,\mathbf{R})$ (Chp 5) have the same maximal torus normalizer. Their maximal torus normalizer pairs are distinct, however, as their component groups are distinct.
(2) More generally [25], let G be any compact connected Lie group and $N(G)$ its maximal torus normalizer. If $N(G)$ is not maximal, there exists a compact Lie group H such that $N(G) \subseteq H \subsetneq G$. The two compact Lie groups, G and H, have isomorphic maximal torus normalizers but distinct maximal torus normalizer pairs as H is nonconnected [6].
3. The Weyl groups for $\mathrm{SL}(2n+1,\mathbf{R})$ (Chp 5) and $\mathrm{GL}(n,\mathbf{H})$, $n \geq 3$, (Chp 6) are isomorphic as reflection groups but $N(\mathrm{SL}(2n+1,\mathbf{R}))$ is a split and $N(\mathrm{GL}(n,\mathbf{H}))$ a nonsplit extension [10, 34] of the Weyl group by the maximal torus. Thus connected 2-compact groups can not be classified by their Weyl group alone.

sec:AM **2.4. The Adams–Mahmud homomorphism.** For a 2-compact group (or extended 2-compact torus [18, 3.12]) X, we let $\mathrm{End}(X) = [BX,*;BX,*]$ denote the monoid of pointed homotopy classes of self-maps of BX. The *automorphism group* $\mathrm{Aut}(X) \subseteq [BX,*;BX,*]$ of X is the group of invertible elements in $\mathrm{End}(X)$ and the *outer automorphism group* $\mathrm{Out}(X) = \pi_0(X)\backslash \mathrm{Aut}(X) \subseteq [BX;BX]$ is the group of conjugacy classes (free homotopy classes [18, 2.1]) of automorphisms of X.

Let X be a 2-compact group with maximal torus normalizer pair (N, N_0). Turn the maximal torus normalizer $Bj\colon BN \to BX$ into a fibration. Any automorphism $f\colon X \to X$ of the 2-compact group X restricts to an automorphism $\mathrm{AM}(f)\colon N \to N$ of the maximal torus normalizer, unique up to the action of the Weyl group $W(X_0) = \pi_1(X/N)$ [37, 3.8, 5.6.(1)] of the identity component X_0 of X, such that the diagram

$$\begin{array}{ccc} BN & \xrightarrow{B(\mathrm{AM}(f))} & BN \\ {\scriptstyle Bj}\downarrow & & \downarrow{\scriptstyle Bj} \\ BX & \xrightarrow[Bf]{} & BX \end{array}$$

commutes up to based homotopy [44, §3]. The Adams–Mahmud homomorphism is the resulting homomorphism

AM (2.5) $$\mathrm{AM}\colon \mathrm{Aut}(X) \to W(X_0)\backslash \mathrm{Aut}(N)$$

of automorphism groups.

The automorphism group of N sits [42, 5.2] in a short exact sequence ($\check{T}(X)$ is the discrete approximation [17, 6.5] to $T(X)$)

eq:H1TW (2.6) $$0 \to H^1(W(X); \check{T}(X)) \to \mathrm{Aut}(N) \xrightarrow{\pi_*} \mathrm{Aut}(W(X), \check{T}(X), e(X)) \to 1$$

where the normal subgroup to the left consists of all automorphisms of N that induce the identity on homotopy groups and the group to the right consists of all pairs $(\alpha, \theta) \in \mathrm{Aut}(W(X)) \times \mathrm{Aut}(\check{T}(X))$ such that θ is α-linear and the induced automorphism $H^2(\alpha^{-1}, \theta)$ [61, 6.7.6] preserves the extension class $e(X) \in H^2(W(X); \check{T}(X))$. The image of $W(X_0)$ in $\mathrm{Aut}(N)$ does not intersect the subgroup $H^1(W(X); \check{T}(X))$ (as $W(X_0)$ is represented faithfully in $\mathrm{Aut}(\check{T}(X))$ [17, 9.7]) so there is an induced short exact sequence

autNW0 (2.7) $$0 \to H^1(W(X); \check{T}(X)) \to W(X_0)\backslash \mathrm{Aut}(N) \xrightarrow{\pi_*} W(X_0)\backslash \mathrm{Aut}(W(X), \check{T}(X), e(X)) \to 1$$

whose middle term is the target of the Adams–Mahmud homomorphism. In particular, if X is *connected*, this short exact sequence

outN (2.8) $$0 \to H^1(W(X); \check{T}(X)) \to \mathrm{Out}(N) \xrightarrow{\pi_*} W(X)\backslash \mathrm{Aut}(W(X), \check{T}(X), e(X)) \to 1$$

has the group $\mathrm{Out}(N) = W(X)\backslash \mathrm{Aut}(N)$ of outer automorphisms of N as its middle term. The group $\mathrm{Aut}(W(X), \check{T}(X), 0)$ may also be described as the normalizer $N_{\mathrm{GL}(L(X))}(W(X))$ of $W(X)$ in $\mathrm{GL}(L(X))$ where $L(X) = \pi_2(BT(X))$. This group evidently fits into an exact sequence [40, §2]

autwNgl (2.9) $$1 \to Z(W(X))\backslash \mathrm{Aut}_{\mathbf{Z}_2 W(X)}(L(X)) \to W(X)\backslash N_{\mathrm{GL}(L(X))}(W(X)) \to \mathrm{Out}_{\mathrm{tr}}(W(X))$$

where $\mathrm{Aut}_{\mathbf{Z}_2 W(X)}(L(X)) = \mathbf{Z}_2^\times$ if X is simple by Schur's lemma, and $\mathrm{Out}_{\mathrm{tr}}(W(X))$ is the group of outer automorphisms of $W(X)$ that preserve the trace taken in $L(X)$.

sec:defns

2.10. Totally N-determined 2-compact groups. We are now ready to formulate the concept of N-determinism that will be used in this paper. The extra complications compared to the odd p case [44, 7.1] stem from the fact that $H^1(W; \check{T})$, the first cohomology group of the Weyl group with coefficients in the discrete maximal torus, is trivial for any connected p-compact group when p is odd [3] but when $p = 2$ it may very well be nontrivial [24].

defn:det 2.11. Definition. *Let X be a 2-compact group with maximal torus normalizer pair $(N, N_0) \xrightarrow{(j, j_0)} (X, X_0)$ (2.2).*

det1 (1) X has N-determined ($\pi_*(N)$-determined) automorphisms *if*

$$\mathrm{AM}\colon \mathrm{Aut}(X) \to W_0\backslash \mathrm{Aut}(N) \quad \big(\pi_* \circ \mathrm{AM}\colon \mathrm{Aut}(X) \to W_0\backslash \mathrm{Aut}(W, \check{T}, e)\big)$$

is injective.

det2 (2) X *is* N-determined *if, for any other 2-compact group X' with maximal torus normalizer pair $(N, N_0) \xrightarrow{(j', j'_0)} (X', X'_0)$, there exist an isomorphism $f\colon X \to X'$ and an automorphism $\alpha \in H^1(W; \check{T}) \subset W_0\backslash \mathrm{Aut}(N)$ such that the diagram*

dia:BaBf (2.12)
$$\begin{array}{ccc} BN & \xrightarrow[\cong]{B\alpha} & BN \\ {\scriptstyle Bj}\downarrow & & \downarrow{\scriptstyle Bj'} \\ BX & \xrightarrow[Bf]{\cong} & BX' \end{array}$$

commutes up to based homotopy.

Furthermore, we say that

- X is *totally N-determined* if X has N-determined automorphisms and is N-determined,
- X is *uniquely* N-determined if X is totally N-determined and X has $\pi_*(N)$-determined automorphisms.

If X is a totally N-determined 2-compact group then

$$X \text{ is uniquely } N\text{-determined} \iff H^1(W;\check{T}) \cap \operatorname{Aut}(X) = 0$$

as we see from the short exact sequence (2.7).

defcons

2.13. Lemma. *Let X be a 2-compact group as in Definition 2.11.*

defcons1

(1) *X has N-determined automorphisms if and only if for any $\alpha \in W_0\backslash\operatorname{Aut}(N)$ with $\pi_*(B\alpha) = \mathrm{Id}$ and for any 2-compact group X' as in 2.11.(2) there is at most one isomorphism $f\colon X \to X'$ such that diagram (2.12) commutes up to based homotopy.*

defcons2

(2) *X has $\pi_*(N)$-determined automorphisms if and only if for any given X' as in 2.11.(2), diagram (2.12) has at most one solution (f,α) with $\pi_*(B\alpha) = \mathrm{Id}$.*

Proof. (1) Suppose that X has N-determined automorphisms. Let (f_1,α) and (f_2,α) be two solutions to diagram (2.12) with the same $\alpha \in H^1(W;\check{T}) \subset W(X_0)\backslash\operatorname{Aut}(N)$. Then $\mathrm{AM}(f_2^{-1}f_1)$ is the identity of $W(X_0)\backslash\operatorname{Aut}(N)$ and since $\mathrm{AM}\colon \operatorname{Aut}(X) \to W(X_0)\backslash\operatorname{Aut}(N)$ is injective, $f_1 = f_2$. For the converse, take $B\alpha$ to be the identity of BN and take X' to be X. Then the assumption is precisely that AM is injective.
(2) Suppose that X has $\pi_*(N)$-determined automorphisms and let (f_1,α_1) and (f_2,α_2) be two solutions to diagram (2.12). Then $\mathrm{AM}(f_2^{-1}f_1) = \alpha_2^{-1}\alpha_1 \in \mathrm{AM}(\operatorname{Aut}(X)) \cap H^1(W(X);\check{T}(X))$ and this intersection is trivial by hypothesis. Thus $\mathrm{AM}(f_2^{-1}f_1) = 1$ and $f_2 = f_1$ as AM is injective. If X does not have $\pi_*(N)$-determined automorphisms, then $\mathrm{AM}(f)$ lies in $H^1(W(X);\check{T}(X)) \subset W(X_0)\backslash\operatorname{Aut}(N)$ for some nontrivial $f \in \operatorname{Aut}(X)$ so that $(f,\mathrm{AM}(f))$ and $(1,0)$ are two solutions to diagram (2.12) with $X' = X$ and $j' = j$. □

2.14. Example. For (the 2-compact group associated to) a connected Lie group G, the cohomology group $H^1(W(G);\check{T}(G))$ is always an elementary abelian 2-group [35, 1.1] (2.21). For instance, this first cohomology group has order two for $G = \mathrm{PGL}(4,\mathbf{C})$ [34, Appendix B]. Let α be an isomorphism of $N(\mathrm{PGL}(4,\mathbf{C}))$ representing the nontrivial element of $H^1(W;\check{T})$. The unique solution (2.13.(2)) to diagram (2.12) is

$$\begin{array}{ccc} N(\mathrm{PGL}(4,\mathbf{C})) & \xrightarrow{\alpha} & N(\mathrm{PGL}(4,\mathbf{C})) \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j'} \\ \mathrm{PGL}(4,\mathbf{C}) & = & \mathrm{PGL}(4,\mathbf{C}) \end{array}$$

when we use the morphisms j, induced by an inclusion of Lie groups, and $j' = j\alpha$ for maximal torus normalizers. This example demonstrates that, in contrast with the p odd case [44, 7.1] [47, 2], diagram (2.12) can not always be solved with α the identity.

underT

2.15. Lemma. *Let X be a connected 2-compact group with maximal torus normalizer $j\colon N \to X$ and maximal torus $T \hookrightarrow N \xrightarrow{j} X$.*

underT1

(1) *X is N-determined if and only if for any other connected 2-compact group X' with maximal torus normalizer $j'\colon N \to X'$ there exists a morphism $f\colon X \to X'$ such that*

dia:underT

$$\begin{array}{ccc} & BT & \\ {\scriptstyle Bj|BT}\swarrow & & \searrow{\scriptstyle Bj'|BT} \\ BX & \xrightarrow[Bf]{} & BX' \end{array} \tag{2.16}$$

commutes up to conjugacy.

underT2

(2) *X is uniquely N-determined if and only if for any other connected 2-compact group X' with maximal torus normalizer $Bj'\colon BN \to BX'$ there exists a unique morphism $Bf\colon BX \to BX'$ such that (2.16) commutes up to homotopy.*

PROOF. (1) Suppose that the connected 2-compact group X is N-determined and let X' be another connected 2-compact group with the same maximal torus normalizer. Then X and X' have the same maximal torus normalizer pair, (N,N), and therefore diagram (2.12) admits a solution (f,α) such that $\pi_*(B\alpha)$ is the identity. In particular, $\pi_2(B\alpha)$ is the identity of $\pi_2(BT)$ which means that $B\alpha$ restricts to the identity on the identity component BT of BN.

Conversely, under the existence assumption of point (1), we shall show that X is N-determined. Let X' be another 2-compact group with the same maximal torus normalizer pair as X. Since the maximal torus normalizer pair informs about component groups (see the remarks just below 2.2), X' is connected. By assumption, there exists a morphism, in fact [**19**, 5.6] [**45**, 3.11] an isomorphism, $Bf\colon BX \to BX'$ under BT. Let $B\alpha\colon BN \to BN$, $B\alpha \in \mathrm{Out}(N) = W\backslash \mathrm{Aut}(N)$, be the restriction of Bf to BN [**47**, §3] so that

$$\begin{array}{ccc} BN & \xrightarrow[\cong]{B\alpha} & BN \\ {\scriptstyle Bj}\downarrow & & \downarrow{\scriptstyle Bj'} \\ BX & \xrightarrow[Bf]{\cong} & BX' \end{array}$$

commutes up to based homotopy as in the definition of the Adams–Mahmud homomorphism (§2.4). The further restriction of $B\alpha$ to the maximal torus BT agrees with the restriction of Bf to BT, the identity of BT, up to the action of a Weyl group element $w \in W$ because $W\backslash[BT,BT] = [BT,BX']$ [**43**, 3.4] [**19**, 3.4]. Since $\pi_1(BN) = W$ is faithfully represented in $\pi_2(BT)$ for the connected 2-compact group X' [**17**, 9.7], it follows that $\pi_1(B\alpha)$ is conjugation by w. Thus $B\alpha$ belongs (2.8) to the subgroup $H^1(W;\check{T})$ of $\mathrm{Out}(N)$ so that (f,α) is a legitimate solution to diagram (2.12).

(2) Suppose that X is uniquely N-determined and let X' be another connected 2-compact group with the same maximal torus normalizer as X. From point (1) we already know that there exists at least one isomorphism $f\colon X \to X'$ under T. Suppose $f_1, f_2\colon X \to X'$ are two such isomorphisms under T. Then $f_2^{-1}f_1$ is an automorphism of X under T, i.e. $\pi_*(\mathrm{BAM}(f_2^{-1}f_1)) \in W\backslash \mathrm{Aut}(W,T)$ is the identity. As $\pi_* \circ \mathrm{AM}$ is injective, $f_2^{-1}f_1$ is the identity of X, so $f_1 = f_2$.

Conversely, under the existence and uniqueness assumption of point (2), we shall show that X is uniquely N-determined. By point (1), X is N-determined, so we only need to show that $\pi_* \circ \mathrm{AM}$ is injective. Let $f\colon X \to X$ be an automorphism of X such that $\pi_*(\mathrm{BAM}(f)) \in W\backslash \mathrm{Aut}(W,T)$ is the identity. Since $\mathrm{BAM}(f)$ is determined only up to conjugacy, we may assume that $\pi_*(\mathrm{BAM}(f))$ *is* the identity of $\pi_*(BN)$. In particular, $\pi_2(\mathrm{BAM}(f))$ is the identity of $\pi_2(\mathrm{B}T)$ meaning that f is an automorphism under T. The identity of X is also an automorphism under T, so f is the identity automorphism of X by the uniqueness hypothesis. This shows that $\pi_* \circ \mathrm{AM}$ is injective. □

lemma:autX

2.17. LEMMA. *Let X be a connected 2-compact group with maximal torus normalizer $N \to X$.*

(1) $\mathrm{Out}(N) = H^1(W(X);\check{T}(X)) \cdot \mathrm{AM}(\mathrm{Aut}(X))$ *if X is N-determined.*
(2) $\mathrm{Out}(N) \cong H^1(W(X);\check{T}(X)) \rtimes \mathrm{Aut}(X)$ *and* $\mathrm{Aut}(X) \cong W(X)\backslash \mathrm{Aut}(W(X),\check{T}(X),e(X))$ *if X is uniquely N-determined. The group* $\mathrm{Aut}(W(X),\check{T}(X),e(X))$ *is a subgroup of* $N_{\mathrm{GL}(L(X))}(W(X))$ *(2.9) and isomorphic to this group if* $e(X) = 0$.

PROOF. (1) For any $\beta \in \mathrm{Out}(N)$ there exist an automorphism $\alpha \in H^1(W(X);\check{T}(X)) \subset \mathrm{Out}(N)$ and an automorphism $f \in \mathrm{Aut}(X)$ such that the diagram

$$\begin{array}{ccc} BN & \xrightarrow{B\alpha} & BN \\ {\scriptstyle Bj}\downarrow & & \downarrow{\scriptstyle Bj\circ B\beta} \\ BX & \xrightarrow[Bf]{} & BX \end{array}$$

commutes up to homotopy (2.11.(2)). Thus $\mathrm{AM}(f) = \beta\alpha$ in $\mathrm{Out}(N)$ (§2.4).

(2) If the connected 2-compact group X is uniquely N-determined, then there is commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(W(X);\check{T}(X)) & \longrightarrow & \mathrm{Out}(N) & \xrightarrow{\pi_*} & W(X)\backslash\,\mathrm{Aut}(W(X),\check{T}(X),e(X)) & \longrightarrow & 1 \\ & & & & \uparrow{\scriptstyle \mathrm{AM}} & \nearrow & & & \\ & & & & \mathrm{Aut}(X) & & & & \end{array}$$

where the top row is the short exact sequence (2.8). The composite homomorphism $\pi_* \circ \mathrm{AM}$ is injectice by assumption (2.11.(1)). It is surjective since $\mathrm{Out}(N)$ is generated by $H^1(W(X);\check{T}(X))$ by item (1) of this lemma. Thus $\pi_* \circ \mathrm{AM}$ is an isomorphism and AM is a splitting of the short exact sequence (2.8). □

As evidence of the conjecture that all connected 2-compact groups are uniquely N-determined we note that all compact connected Lie groups have $\pi_*(N)$-determined automorphisms [**31**, 2.5] and satisfy the above two formulas for automorphism groups [**25**, 3.10].

With a view to the situation for possibly nonconnected 2-compact groups, let $\mathrm{Aut}(N,N_0)$ denote the subgroup of $\mathrm{Aut}(N)$ consissting of all automorphism $\phi \in \mathrm{Aut}(N)$ such that $\pi_0(\phi)$ takes $\pi_0(N_0)$ to itself inducing an isomorphism

$$\begin{array}{ccccc} \pi_0(N_0) & \longrightarrow & \pi_0(N) & \longrightarrow & \pi \\ \downarrow\cong & & \cong\downarrow{\scriptstyle \pi_0(\phi)} & & \downarrow\cong \\ \pi_0(N_0) & \longrightarrow & \pi_0(N) & \longrightarrow & \pi \end{array}$$

of short exact sequences. Since $H^1(W;\check{T})$ is contained in $\mathrm{Aut}(N,N_0)$, there are short exact sequences similar to (2.6) and (2.7) except that $\mathrm{Aut}(W,\check{T},e)$ has been replaced by its subgroup $\mathrm{Aut}(\check{T},W,W_0,e)$ of all $(\alpha,\theta) \in \mathrm{Aut}(W,\check{T},e)$ for which $\alpha(W_0) = W_0$. (If $N = N_0$, then $\mathrm{Aut}(N) = \mathrm{Aut}(N,N_0)$.) Observe that the Adams–Mahmud homomorphism for a nonconnected 2-compact group actually takes values in the subgroup $W(X_0)\backslash\,\mathrm{Aut}(N,N_0)$ of $W(X_0)\backslash\,\mathrm{Aut}(N)$.

lemma:autXnoncon

2.18. LEMMA. *Let X be a 2-compact group with maximal torus normalizer pair $(N,N_0) \to (X,X_0)$.*

(1) $W(X_0)\backslash\,\mathrm{Aut}(N,N_0) = H^1(W(X);\check{T}(X)) \cdot \mathrm{AM}(\mathrm{Aut}(X))$ *if X is N-determined.*
(2) $W(X_0)\backslash\,\mathrm{Aut}(N,N_0) \cong H^1(W(X);\check{T}(X)) \rtimes_{H^1(\pi_0(X);\check{Z}(X_0))} \mathrm{Aut}(X)$ *if X is totally N-determined.*

PROOF. The first item is proved like the first item in 2.17. The claim of the second item is that

$$\begin{array}{ccc} H^1(\pi_0(X);\check{Z}(X_0)) & \hookrightarrow & \mathrm{Aut}(X) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{AM}} \\ H^1(W(X);\check{T}(X)) & \hookrightarrow & W(X_0)\backslash\,\mathrm{Aut}(N,N_0) \end{array}$$

is a push-out diagram. This is proved in 2.37 (allowing ourselves to refer ahead!). □

unbasedOut

2.19. REMARK. When the 2-compact group X has N-determined automorphisms, also the unbased Adams–Mahmud homomorphism

$$\mathrm{Out}(X) = \pi_0(X)\backslash\,\mathrm{Aut}(X) \to \mathrm{Out}(N) = \pi_0(N)\backslash\,\mathrm{Aut}(N) = \pi_0(X)\backslash W(X_0)\backslash\,\mathrm{Aut}(N)$$

is injective.

subsec:regular

2.20. Regular 2-compact groups. For a *connected* 2-compact group X with maximal torus $T \to X$ and Weyl group W, let

eq:theta

$$\theta = \theta(X)\colon\ \mathrm{Hom}(W,\check{T}^W) = H^1(W;\check{T}^W) \to H^1(W;\check{T}) \qquad (2.20)$$

be the homomorphism induced by the inclusion $\check{T}^W \hookrightarrow \check{T}$. Following [**24**, 5.3] we say that X is *regular* if (2.20) is surjective. See [**35**] for a thorough investigation of θ.

kerneltheta 2.21. LEMMA. [35] *Let X be the connected 2-compact group associated to a connected Lie group. Assume that X contains no direct factors isomorphic to an odd orthogonal group* $\mathrm{SO}(2n+1)$, $n \geq 1$. *Consider the homomorphism $\theta = \theta(X)$ (2.20) associated to X.*

(1) $\mathrm{Hom}(W, \check{T}^W)$ *and* $H^1(W; \check{T})$ *are $\mathbf{F}_2$-vector spaces, and the kernel of θ, consisting of those homomorphisms $W \to \check{T}^W$ that are principal crossed homomorphisms $W \to \check{T}$, is an $\mathbf{F}_2$-vector space of dimension equal to the number of direct factors of PX isomorphic to*
kerneltheta.1 $\mathrm{SO}(2n+1)$, $n \geq 1$.

(2) *Suppose that the projective group PX contains no direct factors isomorphic to an odd orthogonal group* $\mathrm{SO}(2n+1)$, $n \geq 1$, $\mathrm{PSU}(4)$, $\mathrm{PSp}(3)$, $\mathrm{PSp}(4)$, *or* $\mathrm{PS0}(8)$. *Then X is*
kerneltheta.2 *regular.*

PROOF. (1) $\mathrm{Hom}(W, \check{T})$ and its subgroup $\mathrm{Hom}(W, \check{T}^W)$ are elementary abelian 2-groups since the abelianization W_{ab} of W is an elementary abelian 2-group of finite rank. The cohomology group $H^1(W; \check{T})$ is isomorphic to $H^2(W; L \otimes \mathbf{Z}_2)$ where L is the fundamental group of the Lie group maximal torus of the Lie group underlying the 2-compact group X. Homological algebra shows that $H^2(W; L \otimes \mathbf{Z}_2) \cong H^2(W; L) \otimes \mathbf{Z}_2$ where $H^2(W; L)$ is an elementary abelian 2-group by [35, 1.1]. The injection $\check{T}^W \to \check{T}$ of W-modules gives a coefficient group long exact sequence

$$0 \to \left(\check{T}/\check{T}^W\right)^W \to \mathrm{Hom}(W, \check{T}^W) \xrightarrow{\theta} H^1(W; \check{T}) \to H^1(W; \check{T}/\check{T}^W) \to H^2(W; \check{T}^W) \to \cdots$$

in cohomology. Thus the kernel of θ is isomorphic to $\left(\check{T}/\check{T}^W\right)^W$ in general. If X is without direct factors isomorphic to $\mathrm{SO}(2n+1)$, then $\check{T}^W$ is the center of X, $\check{T}/\check{T}^W$ is the maximal torus of the adjoint 2-compact group PX, and $\left(\check{T}/\check{T}^W\right)^W$ is isomorphic to $(\mathbf{Z}/2)^s$ where s is the number of direct factors isomorphic to an odd special orthogonal group in the adjoint 2-compact group PX [35, 1.6] [37, 4.6, 4.7]. (See 2.25 for the general case.)
(2) The discrete maximal torus of $PX = X/Z(X)$ is $\check{T}(PX) = \check{T}/\check{T}^W$ for $\check{Z}(X) = \check{T}^W$ as X contains no direct factors isomorphic to an odd orthogonal group. The projective group $PX = \prod G_i$ splits as a product of simple and centerfree compact Lie groups G_i all of which satisfy $\check{T}^W(G_i) = 0$ since they are not odd orthogonal groups. Therefore $H^1(W; \check{T}/\check{T}^W) = H^1(\prod W(G_i); \prod \check{T}(G_i)) = \prod H^1(W(G_i); \check{T}(G_i))$ and these cohomology groups are trivial except in the excluded cases [24]. By the above exact sequence, θ is surjective. □

For a compact connected Lie group X, let $s(X)$ denote the number of direct factors of X isomorphic to $\mathrm{SO}(2n+1)$ with $n \geq 1$. (Keep the low degree identifications (9.25) in mind.)

lemma:kerntheta 2.22. LEMMA. *Let X be a compact connected Lie group and PX its adjoint form. The kernel of* $\theta(X)\colon H^1(W; \check{Z})(X) \to H^1(W; \check{T})(X)$ *is an $\mathbf{F}_2$-vector space of dimension* $s(PX) - s(X)$.

PROOF. In the exact sequence

$$0 \to \check{Z} \to \check{T}^W \to \left(\check{T}/\check{Z}\right)^W \to H^1(W; \check{Z}) \to H^1(W; \check{T})$$

induced from the inclusion $\check{Z} \to \check{T}$ of W-modules, the fixed point groups $\check{T}^W = \check{Z}(X) \times 2^{s(X)}$ and $\left(\check{T}/\check{Z}\right)^W = \check{Z}(X/Z) \times 2^{s(X/Z)} = 2^{s(X/Z)}$ [35, 1.6]. □

regquot 2.23. LEMMA. *Let X be a connected 2-compact group with maximal torus $T \to X$ and Weyl group W, and let $Z \to T \to X$ be a central monomorphism. If X is regular and $H^2(W; \check{Z}) \to H^2(W; \check{T})$ is injective, then the quotient 2-compact group X/Z is regular.*

PROOF. Since the hypothesis implies that $H^1(W; \check{T}) \to H^1(W; \check{T}/\check{Z})$ is surjective, the claim follows from the commutative square

$$\begin{array}{ccc} \mathrm{Hom}(W, \check{T}^W) & \longrightarrow & \mathrm{Hom}(W, (\check{T}/\check{Z})^W) \\ {\scriptstyle\theta(X)}\downarrow & & \downarrow{\scriptstyle\theta(X/Z)} \\ H^1(W; \check{T}) & \twoheadrightarrow & H^1(W; \check{T}/\check{Z}) \end{array}$$

induced by the projection $\check{T} \to \check{T}/\check{Z}$ of W-modules [37, 4.6]. □

exmp:glmCreg

2.24. EXAMPLE. (1) $\mathrm{GL}(m,\mathbf{C})$ is regular for all $m \geq 1$. For $m = 1$, this is obvious. For $m > 2$, the restriction homomorphism ($\check{S} = \mathbf{Z}/2^\infty$)

$$\mathrm{Hom}(\Sigma_m, \check{S}) = H^1(\Sigma_m; \check{S}) \xrightarrow{\mathrm{res}=\theta(\mathrm{GL}(m,\mathbf{C}))} H^1(\Sigma_m; \check{S}^m) \stackrel{\text{Shapiro}}{=} H^1(\Sigma_{m-1}; \check{S}) = \mathrm{Hom}(\Sigma_{m-1}, \check{S})$$

is bijective and for $m = 2$ it is surjective. It now follows [24, 5.7] that all products $\prod \mathrm{GL}(m_j, \mathbf{C})$ are regular.
(2) $\mathrm{PGL}(m,\mathbf{C})$, $2 \leq m$, is regular for $m \neq 4$ since (2.23)

$$\mathrm{Hom}(H_2(\Sigma_m), \check{S}) = H^2(\Sigma_m; \check{S}) \xrightarrow{\mathrm{res}} H^2(\Sigma_m; \check{S}^m) \stackrel{\text{Shapiro}}{=} H^2(\Sigma_{m-1}; \check{S}) = \mathrm{Hom}(H_2(\Sigma_{m-1}), \check{S})$$

is then an isomorphism. The 2-compact group $\mathrm{PGL}(4,\mathbf{C})$ is not regular as $H^1(W; \check{T}) = \mathbf{Z}/2$ is nontrivial while the discrete center $\check{T}^W$ is trivial.

rmk:T/TW

2.25. REMARK. If $X = \mathrm{SO}(2n+1)$, $n \geq 1$, then $\check{T}^W = \mathbf{Z}/2$, W_{ab} is $\mathbf{Z}/2$ for $n = 1$ and $(\mathbf{Z}/2)^2$ for $n \geq 2$, $\theta\colon \mathrm{Hom}(W, \check{T}^W) \to H^1(W; \check{T})$ is surjective [24, 5.5], and $H^1(W; \check{T})$ is trivial for $n = 1$, $\mathbf{Z}/2$ for $n = 2$, and $(\mathbf{Z}/2)^2$ for $n > 2$ [24, Main Theorem, 5.5]. Thus the kernel of θ is

$$\left(\check{T}/\check{T}^W\right)^W = \begin{cases} \mathbf{Z}/2 & n = 1,2 \\ 0 & n > 2 \end{cases}$$

In general, write the connected Lie group $X = X_1 \times X_2$ where X_1 is the product of all direct factors of X isomorphic to $\mathrm{SO}(2n+1)$ for some $n \geq 1$ and X_2 is without such direct factors. Then

$$\left(\check{T}/\check{T}^W\right)^W = \left(\check{T}_1/\check{T}_1^{W_1}\right)^{W_1} \times \left(\check{T}_2/\check{T}_2^{W_2}\right)^{W_2} = (\mathbf{Z}/2)^{s_{\leq 2}(X)} \times (\mathbf{Z}/2)^{s(PX_2)}$$

where $s_{\leq 2}(X)$ is the number of direct factors of X isomorphic to $\mathrm{SO}(3)$ or $\mathrm{SO}(5)$ and $s(PX_2)$ is the number of direct factors of PX_2 isomorphic to $\mathrm{SO}(2n+1)$ for some $n \geq 1$.

subsec:LHS

2.26. LHS 2-compact groups. Let $N_0 \to N$ be maximal rank normal monomorphism between two extended 2-compact tori, i.e. a commutative diagram with rows and columns that are short exact sequences of loop spaces [17, 3.2]

$$\begin{array}{ccccc} T & = & T & \longrightarrow & \{1\} \\ \downarrow & & \downarrow & & \downarrow \\ N_0 & \longrightarrow & N & \longrightarrow & W/W_0 \\ \downarrow & & \downarrow & & \| \\ W_0 & \longrightarrow & W & \longrightarrow & W/W_0 \end{array}$$

where T is a 2-compact torus and $W_0 = \pi_0(N_0)$ a normal subgroup of the finite group $W = \pi_0(N)$. The 5-term exact sequence

$$0 \to H^1(W/W_0; \check{T}^{W_0}) \xrightarrow{\mathrm{inf}} H^1(W; \check{T}) \xrightarrow{\mathrm{res}} H^1(W_0; \check{T})^{W/W_0} \xrightarrow{d_2} H^2(W/W_0; \check{T}^{W_0}) \xrightarrow{\mathrm{inf}} H^2(W; \check{T})$$

is part of the Lyndon–Hochschild–Serre spectral sequence [27] converging to $H^*(W; \check{T})$.

LHSinit

2.27. DEFINITION. *A 2-compact group with maximal torus normalizer pair (N, N_0) is LHS if the restriction homomorphism* $\mathrm{res}\colon H^1(W; \check{T}) \to H^1(W_0; \check{T})^{W/W_0}$ *is surjective.*

Thus X is LHS if and only if the initial segment of the Lyndon–Hochschild–Serre spectral sequence

$$0 \to H^1(W/W_0; \check{T}^{W_0}) \xrightarrow{\mathrm{inf}} H^1(W; \check{T}) \xrightarrow{\mathrm{res}} H^1(W_0; \check{T})^{W/W_0} \to 0$$

is exact. If $\check{T}^{W_0} = 0$ or $W = W_0 \times W/W_0$ is a direct product, then X is LHS. Note that the Weyl group of a compact Lie group G is always the semi-direct product $W(G) = W(G_0) \rtimes \pi_0(G)$ for the action of the component group $\pi_0(G)$ on the Weyl group $W(G_0)$ of the identity component [25, §2.5]. (In fact, It is not so easy to find a nonconnected compact Lie group G for which the extension $G_0 \to G \to G/G_0 = \pi$ is nonsplit [26].)

lhscrit1 2.28. LEMMA. *Let $W = W(X)$ be the Weyl group of the 2-compact group X, $W_0 = W(X_0)$ the Weyl group of the identity component, and $\pi = W/W_0$ the component group* [**37**, 3.8] *of X. If $W = W_0 \rtimes \pi$ is a semi-direct product and*

$$\theta(X_0)^\pi \colon \operatorname{Hom}(W_0, \check{T}^{W_0})^\pi \to H^1(W_0; \check{T})^\pi$$

is surjective, then X is LHS.

PROOF. Assume that the group $G = H \rtimes Q$ is the semi-direct product for a group action $Q \to \operatorname{Aut}(H)$, and let A be a G-module. We show that the image of the restriction homomorphism res: $H^1(G; A) \to H^1(H; A)^Q$ contains the image of $\theta^Q\colon \operatorname{Hom}(H, A^H)^Q \to H^1(H; A)^Q$. Let $\phi \in \operatorname{Hom}(H, A^H)^Q$ be a Q-equivariant homomorphism of H into the fixed point module A^H. Then $\theta(\phi) \in H^1(H; A)^Q$ is the cohomology class represented by the crossed homomorphism $\phi\colon H \to A^H \subset A$. If we define $\overline{\phi}\colon H \rtimes Q \to A$ by $\overline{\phi}(nq) = \phi(n)$, $n \in H$, $q \in Q$, then

$$\begin{aligned}\overline{\phi}(n_1q_1n_2q_2) = \overline{\phi}(n_1(q_1n_2q_1^{-1})q_1q_2) = \overline{\phi}(n_1(q_1\cdot n_2)q_1q_2) \overset{\text{def}}{=} \phi(n_1(q_1\cdot n_2)) = \phi(n_1) + \phi(q_1\cdot n_2)\\ = \phi(n_1) + q_1\phi(n_2)\end{aligned}$$

and also

$$\overline{\phi}(n_1q_1) + n_1q_1\overline{\phi}(n_2q_2) \overset{\text{def}}{=} \phi(n_1) + n_1q_1\phi(n_2) = \phi(n_1) + q_1\phi(n_2)$$

as $q_1\phi(n_2) \in A^H$. This shows that the crossed homomorphism ϕ defined on H extends to a crossed homomorphism $\overline{\phi}$ defined on $G = H \rtimes Q$. (I do not know if the LHS spectral sequence differential $d_2\colon H^1(H; A)^Q \to H^2(Q; A^H)$ is always trivial for a semi-direct product $H \rtimes Q$ of finite groups.) □

The next example demonstrates that condition 2.28 is not necessary.

exmp:sl2C 2.29. EXAMPLE. (1) $X = \mathrm{PGL}(6, \mathbf{R}) = \mathrm{PSL}(6, \mathbf{R}) \rtimes C_2$ does not satisfy the condition of 2.28 for $H^1(W_0; \check{T}) = \mathbf{Z}/2$ [**24**, Main Theorem] while $\check{T}^{W_0} = \check{Z}(X_0) = 0$. Nevertheless, X is LHS because also $H^1(W; \check{T}) = \mathbf{Z}/2$ (computer computation).
(2) $X = \mathrm{PGL}(8, \mathbf{R}) = \mathrm{PSL}(8, \mathbf{R}) \rtimes C_2$ does not satisfy the condition of 2.28 for $H^1(W_0; \check{T}) = \mathbf{Z}/2 \oplus \mathbf{Z}/2$ [**24**, Main Theorem] while $\check{T}^{W_0} = \check{Z}(X_0) = 0$. Nevertheless, X is LHS because $H^1(W; \check{T}) = \mathbf{Z}/2$ and the outer automorphism group C_2 acts nontrivially on $H^1(W_0; \check{T})$ (computer computation).
(3) When $X_0 = \mathrm{SL}(2, \mathbf{C})$, the Weyl group $W_0 = \Sigma_2$ has order two, the center $\check{Z} = \check{T}^{W_0}$ also has order two, and $H^1(W_0; \check{T}) = 0$ is trivial, so the homomorphism $\theta(X_0)$ is trivial as well, of course. Indeed, the nontrivial homomorphism $W_0 \to \check{Z} \subset \check{T}$ is the principal crossed homomorphism corresponding to the element $\mathrm{diag}(i, -i)$ of the maximal torus. More generally, the direct product $X_0^r = \mathrm{SL}(2, \mathbf{C})^r$ is regular [**24**, 5.7], has Weyl group W_0^r, center $\check{Z}^r$, and 2.21.(1) identifies the kernel of $\theta(X_0)$ enabling us to conclude that

H1W0 (2.30)
$$H^1(W; \check{T})(X_0^r) = \frac{\operatorname{Hom}(W_0^r, \check{Z}^r)}{\operatorname{Hom}(W_0, \check{Z})^r}$$

is an $\mathbf{F}_2$-vector space of dimension $r^2 - r$ as in [**24**, 5.8]. Let $X = X_0 \rtimes C_2$ be the semi-direct product for the nontrivial outer automorphism of X_0. The component group C_2^r of X^r acts trivially on (2.30) and as $H^1(W; \check{T})(X^r)$ has dimension $2r^2 - r$ (by induction) and $H^1(C_2^r; \check{Z}^r)$ dimension r^2, the direct product X^r is LHS for all $r \geq 1$.
(4) When $X = \mathrm{SL}(4, \mathbf{R})$, the Weyl group $W = \langle \sigma, c_1c_2 \rangle = \mathbf{Z}/2 \times \mathbf{Z}/2$ is elementary abelian generated by $\sigma = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$ and $c_1c_2 = \mathrm{diag}(-1, 1, -1, 1)$. The center $\check{Z} = \check{T}^W = \langle \mathrm{diag}(-1, -1, -1, -1) \rangle = \mathbf{Z}/2$ has order 2, and the first cohomology group $H^1(W; \check{T}) = 0$ is trivial, so the homomorphism $\theta\colon \operatorname{Hom}(W; \check{T}^W) \to H^1(W; \check{T})$ is also trivial, of course. Indeed, the principal homomorphism $\varphi(w) = (w \cdot t) \cdot t^{-1}\colon W \to \check{T}$, is the first coordinate function $W \to \check{Z}(X)$ when $t = \mathrm{diag}(-E, E)$ and the second coordinate function when $t = \mathrm{diag}(I, I)$. The outer automorphism, conjugation with $D = \mathrm{diag}(-1, 1, 1, 1) \in \mathrm{GL}(4, \mathbf{R})$, sends σ to $\sigma^D = \sigma(c_1c_2)$ and c_1c_2 to itself.

More generally, when X^r is a product of r copies of $\mathrm{SL}(4, \mathbf{R})$, the Weyl group W^r is a product of r copies $W = W(\mathrm{SL}(4, \mathbf{R})) = \mathbf{Z}/2 \times \mathbf{Z}/2$, the center $\check{Z}(X) = \check{Z}^r$ is a product of r copies of

$\check{Z} = \check{Z}(\mathrm{SL}(4,\mathbf{R})) = \mathbf{Z}/2$ and as $\theta\colon \mathrm{Hom}(W^r, \check{Z}^r) \to H^1(W;\check{T})(X^r)$ is surjective [**24**, 5.5, 5.7], the first cohomology group

$$H^1(W;\check{T})(X^r) = \frac{\mathrm{Hom}(W^r,\check{Z}^r)}{\mathrm{Hom}(W,\check{Z})^r}$$

has dimension $2r^2 - 2r$ over $\mathbf{F}_2$ (2.21). The component group $\pi_0(\mathrm{GL}(4,\mathbf{R})^r) = C_2^r$ acts on this $\mathbf{F}_2$-vector space such that the space of fixed vectors has dimension $r^2 - r$. By induction we see that $H^1(W;\check{T})(\mathrm{GL}(4,\mathbf{R})^r)$ is an $\mathbf{F}_2$-vector space of dimension $2r^2 - r$ and clearly $H^1(C_2^r;\check{Z}^r)$ has dimension r^2. Thus $\mathrm{GL}(4,\mathbf{R})^r$ is LHS for all $r \geq 1$.
(5) The homomorphisms θ is surjective for $\mathrm{SL}(2n,\mathbf{R})$ for all $n \geq 1$ [**24**, Main Theorem, 5.4] and $H^1(W;\check{T})(\mathrm{SL}(2n,\mathbf{R})) = 0$ for $n = 1,2$ and $H^1(W;\check{T})(\mathrm{SL}(2n,\mathbf{R})) = \mathbf{Z}/2$ for $n \geq 3$. Hence $\mathrm{GL}(2n,\mathbf{R})$ is LHS for all $n \geq 1$ by 2.28.

I do not know any examples of 2-compact groups that are not LHS.

The coefficient group short exact sequence $0 \to L \to L \otimes \mathbf{Q} \to \check{T} \to 0$ gives the exact sequence

$$0 \to H^0(W;L) \to H^0(W;L\otimes\mathbf{Q}) \to H^0(W;\check{T}) \to H^1(W;L) \to 0$$

form which we see that

eq:hiwt

$$H^i(W;\check{T}) = \begin{cases} H^0(W;L\otimes\mathbf{Q})/H^0(W;L) \oplus H^1(W;L) & i = 0 \\ H^{i+1}(W;L) & i > 0 \end{cases} \tag{2.31}$$

sec:centermtn

2.32. The center of the maximal torus normalizer. We need criteria to ensure that the center of the 2-compact group X agrees with the center of its maximal torus normalizer. (This is automatic when $p > 2$ [**44**, 3.4] but not when $p = 2$ [**18**, §7].)

prop:centerofN

2.33. PROPOSITION. *Let X be a 2-compact group with identity component X_0. If $Z(X_0) = Z(N(X_0))$ and X_0 has N-determined automorphisms, then $Z(X) = Z(N(X))$.*

PROOF. This is proved in [**47**, 4.12] for p-compact groups where p is odd. If we replace the assumption that p is odd by the assumption that $Z(X_0) = Z(N(X_0))$ (which always holds when $p > 2$ [**18**, 7.1]), then the same proof works also for 2-compact groups. □

Assume now that X is a *connected* 2-compact group. Then $\check{Z}(N(X)) = \check{T}(X)^{W(X)}$ and there is an injection $\check{Z}(X) \hookrightarrow \check{Z}(N(X))$ which is not necessarily an isomorphism [**18**, §7].

Inspection shows that $Z(G) = ZN(G)$ for any *simply connected* compact Lie group G; see [**13**, 1.4] for a conceptual proof of this fact. In fact, $Z(G) = ZN(G)$ for any connected compact Lie group G containing no direct factors isomorphic to $\mathrm{SO}(2n+1)$ [**35**, 1.6].

Let $Z \to N(X)$ be a central monomorphism such that also the composition $Z \to N(X) \to X$ is central. Under these assumptions, the quotient loop spaces $N(X)/Z$ and X/Z can be defined [**18**, 2.8]. The action map [**17**, 8.6] $BZ \times BN(X) \to BN(X)$ induces an action $[BN(X), BZ] \times \mathrm{Out}(N(X)) \to \mathrm{Out}(N(X))$ of the group $[BN(X), BZ] \cong H^1(\check{N}(X);\check{Z})$ on the set $\mathrm{Out}(N(X))$. Let $[BN(X), BZ]_{(1)}$ denote the isotropy subgroup at $(1) \in \mathrm{Out}(N(X))$.

2.34. LEMMA. *If $Z(X) = Z(N(X))$ and $[BN(X), BZ]_{(1)} = 0$, then $Z(X/Z) = ZN(X/Z)$.*

PROOF. Using [**37**, 4.6.4], the assumption of the lemma, and [**47**, 5.11], we get $Z(X/Z) = Z(X)/Z = Z(N(X))/Z = Z(N(X)/Z) = ZN(X/Z)$. □

sec:red

2. Reduction to the connected, centerless (simple) case

In this section we reduce the general classification problem first to the connected case and next to the connected and centerless case. We first show (2.35, 2.40) that if X is any nonconnected 2-compact group with identity component X_0 then

$$\left.\begin{array}{l} X_0 \text{ is uniquely } N\text{-determined} \\ X \text{ is LHS} \\ H^i(W/W_0;\check{Z}(X_0)) \to H^i(W/W_0;\check{T}^{W_0}) \text{ is injective for } i=1,2 \end{array}\right\} \Longrightarrow X \text{ is totally } N\text{-determined}$$

The H^i-injectivity conditions holds when X_0 is a connected Lie group [**35**, 1.6] or equals DI(4) [**16, 52**]. To see this observe that the condition obviously holds when $\check{Z}(X_0)=\check{T}(X_0)^{W_0}$ or $\check{Z}(X_0)$ is trivial. If the conjecture [**11**, 5.1] that any connected 2-compact group splits as a product of a compact connected Lie group and a finite number of DI(4) is true, then this condition is always satisfied. Indeed, if the splitting conjecture is true then $X_0=G'\times G''\times \mathrm{DI}(4)^s$ where G' is a connected compact Lie group with no direct factors isomorphic to $\mathrm{SO}(2n+1)$, G'' is a direct product of $\mathrm{SO}(2n+1)$s, and $s\geq 0$. The $\pi_0(X)$-equivariant group homomorphism $\check{Z}(G')=\check{Z}(X_0)\to\check{T}(G')^{W(G')}\times\check{T}(G'')^{W(G'')}$ has a left inverse since it takes $\check{Z}(G')$ isomorphically to the $\pi_0(X)$-subgroup $\{1\}\times\check{T}(G'')^{W(G'')}$ of the left hand side. The induced map on cohomology therefore also has a left inverse. However, it is not at present clear if all nonconnected 2-compact groups are LHS.

Next we consider a connected 2-compact group X with adjoint form $PX=X/Z(X)$ [**18**, 2.8] and show (2.38, 2.42) that

$$PX \text{ is uniquely } N\text{-determined} \Longrightarrow X \text{ is uniquely } N\text{-determined}$$

This reduces in principle the problem to the connected and centerless case. One can go a little further since connected, centerless 2-compact groups split into products of simple factors [**19, 48**]. We show (2.39, 2.43) that

$$X_1 \text{ and } X_2 \text{ are uniquely } N\text{-determined} \Longrightarrow X_1\times X_2 \text{ is uniquely } N\text{-determined}$$

when X_1 and X_2 are connected. Therefore it suffices to show that all *connected, centerless and simple* 2-compact groups are uniquely N-determined. It is already known that all connected compact Lie groups as well as DI(4) have $\pi_*(N)$-determined automorphisms [**31, 52**].

Let X be a 2-compact group with maximal torus normalizer pair $(N,N_0)(X)=(N,N_0)$.

lemma:autononcon

2.35. Lemma. [**44**, 4.2] *Suppose that X_0 has N-determined automorphisms. Then*

$$X \text{ has } N\text{-determined automorphisms} \iff H^1(W/W_0;\check{Z}(X_0))\to H^1(W/W_0;\check{T}^{W_0}) \text{ is injective}$$

Proof. The restriction of AM to the subgroup $H^1(W/W_0;\check{Z}(X_0))\subset \mathrm{Aut}(X)$ is the homomorphism

AMres

$$H^1(W/W_0;\check{Z}(X_0))\to H^1(W/W_0;\check{T}^{W_0})\xrightarrow{\inf} H^1(W;\check{T}) \tag{2.36}$$

where inf is the inflation monomorphism. If the first homomorphism has a nontrivial kernel, X does not have N-determined automorphisms. Conversely, assume that the first homomorphism is injective, and let $f\in\mathrm{Aut}(X)$ be an automorphism such that $\mathrm{AM}(f)\in W_0\backslash\mathrm{Aut}(N)$ is the identity. Then $\mathrm{AM}(f_0)\in W_0\backslash\mathrm{Aut}(N_0)$ and $\pi_0(f)$ equal the respective identity maps. Since X_0 has N-determined automorphisms by assumption, f_0 is the identity. Thus f belongs to the subgroup $H^1(W/W_0;\check{Z}(X_0))$ of $\mathrm{Aut}(X)$ [**42**, 5.2] where AM is injective, so f is the identity automorphism of X. (The description of the kernel in the short exact sequence of [**42**, 5.2] holds for all p-compact groups, not just those with a completely reducible identity component.) □

idcomp

2.37. Lemma. *Suppose that X has N-determined automorphisms and that X_0 has $\pi_*(N)$-determined automorphisms. Then* $\mathrm{Aut}(X)\cap H^1(W;\check{T})=H^1(W/W_0;\check{Z}(X_0))$ *so that*

$$X \text{ has } \pi_*(N)\text{-determined automorphisms} \iff H^1(W/W_0;\check{Z}(X_0))=0$$

Proof. Let $f\in\mathrm{Aut}(X)$ be an automorphism such that $\pi_*\mathrm{AM}(f)$ is the identity. Then also $\pi_*\mathrm{AM}(f_0)$ and $\pi_0(f)$ equal the respective identity maps. Since X_0 is assumed to have $\pi_*(N)$-determined automorphisms, f_0 is the identity. Thus f belongs to the subgroup $H^1(W/W_0;\check{Z}(X_0))$

of $\mathrm{Aut}(X)$ [42, 5.2]. This shows that $\mathrm{Aut}(X) \cap H^1(W;\check{T}) \subset H^1(W/W_0;\check{Z}(X_0))$. The opposite inclusion is immediate from (2.36). ☐

autondetcenter

2.38. Lemma. [44, 4.8] *Suppose that X is connected. If the adjoint form $PX = X/Z(X)$ has $\pi_*(N)$-determined automorphisms, so does X.*

Proof. If $f \in \mathrm{Aut}(X)$ is an automorphism under $T(X)$, the induced automorphism $Pf \in \mathrm{Aut}(PX)$ is an automorphism under $T(PX)$, hence equals the identity, and the induced automorphism $Z(f) \in \mathrm{Aut}(ZX)$ is also the identity since the center $ZX \to X$ factors through the maximal torus $T(X) \to X$ [18, 7.5] [37, 4.3]. But then f itself is the identity for $\mathrm{Aut}(X)$ embeds into $\mathrm{Aut}(PX) \times \mathrm{Aut}(ZX)$ [43, 4.3]. ☐

prodauto

2.39. Lemma. [47, 9.4] *If the two connected 2-compact groups X_1 and X_2 have N-determined (resp. $\pi_*(N)$-determined) automorphisms, so does the product $X_1 \times X_2$.*

Proof. Since the statement concerning N-determined automorphisms is proved in [47, 9.4] we deal here only with the case of $\pi_*(N)$-determined automorphisms. Let f be an automorphism under $T_1 \times T_2$ of the product 2-compact group $X_1 \times X_2$. Then

$$f_1\colon X_1 \to X_1 \times X_2 \xrightarrow{f} X_1 \times X_2 \to X_1$$
$$f_2\colon X_2 \to X_1 \times X_2 \xrightarrow{f} X_1 \times X_2 \to X_2$$

are endomorphisms under the maximal tori and therefore conjugate to the respective identity maps. But f is [47, 9.3] in fact conjugate to the product morphism (f_1, f_2) which is the identity. ☐

redtoconnected

2.40. Lemma. *(Cf [44, 7.8]) Suppose that*

(1) *X_0 is uniquely N-determined.*

edtoconnected.2 (2) *X is LHS.*

edtoconnected.3 (3) *$H^2(W/W_0, \check{Z}(X_0)) \to H^2(W/W_0, \check{T}^{W_0})$ is injective.*

Then X is N-determined.

Proof. Let X' be another 2-compact group with maximal torus normalizer pair (N, N_0). The assumption on the identity component X_0 means (2.15) that there exists an isomorphism $f_0\colon X_0 \to X_0'$ under T. For any $\xi \in W/W_0 = N/N_0 = X/X_0 = X'/X_0'$, the isomorphism $\xi f_0 \xi^{-1}$ is also an isomorphism under T and thus $\xi f_0 = f_0 \xi$ as X_0 is uniquely N-determined. By the second assumption, the automorphism $\alpha_0 = \mathrm{AM}(f_0)\colon N_0 \to N_0$ with $\pi_*(B\alpha_0) = \mathrm{Id}$ extends to an isomorphism $\alpha\colon N \to N$ with $\pi_*(B\alpha) = \mathrm{Id}$.

The situation is now as shown in the commutative diagram

$$\begin{tikzcd} BX_0 \arrow[rrr, bend left, "Bf_0"] \arrow[d] & BN_0 \arrow[l, "Bj_0"'] \arrow[r, "B\alpha_0"'] \arrow[d] & BN_0 \arrow[r, "Bj_0'"'] \arrow[d] & BX_0' \arrow[d] \\ BX \arrow[d] & BN \arrow[l, "Bj"'] \arrow[r, "B\alpha"] \arrow[d] & BN \arrow[r, "Bj'"] \arrow[d] & BX' \arrow[d] \\ B\pi_0(X) & B(W/W_0) \arrow[l, "\cong"'] \arrow[r, equal] & B(W/W_0) \arrow[r, "\cong"] & B\pi_0(X') \end{tikzcd}$$

Our aim is to find an isomorphism $f\colon X \to X'$ to fill in the based homotopy commutative diagram

$$\begin{tikzcd} BX_0 \arrow[r, "Bf_0", "\cong"'] \arrow[d] & BX_0' \arrow[d] \\ BX \arrow[r, dotted] \arrow[d] & BX' \arrow[d] \\ B\pi_0(X) \arrow[r, "\cong"'] & B\pi_0(X') \end{tikzcd}$$

where the isomorphism between the base 2-compact groups is given by the isomorphisms $\pi_0(X) \leftarrow N/N_0 \to \pi_0(X')$. Since f_0 is $\pi_0(X)$-equivariant up to homotopy, $\mathrm{map}(BX_0, BX_0')_{Bf_0}$ is a $\pi_0(X)$-space in the sense that there exists a fibration

$$\mathrm{map}(BX_0, BX_0'; Bf_0) \to \mathrm{map}(BX_0, BX_0'; Bf_0)_{h\pi_0(X)} \to B\pi_0(X)$$

over $B\pi_0(X)$ with $\mathrm{map}(BX_0, BX_0')_{Bf_0}$, here written as $\mathrm{map}(BX_0, BX_0'; Bf_0)$, as fibre. The space of sections of this fibration, $\mathrm{map}(BX_0, BX_0')_{Bf_0}^{h\pi_0(X)}$, is a space of fibre maps of BX to BX'. This fibration sits to the left in the commutative diagram

$$\begin{array}{ccccc}
\mathrm{map}(BX_0, BX_0'; Bf_0) & \longrightarrow & \mathrm{map}(BN_0, BX_0'; B(j_0'\alpha)) & \xleftarrow{\simeq} & \mathrm{map}(BN_0, BN_0; B\alpha_0) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{map}(BX_0, BX_0'; Bf_0)_{h\pi_0(X)} & \to & \mathrm{map}(BN_0, BX_0'; B(j_0'\alpha))_{h(W/W_0)} & \leftarrow & \mathrm{map}(BN_0, BN_0; B\alpha_0)_{h(W/W_0)} \\
\downarrow & & \downarrow & & \downarrow \\
B\pi_0(X) & \longleftarrow & B(W/W_0) & = & B(W/W_0)
\end{array}$$

where the columns are fibrations and the horizontal maps are defined as composition with Bj and Bj', respectively. The fibre map from the right column to the central one is actually a fibre homotopy equivalence because the centralizer of the maximal torus in X_0' and in N_0 are isomorphic in that they are both isomorphic to the maximal torus.

The middle fibration admits a section corresponding to the fibrewise map $Bj' \circ B\alpha$. But then the left fibration also admits a section: The obstruction to a section of the left fibration is a cohomology class in $H^2(\pi_0(X); \check{Z}(X_0))$. Since the middle fibration does admit a section, this obstruction class is in the kernel of the coefficient group homomorphism $H^2(\pi_0(X); \check{Z}(X_0)) \to H^2\big(W/W_0, \check{T}^{W_0}\big)$. But the assumption is that this is an injection and therefore the obstruction must vanish. (We are here tacitly replacing the three fibrations above by their fibrewise discrete approximations [**42**, 4.3].)

A section of the left fibration corresponds to a morphism $Bf\colon BX \to BX'$ under the isomorphism $Bf_0\colon BX_0 \to BX_0'$ and over $B\pi_0(X) \xrightarrow{\cong} B\pi_0(X')$ such that $Bf \circ Bj$ and $Bj \circ B\alpha$ are homotopic over $B(N/N_0) \to B\pi_0(X')$. But since the fibre BX_0' of $BX' \to B\pi_0(X')$ is simply connected this means that $Bf \circ Bj$ and $Bj \circ B\alpha$ are based homotopic maps $BN \to BX'$. □

ex:toral

2.41. Example. 1. Any 2-compact torus T is uniquely N-determined for if $j\colon T \to X$ is the maximal torus normalizer for the connected 2-compact group X, then j is an isomorphism. Indeed, $H^*(BT; \mathbf{Q}_2) \cong H^*(BX; \mathbf{Q}_2)$ [**17**, 9.7.(3)] and the connected space X/T has cohomological dimension $\mathrm{cd}_{\mathbf{F}_2}(X/T) = 0$ [**18**, 4.5, 5.6] so is a point.
2. Any 2-compact toral group G is totally N-determined: G clearly has N-determined automorphisms as G is its own maximal torus normalizer. If the 2-compact group X has the same maximal torus normalizer pair (G, T) as G, then X is a 2-compact toral group and $j'\colon G \to X$ is an isomorphism. G is uniquely N-determined if and only if $H^1(\pi_0(G); \check{T}) = 0$. In particular, $\mathrm{GL}(2, \mathbf{R})$ is uniquely N-determined.

ndetcenter

2.42. Lemma. *(Cf [**44**, 7.10]) Let X be a connected 2-compact group and $Z \to X$ its center. If X/Z is N-determined, so is X.*

Proof. Let $j\colon N \to X$ be the maximal torus normalizer for X and $j'\colon N \to X'$ the maximal torus normalizer for some other connected 2-compact group X'. It suffices (2.15) to find a morphism $f\colon X \to X'$ under the maximal tori $X \xleftarrow{i} T \xrightarrow{i'} X'$. The 2-discrete center $\check{Z}$ of X and X' is contained in the the 2-discrete maximal torus $\check{T}$ [**18**, 7.5]. Factoring out [**17**, 8.3] these central monomorphisms we obtain the commutative diagram

$$\begin{array}{ccccc} B\check{X} & \xleftarrow{Bi} & B\check{T} & \xrightarrow{Bi'} & B\check{X}' \\ \downarrow & & \downarrow & & \downarrow \\ B(X/Z) & \xleftarrow{B(i/Z)} & B(T/Z) & \xrightarrow{B(i'/Z)} & B(X'/Z) \\ B(X/Z) & & \xrightarrow{\quad B(f/Z) \quad} & & B(X'/Z) \end{array}$$

where the vertical maps are fibrations with fibre $B\check{Z}$, the total spaces, such as $B\check{X}$, are the fibre-wise discrete approximations, and $f/Z\colon X/Z \to X'/Z$ is the isomorphism under T/Z that exists because X/Z is N-determined. Construct the fibration

$$\operatorname{map}(B\check{Z}, B\check{Z}; B1) \to B\check{Z}_{h(X/Z)} \to B(X/Z)$$

whose sections are maps $BX \to BX'$ over $B(f/Z)$ and under $B\check{Z}$. There are two other such fibrations related to this one as shown in the commutative diagram

$$\begin{array}{ccccc} \operatorname{map}(B\check{Z}, B\check{Z}; B1) & = & \operatorname{map}(B\check{Z}, B\check{Z}; B1) & = & \operatorname{map}(B\check{Z}, B\check{Z}; B1) \\ \downarrow & & \downarrow & & \downarrow \\ B\check{Z}_{h(X/Z)} & \longleftarrow & B\check{Z}_{h(T/Z)} & \xrightarrow{Bi^*} & B\check{Z}_{h(T/Z)} \\ \downarrow & & \downarrow & & \downarrow \\ B(X/Z) & \xleftarrow[B(i/Z)]{} & B(T/Z) & = & B(T/Z) \end{array}$$

where the middle fibration is the pull-back along $B(i/Z)$ of the left fibration and the fibre over $b \in B(T/Z)$ of the right fibration consists of one component of the space of maps of the fibre $B\check{T}_b$ over b into the fibre $B\check{X}'_{B(i'/Z)(b)}$ over $B(i'/Z)(b)$. The fibre equivalence Bi^* is induced by $Bi\colon B\check{T} \to B\check{X}$. The middle fibration has a section u' such that $Bi^* \circ u'$ is the section $Bi'\colon B\check{T} \to B\check{X}'$ of the right fibration. We now have fibre maps

$$\begin{array}{ccc} X/T & \xrightarrow{u|X/T} & B\check{Z} \\ \downarrow & & \downarrow \\ B\check{T} & \longrightarrow & B\check{Z}_{h(X/Z)} \\ & {\scriptstyle B(i/Z)}\searrow \quad \swarrow & \\ & B(X/Z) & \end{array}$$

where u is the composition of u' and $B\check{Z}_{h(T/Z)} \to B\check{Z}_{h(X/Z)}$. The canonical map, given by constants, $B\check{Z} \to \operatorname{map}(X/T, B\check{Z})$ is a homotopy equivalence since X/T is simply connected [37, 5.6] and hence a version [44, 6.6] of the Zabrodsky lemma implies that $u = v \circ B(i/Z)$ for some section $v\colon B(X/Z) \to B\check{Z}_{h(X/Z)}$ of the left fibration. The section v is, after fibre-wise completion, a fibre map $BX \to BX'$ under BT. □

Let $j_1\colon N_1 \to X_1$ and $j_2\colon N_2 \to X_2$ be maximal torus normalizers for the connected 2-compact groups X_1 and X_2 and suppose that X' is some connected 2-compact group that admits a maximal torus normalizer of the form $j'\colon N_1 \times N_2 \to X'$. The Splitting Theorem [19, 1.4], or more explicitly in the form of [48, 5.5], says that there exist 2-compact groups X'_1 and X'_2 and an isomorphism $X'_1 \times X'_2 \to X'$ such that

$$\begin{array}{ccc} & BN_1 \times BN_2 & \\ {\scriptstyle Bj'_1 \times Bj'_2}\swarrow & & \searrow{\scriptstyle Bj'} \\ BX'_1 \times BX'_2 & \xrightarrow{\simeq} & BX' \end{array}$$

commutes where $j_1'\colon N_1 \to X_1'$ and $j_2'\colon N_2 \to X_2'$ are maximal torus normalizers. The following lemma is an immediate consequence

ndetprod

2.43. LEMMA. *The product of two N-determined connected 2-compact groups is N-determined.*

PROOF. Since X_1, X_2 are N-determined there exist isomorphisms $f_1\colon X_1 \to X_1'$, $f_2\colon X_2 \to X_2'$ and automorphisms $\alpha_1 \in H^1(W_1; \check{T}_1) \subset \mathrm{Out}(N_1)$, $\alpha_2 \in H^1(W_2; \check{T}_2) \subset \mathrm{Out}(N_2)$ such that

$$\begin{array}{ccccc} BN_1 \times BN_2 & \xrightarrow{B\alpha_1 \times B\alpha_2} & BN_1 \times BN_2 & \overset{Bj'}{\searrow} & \\ {\scriptstyle Bj_1 \times Bj_2}\downarrow & & \downarrow{\scriptstyle Bj_1' \times Bj_2'} & & \\ BX_1 \times BX_2 & \xrightarrow[Bf_1 \times Bf_2]{} & BX_1' \times BX_2' & \xrightarrow{\simeq} & BX' \end{array}$$

commutes up to based homotopy. $\square$

:conncenterless

3. N-determined connected, centerless 2-compact groups

In this section we formulate inductive criteria that, at least in favorable cases, can be used to show total N-determinacy for connected, centerless (simple) 2-compact groups X. The key tool is the homology decomposition [18, 8.1]

hodecomp (2.44)
$$\mathrm{hocolim}_{\mathbf{A}(X)^{\mathrm{op}}} BC_X \to BX$$

of BX in terms of centralizers of elementary abelian subgroups. Since X has no center, the cohomological dimension of each centralizer $C_X(V,\nu)$ is smaller than the cohomological dimension of X. As part of an inductive argument we will therefore assume that all centralizers are totally N-determined and formulate criteria (2.48, 2.51) that imply that also X is totally N-determined.

defn:AX 2.45. Definition. [18, §8] *The objects of the Quillen category $\mathbf{A}(X)$ are conjugacy classes of monomorphisms $\nu\colon V \to X$ of nontrivial elementary abelian 2-groups into X; the morphisms $\alpha\colon (V_1,\nu_1) \to (V_2,\nu_2)$ are injective group homomorphisms $\alpha\colon V_1 \to V_2$ such that ν_1 and $\nu_2\alpha$ are conjugate monomorphisms $V_1 \to X$. We shall write $\mathbf{A}(X)(V_1,V_2)$ for the set of morphism $V_1 \to V_2$ and $\mathbf{A}(X)(V)$ for the group of all endomorphisms (which are all isomorphisms) of V.*

The functor

eq:defnBCX (2.46)
$$BC_X\colon \mathbf{A}(X)^{\mathrm{op}} \to \mathbf{Top} \qquad \text{(topological spaces)}$$

takes an object (V,ν) of the Quillen category $\mathbf{A}(X)$ to its centralizer $BC_X(V,\nu) = \mathrm{map}(BV,BX)_{B\nu}$. The covariant functor

eq:defnpiiBZC (2.47)
$$\pi_i(BZC_X)\colon \mathbf{A}(X) \to \mathbf{Ab} \qquad \text{(abelian groups)}$$

takes (V,ν) into the abelian homotopy group $\pi_i(\mathrm{map}(BC_X(V,\nu),BX),e(\nu))$ based at the evaluation map $e(\nu)\colon BC_X(V,\nu) \to BX$. The space $\mathrm{map}(BC_X(V,\nu),BX)$ is homotopy equivalent to $BZC_X(V,\nu)$ [12]).

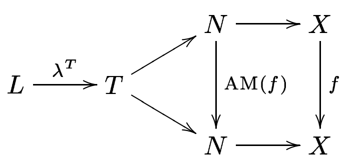
ndetauto 2.48. Lemma. [44, 4.9] *Suppose that X is connected and centerless. If*

ndetauto1 (1) *$C_X(L,\lambda)$ has N-determined (resp. $\pi_*(N)$-determined) automorphisms for each rank 1 object (L,λ) of $\mathbf{A}(X)$ and*

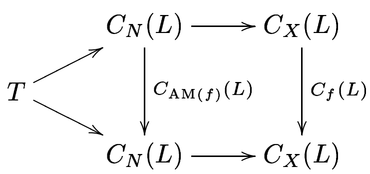
ndetauto2 (2) $\lim^1(\mathbf{A}(X);\pi_1(BZC_X)) = 0 = \lim^2(\mathbf{A}(X);\pi_2(BZC_X))$

Then X has N-determined (resp. $\pi_*(N)$-determined) automorphisms.

Proof. Suppose first that each line centralizer has $\pi_*(N)$-determined automorphisms. Let $f\colon X \to X$ be an automorphism under the maximal torus $T \to X$. Since any monomorphism $\lambda\colon L \to X$, $L = \mathbf{Z}/2$, factors through the maximal torus, the commutative diagram



shows that $f\lambda = \lambda$ and gives a commutative diagram



of automorphisms under T. Thus $\mathrm{AM}(C_f(L)) = C_{\mathrm{AM}(f)}(L)\colon C_N(L) \to C_N(L)$. Now, $\pi_*(C_N(L))$ is a subgroup of $\pi_*(N)$ (for $\pi_1(C_N(L)) = \pi_1(N)$ and $\pi_0(C_N(L)) = W(X)(L)$ is [18, 7.6] [43, 3.2.(1)] the stabilizer subgroup at $L < \check{T}$ for the action of $W(X)$ on $\check{T}$) so $\pi_*(C_{\mathrm{AM}(f)}(L)) = 1$ and $C_f(L) \simeq 1_{C_X(L)}$ since $C_X(L)$ has $\pi_*(N)$-determined automorphisms. For any other object (V,ν)

of $\mathbf{A}(X)$ of rank > 1, choose a line L in V. Since the monomorphism $\nu\colon V \to X$ canonically factors through $C_X(L)$ [17, 8.2] [47, 3.18], the commutative diagram

$$\begin{array}{ccccc} & & \xrightarrow{\nu} & & X \\ V & \longrightarrow & C_X(L) & \nearrow & \downarrow f \\ & & \xrightarrow[\nu]{} & \searrow & X \end{array}$$

shows that $f\nu = \nu$ and the induced diagram

$$\begin{array}{ccc} & \overset{\cong}{\nearrow} & C_X(V) \\ C_{C_X(L)}(V) & & \downarrow C_f(V) \\ & \underset{\cong}{\searrow} & C_X(V) \end{array}$$

that $C_f(V)\colon C_X(V) \to C_X(V)$ is conjugate to the identity. The third assumption of the lemma assures that there are no obstructions to conjugating f to the the identity now that we know that the restriction of f to each of the centralizers is conjugate to the identity, see [44, 4.9].

Suppose next that each line centralizer has N-determined automorphisms. Let $f\colon X \to X$ be an automorphism such that the diagram

$$\begin{array}{ccc} & \nearrow & X \\ N & & \downarrow f \\ & \searrow & X \end{array}$$

commutes up to conjugacy. For each line L in T, the induced diagram

$$\begin{array}{ccc} & \nearrow & C_X(L) \\ C_N(L) & & \downarrow C_f(L) \\ & \searrow & C_X(L) \end{array}$$

also commutes up to conjugacy. By assumption, this means (2.19) that the induced automorphisms $C_f(L)$ of line centralizers are conjugate to the identity. As above, this implies that the induced map $C_f(V)\colon C_X(V) \to C_X(V)$ is conjugate to the identity for any object (V, ν) of the Quillen category for X and that f is conjugate to the identity. □

Consider next an extended 2-compact torus N and two connected, centerless 2-compact groups X and X' both having N as their maximal torus normalizer

eq:situation (2.49)
$$X \xleftarrow{j} N \xrightarrow{j'} X'$$

Our task is (2.15.1) to construct an isomorphism $X \to X'$ under the maximal torus.

defn:toralAX 2.50. Definition. *An object (V, ν) of $\mathbf{A}(X)$ is* toral *if the monomorphism $\nu\colon V \to X$ factors through the maximal torus $T \to X$. Let $\mathbf{A}(X)^{\leq t}$ denote the full subcategory of toral objects, and $\mathbf{A}(X)^{\leq t}_{\leq 2}$ the full subcategory of toral objects of rank ≤ 2.*

For each toral object (V, ν) of $\mathbf{A}(X)^{\leq t}$, let $\nu^N\colon V \to N$ be the unique preferred lift [45, 4.10] of ν (which factors through the identity component of N) and let (V, ν') be the toral object of $\mathbf{A}(X')$ defined by $\nu' = j \circ \nu^N\colon V \to X'$ as in the commutative diagram

$$\begin{array}{ccccc} & \overset{\nu}{\swarrow} & V & \overset{\nu'}{\searrow} & \\ & & \downarrow \nu^N & & \\ X & \xleftarrow[j]{} & N & \xrightarrow[j']{} & X' \end{array}$$

The functor $\mathbf{A}(X)^{\leq t} \to \mathbf{A}(X')^{\leq t}$ that takes the object (V,ν) to the object (V,ν') and is the identity on morphisms is an equivalence of toral Quillen categories [47, 2.8].

indstepalt

THEOREM 2.51. *(Cf [47, 3.8]) In the situation of (2.49), assume the following:*

indstepalt3 (1) *The centralizer $C_X(V,\nu)$ of any $(V,\nu) \in \mathrm{Ob}(\mathbf{A}(X)^{\leq t}_{\leq 2})$ has N-determined automorphisms.*

indstepalt1 (2) *There exists a self-homotopy equivalence $\alpha \in H^1(W;\check{T}) \subseteq \mathrm{Out}(N)$ such that for every object $(L,\lambda) \in \mathrm{Ob}(\mathbf{A}(X)^{\leq t}_{\leq 1})$ the diagram*

$$\begin{array}{ccc} C_N(L,\lambda^N) & \xrightarrow{\alpha|C_N(\lambda^N)} & C_N(L,\lambda^N) \\ {\scriptstyle j|C_N(\lambda^N)}\downarrow & & \downarrow{\scriptstyle j'|C_N(\lambda^N)} \\ C_X(L,\lambda) & \xrightarrow[f_\lambda]{} & C_{X'}(L,\lambda') \end{array}$$

commutes for some isomorphism f_λ.

indstepalt4 (3) *For any nontoral rank two object (V,ν) of $\mathbf{A}(X)$ the composite monomorphism*

$$\nu'_L\colon\ V \xrightarrow{\overline{\nu}(L)} C_X(L,\nu|L) \xrightarrow[\cong]{f_{\nu|L}} C_{X'}(L,(\nu|L)') \xrightarrow{\mathrm{res}} X'$$

and the induced isomorphism $f_{\nu,L}\colon C_X(V,\nu) \to C_{X'}(V,\nu'_L)$ defined by the commutative diagram

$$\begin{array}{ccc} C_{C_X(L,\nu|L)}(V,\overline{\nu}(L)) & \xrightarrow{C_{f_{\nu|L}}} & C_{C_{X'}(L,(\nu|L)')}(V,f_{\nu|L}\circ\overline{\nu}(L)) \\ \cong\downarrow & & \downarrow\cong \\ C_X(V,\nu) & \xrightarrow[f_{\nu,L}]{} & C_{X'}(V,\nu'_L) \end{array}$$

do not depend on the choice of line $L < V$. (See 2.65 for the definition of the canonical factorization $\overline{\nu}(L)$.)

indstepalt5 (4) $\lim^2(\mathbf{A}(X);\pi_1(BZC_X)) = 0 = \lim^3(\mathbf{A}(X);\pi_2(BZC_X))$.

Then there exists an isomorphism $f\colon X \to X'$ under T (2.15).

PROOF. The idea is that the isomorphisms $f_\lambda\colon C_X(\lambda) \to C_{X'}(\lambda')$ on the line centralizers restrict to isomorphisms $f_\nu\colon C_X(\nu) \to C_{X'}(\nu')$ for all centralizers in the $\mathbf{F}_2$-homology decomposition (2.44) of BX. These locally defined isomorphisms combine to a globally defined isomorphism $BX \to BX'$.

First observe that the isomorphisms f_λ on the line centralizers are uniquely determined by the cohomology class $\alpha \in H^1(W;\check{T})$ (2.13.(1)).

Let now (V,ν) be a rank two object of $\mathbf{A}(X)$ and L a line in the plane V. If (V,ν) is *toral*, define $f_\nu\colon C_X(V,\nu) \to C_{X'}(V,\nu')$ to be the isomorphism induced by $f_{\nu|L}\colon C_X(L,\nu|L) \to C_{X'}(L,(\nu|L)'$. Since f_ν is an isomorphism under $\alpha|C_N(V,\nu^N)$ it does not depend on the choice of L in V (2.13.(1)). If (V,ν) is *nontoral*, define ν' to be ν'_L and define $f_\nu\colon C_X(V,\nu) \to C_{X'}(V,\nu')$ to be $f_{\nu,L}$. By assumption 2.51.(3), the monomorphism ν' and the isomorphism $f_{\nu,L}$ are independent of the choice of L.

This construction respects morphisms in $\mathbf{A}(X)$. Consider first, for instance, a morphism $\beta\colon(L_1,\lambda_1) \to (L_2,\lambda_2)$ between two lines in X. Then $\lambda_1 = \lambda_2\beta$ and $\lambda_1^N = \lambda_2^N\beta$. The commutative diagram of isomorphisms

$$
\begin{array}{ccccccc}
C_X(\lambda_1) & & & \xleftarrow{\quad C_X(\beta)\quad} & & & C_X(\lambda_2) \\
 & \nwarrow & & & & \nearrow & \\
\Big\downarrow{\scriptstyle f_{\lambda_1}} & & C_N(\lambda_1^N) & \xleftarrow{C_N(\beta)} & C_N(\lambda_2^N) & & \Big\downarrow{\scriptstyle f_{\lambda_2}} \\
 & & \Big\downarrow{\scriptstyle \alpha|C_N(\lambda_1^N)} & & \Big\downarrow{\scriptstyle \alpha|C_N(\lambda_2^N)} & & \\
 & & C_N(\lambda_1^N) & \xleftarrow[C_N(\beta)]{} & C_N(\lambda_2^N) & & \\
 & \swarrow & & & & \searrow & \\
C_{X'}(\lambda_1') & & & \xleftarrow[\quad C_{X'}(\beta)\quad]{} & & & C_{X'}(\lambda_2')
\end{array}
$$

shows that $C_{X'}(\beta)^{-1} \circ f_{\lambda_1} \circ C_X(\beta) = f_{\lambda_2}$ for they are both isomorphism under $C_N(\beta)^{-1} \circ \alpha|C_N(\lambda_1^N) \circ C_N(\beta) = \alpha|C_N(\lambda_2^N)$. Second, by the very definition of f_ν, the diagram

$$
\begin{array}{ccc}
C_X(V,\nu) & \xrightarrow{f_\nu} & C_{X'}(V,\nu') \\
\Big\downarrow & & \Big\downarrow \\
C_X(L,\nu|L) & \xrightarrow[f_{\nu|L}]{} & C_{X'}(L,(\nu|L)')
\end{array}
$$

commutes whenever $L < V$ and (V,ν) is (toral or nontoral) rank 2 object of $\mathbf{A}(X)$.

We have now defined natural isomorphisms $f_\nu \colon C_X(V,\nu) \to C_{X'}(V,\nu')$ for all objects $(V,\nu) \in \mathrm{Ob}(\mathbf{A}(X))$ of rank ≤ 2. For any other object (E,ε) of $\mathbf{A}(X)$, choose a line $L < E$ and proceed as for toral rank 2 objects. That is, define $\varepsilon' \colon E \to X'$ to be the monomorphism

$$E \xrightarrow{\overline{\varepsilon}(L)} C_X(E,\varepsilon|L) \xrightarrow{f_{\varepsilon|L}} C_{X'}(E,(\varepsilon|L)') \xrightarrow{\text{res}} X'$$

and define $f_\varepsilon \colon C_X(E,\varepsilon) \to C_{X'}(E,\varepsilon')$ to be the isomorphism

$$
\begin{array}{ccc}
C_{C_X(E,\varepsilon|L)}(\overline{\varepsilon}(L)) & \xrightarrow{(f_{\varepsilon|L})_*} & C_{C_{X'}(E,(\varepsilon|L)')}(f_{\varepsilon|L} \circ \overline{\varepsilon}(L)) \\
{\scriptstyle\cong}\Big\downarrow & & \Big\downarrow{\scriptstyle\cong} \\
C_X(E,\varepsilon) & \xrightarrow[f_\varepsilon]{} & C_{X'}(E,\varepsilon')
\end{array}
$$

induced by $f_{\varepsilon|L}$. If L_1 and L_2 are two distinct lines in E, let $P = \langle L_1, L_2\rangle$ be the plane generated by them. Then the commutative diagram

$$
\begin{array}{ccccccc}
 & & C_X(L_1,\varepsilon|L_1) & \xrightarrow[\cong]{f_{\varepsilon|L_1}} & C_{X'}(L_1,(\varepsilon|L_1)') & & \\
 & \nearrow^{\overline{\varepsilon}(L_1)} & \Big\uparrow & & \Big\uparrow & \searrow^{\text{res}} & \\
P & \xrightarrow{\overline{\varepsilon}(P)} & C_X(P,\varepsilon|P) & \xrightarrow[f_{\varepsilon|P}]{\cong} & C_{X'}(P,(\varepsilon|P)') & \xrightarrow{\text{res}} & X' \\
 & \searrow_{\overline{\varepsilon}(L_2)} & \Big\downarrow & & \Big\downarrow & \nearrow_{\text{res}} & \\
 & & C_X(L_2,\varepsilon|L_2) & \xrightarrow[f_{\varepsilon|L_2}]{\cong} & C_{X'}(L_2,(\varepsilon|L_2)') & &
\end{array}
$$

shows that neither $(E,\varepsilon') \in \mathrm{Ob}(\mathbf{A}(X'))$ nor the isomorphism f_ε depend on the choice of line in E. Thus we have constructed a collection of centric [**12**] maps

eq:collection (2.52) $$\mathrm{B}C_X(V,\nu) \to \mathrm{B}X', \quad (V,\nu) \in \mathrm{Ob}(\mathbf{A}(X)),$$

that are homotopy invariant under $\mathbf{A}(X)$-morphisms. The vanishing (2.51.(4)) of the obstruction groups means [**62**] that these homotopy $\mathbf{A}(X)$-invariant maps can be realized by a map

$$\mathrm{B}f \colon \mathrm{B}X \xleftarrow{\simeq} \operatorname{hocolim} \mathrm{B}C_X \to \mathrm{B}X'$$

such that $f \circ \mathrm{res} = \mathrm{res} \circ f_\nu$ for all $(V,\nu) \in \mathrm{Ob}(\mathbf{A}(X))$. In particular, f is a map under T and an isomorphism (2.15). $\square$

sec:cond2

2.53. Verification of condition 2.51.(2). Let $\mathbf{A}(X)^{\leq t}$ be the toral part of the Quillen category and let $H^1(W_0;\check{T})^{W/W_0} \colon \mathbf{A}(X)^{\leq t} \to \mathbf{Ab}$ the functor with value $H^1(W(C_X(V,\nu)_0);\check{T})^{\pi_0 C_X(V,\nu)}$ on the object (V,ν). If the 2-compact group C satisfies the conditions of Lemma 2.40 and $\check{Z}(C_0) = \check{T}(C_0)^{W(C_0)}$ we say that C satisfies the the conditions of Lemma 2.40 *in the strong sense.*

lemma:altcond2

2.54. LEMMA. *Suppose that*

- *The centralizers $C_X(V,\nu)$ of all $(V,\nu) \in \mathrm{Ob}(\mathbf{A}(X)^{\leq t}_{\leq 2})$ satisfy the conditions of Lemma 2.40 in the strong sense,*
- *$H^1(W;\check{T}) \to \lim^0(\mathbf{A}(X)^{\leq t}_{\leq 2}; H^1(W_0;\check{T})^{W/W_0})$ is surjective*

Then conditions 2.51.(1) and 2.51.(2) are satisfied.

PROOF. Let (V,ν) be an object of $\mathbf{A}(X)^{\leq t}$ of rank ≤ 2. Since (2.40) the centralizer $C_X(V,\nu)$ is N-determined there is a solution $(f(V,\nu),\alpha(V,\nu))$ to the isomorphism problem

$$\begin{array}{ccc} C_N(V,\nu^N) & \xrightarrow{\alpha(V,\nu)} & C_N(V,\nu^N) \\ \downarrow & & \downarrow \\ C_X(V,\nu) & \xrightarrow[f(V,\nu)]{} & C_{X'}(V,\nu') \end{array}$$

and the set of all solutions is (2.35, 2.37) a $H^1(W/W_0;\check{T}^{W_0})(C_X(V,\nu))$-coset. Let

$$\overline{\alpha}(V,\nu) \in H^1(W_0;\check{T}^{W/W_0})(C_X(V,\nu))$$

be the restriction of any solution $\alpha(V,\nu) \in H^1(W;\check{T})(C_X(V,\nu))$ to the above isomorphism problem. Then

limalpha

$$\{\overline{\alpha}(V,\nu)\}_{(V,\nu)\in \mathrm{Ob}(\mathbf{A}(X)^{\leq t}_{\leq 2}} \in \lim{}^0(\mathbf{A}(X)^{\leq t}_{\leq 2}; H^1(W_0;\check{T})^{W/W_0}) \tag{2.55}$$

because the restriction of a solution is a solution. By assumption, there is an element $\alpha \in H^1(W;\check{T})$ that maps to (2.55) and α satisfies 2.51.(2). $\square$

In case $H^1(W;\check{T}) = 0$, the second point reduces to $\lim^0(\mathbf{A}(X)^{\leq t}_{\leq 2}; H^1(W_0;\check{T})^{W/W_0}) = 0$. Alternatively, if $\lim^1(\mathbf{A}(X)^{\leq t}_{\leq 2}; H^1(W/W_0;\check{T}^{W_0})) = 0$, then the short exact sequences (2.27) for $C_X(V,\nu)$, $(V,\nu) \in \mathrm{Ob}(\mathbf{A}(X)^{\leq t}_{\leq 2})$, will produce a short exact sequence

$$0 \to \lim{}^0\left(\mathbf{A}(X)^{\leq t}_{\leq 2}, H^1(W/W_0;\check{T}^{W_0})\right) \to \lim{}^0\left(\mathbf{A}(X)^{\leq t}_{\leq 2}, H^1(W;\check{T})\right) \\ \to \lim{}^0\left(\mathbf{A}(X)^{\leq t}_{\leq 2}, H^1(W_0;\check{T})^{W/W_0}\right) \to 0,$$

in the limit. Since $H^1(W;\check{T})$ is isomorphic to the middle term by [16, 8.1], it maps onto the third term.

sec:cond4

2.56. Verification of condition 2.51.(3). In this subsection we assume that conditions 2.51.(1) and 2.51.(2) are satisfied. The following observations can sometimes be useful in the verification of condition 2.51.(3).

Let (V,ν) be a nontoral rank two object of $\mathbf{A}(X)$ and $L < V$ a rank one subgroup. The commtutative diagram

dia:altdefnuL

$$\begin{array}{ccccc} & N & \xrightarrow{\alpha} & N & \\ {}^{\nu_L^N}\nearrow & \uparrow & & \uparrow & \searrow^{j'} \\ V \xrightarrow{\overline{\nu_L^N}(L)} & C_N(L,\nu_L^N|L) & \xrightarrow{C_\alpha} & C_N(L,\nu_L^N|L) & \quad X' \\ {}_{\overline{\nu}(L)}\searrow & \downarrow & & \downarrow & \nearrow_{\mathrm{res}} \\ & C_X(L,\nu|L) & \xrightarrow[f_{\nu|L}]{} & C_{X'}(L,(\nu|L)') & \end{array} \tag{2.57}$$

shows that ν'_L, which is defined to be $\mathrm{res}\circ f_{\nu|L}\circ\overline{\nu}(L)$, is equal to the composite $\nu'_L = j'\circ\alpha\circ\nu_L^N$. Moreover, we see by taking the centralizer of $\overline{\nu}(L)$ that

dia:undernuL (2.58)

$$\begin{array}{ccccc} & & V & & \\ & \swarrow{\scriptstyle\overline{\nu}(V)} & & {\scriptstyle\overline{\nu}'_L(V)}\searrow & \\ C_X(V,\nu) & & \xrightarrow[f_{\nu,L}]{\cong} & & C_{X'}(V,\nu') \end{array}$$

commutes.

We are looking for criteria that ensure that $\nu'_L\colon V\to X'$ is independent of the choice of $L<V$.

lemma:C3cond 2.59. Lemma. *Let (V,ν) be a nontoral rank two object of $\mathbf{A}(X)$ and $L<V$ a line in V. Write C_3 for the Sylow 3-subgroup of $\mathrm{GL}(V)$. Suppose that*

lemma:C3cond.1 (1) $C_3\subseteq\mathbf{A}(X)(V,\nu)\cap\mathbf{A}(X')(V,\nu'_L)$

lemma:C3cond2 (2) $f_{\nu,L}\colon C_X(V,\nu)\to C_{X'}(V,\nu'_L)$ *is C_3-equivariant*

Then condition 2.51.(3) is satisfied.

Proof. Let β be an automorphism of V. For general reasons, $\nu_L^N\beta = (\nu\beta)^N_{\beta^{-1}L}$ and the diagram

$$\begin{array}{ccc} C_X(V,\nu) & \xrightarrow{f_{\nu,L}} & C_{X'}(V,\nu'_L) \\ {\scriptstyle C_X(\beta)}\downarrow\cong & & \cong\downarrow{\scriptstyle C_{X'}(\beta)} \\ C_X(V,\nu\beta) & \xrightarrow[f_{\nu\beta,\beta^{-1}L}]{} & C_{X'}(V,\nu'_L\beta) \end{array}$$

commutes. Now, if $\beta\in\mathbf{A}(X)(V,\nu)\cap\mathbf{A}(X')(V,\nu'_L)$, then $\nu\beta=\nu$, $\nu'_L\beta=\nu'_L$, and $f_{\nu\beta,\beta^{-1}L}=f_{\nu,\beta^{-1}L}$ so that $f_{\nu,\beta^{-1}L}=C_{X'}(\beta)\circ f_{\nu,L}\circ C_X(\beta)^{-1}$ according to the above diagram. If also $f_{\nu,L}$ commutes with the action of β, we conclude that $f_{\nu,L}=f_{\nu,\beta^{-1}L}$. ☐

The following lemma assures that condition 2.59.(1) holds.

help1 2.60. Lemma. *Let L and V denote elementary abelian 2-groups of rank one and two, respectively. Suppose that*

(1) *There is (up to conjugacy) a unique monomorphism $\lambda\colon L\to X$ with nonconnected centralizer*

(2) *There is (up to conjugacy) a unique nontoral monomorphism $\nu\colon V\to X$*

Then the same holds for X', and $\mathbf{A}(X)(V,\nu)=\mathrm{GL}(V)=\mathbf{A}(X')(V,\nu')$ for the unique nontoral rank two objects (V,ν) of $\mathbf{A}(X)$ and (V,ν') of $\mathbf{A}(X')$.

Proof. Let $\nu'\colon V\to X'$ be a nontoral monomorphism and $i\colon L\to V$ an inclusion. Then $(L,\nu' i)=(L,\lambda')$ for $C_{X'}(L,\nu' i)$ is nonconnected so that $\nu' i$ and λ must correspond under the bijection $\mathbf{A}(X)^{\leq t}\to\mathbf{A}(X')^{\leq t}$ between toral categories. Moreover, the diagram

dia:D1 (2.61)

$$\begin{array}{ccccccc} X & \xleftarrow{\mathrm{res}} & C_X(L,\lambda) & \xrightarrow[\cong]{f_\lambda} & C_{X'}(L,\lambda') & \xrightarrow{\mathrm{res}} & X' \\ & {\scriptstyle\nu}\nwarrow & \nwarrow{\scriptstyle\overline{\nu}(L)} & & \nearrow{\scriptstyle\overline{\nu}'(L)} & \nearrow{\scriptstyle\nu'} & \\ & & & V & & & \end{array}$$

is commutative. To see this, observe that $(V,\mathrm{res}\circ f_\lambda^{-1}\circ\overline{\nu}'(L))$ is a nontoral rank two object of $\mathbf{A}(X)$ (its centralizer is isomorphic to $C_{C_{X'}(L,\lambda')}(V,\overline{\nu}'(L))=C_{X'}(V,\nu'))$ so that $(V,\nu)=(V,\mathrm{res}\circ f_\lambda^{-1}\circ\overline{\nu}'(L))$ by uniqueness of (V,ν). Also, we see from the commutative diagram

$$\begin{array}{ccccc} & & & & L \\ & \swarrow{\scriptstyle\lambda} & & \swarrow{\scriptstyle\overline{\lambda}(L)} & \downarrow{\scriptstyle i} \\ X & \xleftarrow[\mathrm{res}]{} & C_X(L,\lambda) & \underset{f_\lambda^{1}\circ\overline{\nu}'(L)}{\overset{\overline{\nu}(L)}{\leftleftarrows}} & V \end{array}$$

that $\overline{\nu}(L) = f_\lambda^{-1} \circ \overline{\nu}'(L)$ by uniqueness of canonical factorizations under L [46, 3.9]. We conclude that $\nu' = \mathrm{res} \circ \overline{\nu}'(L) = \mathrm{res} \circ f_\lambda \circ \overline{\nu}(L)$. This means (2.57) that $\nu' = \nu'_L$ for any choice of line $L < V$. Since thus ν' is unique up to conjugacy, $\nu'\beta = \nu'$ for any automorphism β of V. □

Note in connection with the verification of condition 2.59.(2), that if 2.59.(1) is satisfied so that $\nu'_L = \nu'$ is independent of L, then (2.58) shows that $f_{\nu,L}$ is a map under V in the sense that

dia:triangle (2.62)

$$\begin{array}{ccc} & V & \\ {\scriptstyle \overline{\nu}(V)} \swarrow & & \searrow {\scriptstyle \overline{\nu}'(V)} \\ C_X(V,\nu) & \xrightarrow[f_{\nu,L}]{\cong} & C_{X'}(V,\nu') \end{array}$$

commutes. Since the canonical monomorphisms, $\overline{\nu}(V)$ and $\overline{\nu}'(V)$, are $\mathrm{GL}(V)$-equivariant, the restriction of $f_{\nu,L}$ to V is C_3-equivariant.

For any nontoral object (not necessarily of rank two) (V,ν) of $\mathbf{A}(X)$ and any rank one subgroup $L \subset V$, let $\nu_L^N\colon V \to N$ be a preferred lift of ν such that $\nu_L^N|L$ is the preferred lift of $\nu|L$, ie $\nu_L^N|L = (\nu|L)^N$. (It is always possible to extend a preferred lift given on the subgroup L to a preferred lift defined on all of V but a preferred lift defined on V may not restrict to a preferred lift on L [45, 4.9].) Also, define $\nu'_L\colon V \to X'$ and $f_{\nu,L}\colon C_X(V,\nu) \to C_{X'}(V,\nu'_L)$ as in 2.51.(3).

lemma:uniquenu 2.63. Lemma. *Let $\nu\colon V \to X$ be any nontoral object of $\mathbf{A}(X)$.*

lemma:uniquenu1
lemma:uniquenu2

(1) *If the centralizer of ν has a nontrivial identity component, then $\nu'_L\colon V \to X'$ is independent up to conjugacy of the choice of $L \subset V$, and $\nu'_L = j' \circ \alpha \circ \nu_L^N$.*
(2) *If also there exist a 2-compact torus T_ν and isomorphisms $T_\nu \to C_N(V,\nu_L^N)_0$ such that the composites $T_\nu \to C_N(V,\nu_L^N)_0 \to T$ are independent up to conjugacy of $L < V$, then $f_{\nu,L}\colon C_X(V,\nu) \to C_{X'}(V,\nu')$ are isomorphisms under the maximal torus T_ν for all $L < V$.*

Proof. (1) Just as in (2.57) we see that $\nu'_L = \mathrm{res} \circ f_{\nu|L} \circ \overline{\nu}(L) = j' \circ \alpha \circ \nu_L^N$. The hypothesis implies that there exists [17, 5.4, 7.3] a morphism $\phi\colon L_1 \times V \to X$ extending $\nu\colon V \to X$ whose adjoint $L_1 \to C_X(\nu)$ factors through the identity component of $C_X(\nu)$. Let $L_1 \to C_N(V,\nu_L^N)$ be the preferred lift of $L_1 \to C_X(V,\nu)$ as in the commutative diagram

$$\begin{array}{ccccc} & \nearrow & C_N(V,\nu_L^N) & \xrightarrow{\mathrm{res}} & N \\ & & \downarrow & & \downarrow{\scriptstyle j} \\ L_1 & \longrightarrow & C_X(V,\nu) & \xrightarrow[\mathrm{res}]{} & X \end{array}$$

This preferred lift will factor through the identity component of $C_N(\nu_L^N)$ (and hence its composition with $C_N(\nu_L^N) \to N$ will factor through the identity component of N) since $L_1 \to C_X(V,\nu)$ factors through the identity component of $C_X(\nu)$ [45, 4.10]. Let $\phi_L^N\colon L_1 \times V \to N$ be the adjoint of the preferred lift $L_1 \to C_N(\nu_L^N)$. Then $\phi_L^N|L_1\colon L_1 \to N$ factors through the identity component of N (the maximal torus) so it is [45, 4.10] the preferred lift of $\phi|L_1\colon L_1 \to X$. In particular, $\phi_L^N|L_1 = (\phi|L_1)^N$ does not depend on the choice of L.

The adjoints, $\phi_2^N\colon V \to C_N(\phi_L^N|L_1)$ and $\phi_2\colon V \to C_X(\phi|L_1)$, of ϕ_L^N and ϕ, respectively, with respect to the second factor, give a commutative diagram

$$\begin{array}{ccccccc} & \nearrow^{\phi_2} & C_X(L_1,\phi|L_1) & \xrightarrow{f_{\phi|L_1}} & C_{X'}(L_1,(\phi|L_1)') & \searrow^{\mathrm{res}} & \\ & & \uparrow & & \uparrow & & \\ V & \xrightarrow{\phi_2^N} & C_N(L_1,\phi_L^N|L_1) & \xrightarrow{C_\alpha} & C_N(L_1,\phi_L^N|L_1) & & X' \\ & & \downarrow & & \downarrow & & \\ & \searrow_{\nu_L^N} & N & \xrightarrow[\alpha]{} & N & \nearrow_{j'} & \end{array}$$

We conclude that $\nu'_L = j' \circ \alpha \circ \nu_L^N = \mathrm{res} \circ f_{\phi|L_1} \circ \phi_2\colon V \to X'$ is independent of the choice of $L < V$.

(2) The upper square in the diagram

$$\begin{array}{ccc} T_\nu & = & T_\nu \\ \downarrow & & \downarrow \\ C_N(V,\nu_L^N) & \xrightarrow{\alpha} & C_N(V,\alpha\nu_L^N) \\ \downarrow & & \downarrow \\ C_X(V,\nu) & \xrightarrow[f_{\nu,L}]{} & C_{X'}(V,\nu') \end{array}$$

commutes because α restricts to the identity on the identity component T of N and hence also on T_ν. That the lower square is commutative is consequence of the commutative diagram

$$\begin{array}{ccccc} & \overset{\overline{\nu}_L^N(L)}{\nearrow} & C_N(L,\nu_L^N|L) & \xrightarrow{C_\alpha} & C_N(L,\alpha\nu_L^N) \\ & & \downarrow & & \downarrow \\ V & \xrightarrow[\overline{\nu}(L)]{} & C_X(L,\nu|L) & \xrightarrow[f_{\nu|L}]{} & C_{X'}(L,(\nu|L)') \end{array}$$

where $\overline{\nu}_L^N(L)$ and $\overline{\nu}(L)$ are the canonical factorizations (2.65). □

Let $\mu\colon U \to X$ be a nontrivial elementary abelian 2-group and $\mu\colon U \to X$ a monomorphism whose centralizer $C_X(U,\mu)$ has nontrivial identity component. Suppose that U contains a nontrivial subgroup $V < U$ such that the restriction of μ to V is nontoral. Choose a rank one subgroup $L \subset V \subset U$. We may choose the preferred lifts μ_L^N and $(\mu|V)_L^N$ such that $\mu_L^N|V = (\mu|V)_L^N$. Since $C_X(U,\mu)$ has nontrivial identity component, the conjugacy classes of the monomorphisms $\mu' = \mu'_L$ and $(\mu|V)'_L = \mu'|L$ are independent of the choice of L by 2.63.(1). Then there is a commutative diagram

dia:Utriangle (2.64)

$$\begin{array}{ccc} & U & \\ \overset{\overline{\mu}(V)}{\swarrow} & & \overset{\overline{\mu}'(V)}{\searrow} \\ C_X(V,\mu|V) & \xrightarrow[f_{\mu|V,L}]{} & C_{X'}(V,\mu'|V) \end{array}$$

similar to (2.62).

rmk:canfact

2.65. Canonical factorizations. Let $\nu\colon V \to X$ be a monomorphism from an elementary abelian p-group to the p-compact group X. The canonical factorization of ν through its centralizer is the central monomorphism $\overline{\nu}(V)\colon V \to C_X(V,\nu)$ whose adjoint is $V \times V \xrightarrow{+} V \xrightarrow{\nu} X$ [**17**, 8.2]. If $\alpha\colon (V_1,\nu_1) \to (V_2,\nu_2)$ is a morphism in $\mathbf{A}(X)$ then the canonical factorizations are related by a commutative diagram

dia:canfact1 (2.66)

$$\begin{array}{ccccc} V_1 & \xrightarrow{\overline{\nu}_1(V_1)} & C_X(V_1,\nu_1) & \xrightarrow{\text{res}} & X \\ {\scriptstyle\alpha}\downarrow & & \uparrow{\scriptstyle C_X(\alpha)} & & \| \\ V_2 & \xrightarrow[\overline{\nu}_2(V_2)]{} & C_X(V_2,\nu_2) & \xrightarrow[\text{res}]{} & X \end{array}$$

and we shall write $\overline{\nu}_2(V_1)\colon V_2 \to C_X(V_1,\nu_1)$ for $C_X(\alpha)\circ\overline{\nu}_2(V_2)$ and call it the canonical factorization of ν_2 through the centralizer of ν_1. The induced diagram

dia:canfact2 (2.67)

$$\begin{array}{ccccc} C_{C_X(V_2,\nu_2)}(V_2,\overline{\nu}_2(V_2)) & \xrightarrow[\cong]{C_{C_X(\alpha)}} & C_{C_X(V_1,\nu_1)}(V_2,\overline{\nu}_2(V_1)) & \xrightarrow{C_{C_X(V_1,\nu_1)}(\alpha)} & C_{C_X(V_1,\nu_1)}(V_1,\overline{\nu}_1(V_1)) \\ \cong\downarrow & & & & \downarrow\cong \\ C_X(V_2,\nu_2) & & \xrightarrow[C_X(\alpha)]{} & & C_X(V_1,\nu_1) \end{array}$$

is a factorization of $C_X(\alpha)$.

sec:limAWt

4. An exact functor

Let W be a finite group, p a prime, and $\rho\colon W \to \mathrm{GL}(t)$ a representation of W in an $\mathbf{F}_p$-vector space t of finite dimension. For any nontrivial subgroup $V \subset t$, let

$$W(V) = \{w \in W \mid \forall v \in V\colon wv = v\}$$

be the subgroup of elements of W that act as the identity on V. For any two nontrivial subgroups $V_1, V_2 \subset t$, let

$$\overline{W}(V_1, V_2) = \{w \in W | wV_1 \subset V_2\}$$

be the transporter set. (Even though suppresed in the notation, these set depend on the representation ρ.)

Suppose that we are given also a $\mathbf{Z}_pW$-module L.

defn:AWt

2.68. Definition. [**47**, 2.2] *$\mathbf{A}(\rho, t)$ is the category whose objects are nontrivial subspaces of V and whose morphisms are group homomorphisms induced by the W-action. The functor $L_i\colon \mathbf{A}(\rho, t) \to \mathbf{Ab}$ is the functor that takes the object $V \subset t$ to $H^i(W(V); L)$ and the morphism $w\colon V_1 \to V_2$ to $H^i(W(V_1); L) \xrightarrow{w^*} H^i(W(V_1)^w; L) \xrightarrow{\mathrm{res}} H^i(W(V_2); L)$ where res is restriction and w^* induced from conjugation with $w \in W$.*

The category $\mathbf{A}(\rho, t)$ depends only on the image of W in $\mathrm{GL}(t)$ but the functor L_i depends on the actual representation. The morphism set in $\mathbf{A}(\rho, t)$ is the set of orbits

$$\mathbf{A}(\rho, t)(V_1, V_2) = \overline{W}(V_1, V_2)/W(V_1)$$

for the action of the group $W(V_1)$ on the set $\overline{W}(V_1, V_2)$. We shall often write $\mathbf{A}(W, t)$ for $\mathbf{A}(\rho, t)$ when the representation ρ is clear from the context and $\mathbf{A}(W, t)(V)$ will be used as an abbreviation for the endomorphism group $\mathbf{A}(W, t)(V, V) = \overline{W}(V, V)/W(V)$.

dw:limits

2.69. Lemma. [**16**, 8.1] *L_i is an exact functor with limit $H^i(W; L)$:*

$$\lim{}^j(\mathbf{A}(W, t), L_i) = \begin{cases} H^i(W; L) & j = 0 \\ 0 & j > 0 \end{cases}$$

Proof. The proof of [**16**, 8.1] also applies to this slightly different setting where the action of W on the $\mathbf{F}_p$-vector space t may not be faithful and L is a $\mathbf{Z}_pW$-module (and not an $\mathbf{F}_pW$-module).

Another possibility is to use the ideas of [**30**]. It suffices to show that the category $\mathbf{A}(W, t)$ satisfies (the dual of) the conditions of [**30**, 5.16] and that L_* is a proto-Mackey functor. Define $L^*\colon \mathbf{A}(W, t) \to \mathbf{Ab}$ to be the contravariant functor that agrees with L_* on objects but takes the $\mathbf{A}(W, t)$-morphism $w\colon E_0 \to E_1$ to the group homomorphism

$$H^*(W(E_0); L) \xleftarrow{(w^{-1})^*} H^*(W(E_0)^w; L) \xleftarrow{\mathrm{tr}} H^*(W(E_1); L)$$

where tr is transfer. To prove the existence of coproducts and push-outs in the multiplicative extension $\mathbf{A}(W, t)_{\prod}$ we follow [**30**, 6.3]. Let E_0, E_1, E_2 be elementary abelian subgroups of t where $E_0 \subset E_1$ and there is a morphism $E_0 \to E_2$ represented by an element $w \in \overline{W}(E_0, E_2) \subset W$. ($E_0$ is possibly empty to allow for the construction of coproducts.) Each coset $gW(E_1) \in W(E_0)/W(E_1)$ has an associated special diagram

$$\begin{array}{ccc} E_0 & \hookrightarrow & E_1 \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle wg} \\ E_2 & \hookrightarrow & E_2 + wgE_1 \end{array}$$

where we note that $W(E_2 + wgE_1) = W(E_2) \cap W(E_1)^{wg}$. This construction determines a bijection between the double coset $w^{-1}W(E_2)w\backslash W(E_0)/W(E_1)$ and the set of isomorphism classes of special diagrams, cf. [**30**, 7.3], and therefore

$$\begin{array}{ccc} E_0 & \hookrightarrow & E_1 \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle \prod wg} \\ E_2 & \hookrightarrow & \prod(E_2 + wgE_1) \end{array}$$

where the product is taken over all $g \in w^{-1}W(E_2)w\backslash W(E_0)/W(E_1)$, is a push-out diagram in $\mathbf{A}(W,t)_{\prod}$ [30, 6.3]. By [30, 5.13], we need to show that the diagram

$$\begin{array}{ccc} H^*(W(E_0);L) & \xleftarrow{L^*(E_0\subseteq E_1)} & H^*(W(E_1);L) \\ {\scriptstyle L_*(w)}\downarrow & & \downarrow{\scriptstyle \prod L_*(wg)} \\ H^*(W(E_2);L) & \xleftarrow[\sum L^*(E_2\subseteq E_2+gwE_1)]{} & \prod H^*(W(E_2)\cap W(E_1)^{wg};L) \end{array}$$

commutes. But this is precisely the content of the Cartan–Eilenberg double coset formula relating the restriction and transfer homomorphisms in group cohomology [8] [21, 4.2.6].

The restriction homomorphism $H^*(W;L) \to \lim^0(\mathbf{A}(W,t);L_*)$ is injective since t contains an elementary abelian subgroup $E \subset t$ such that the index of $W(E)$ in W is prime to p. To show surjectivity, we use the argument from the proof of [30, 7.2]. □

CHAPTER 3

The A-family

cha:afam

The A-family consists of the matrix groups

$$\mathrm{PGL}(n+1,\mathbf{C}) = \frac{\mathrm{GL}(n+1,\mathbf{C})}{\mathrm{GL}(1,\mathbf{C})}, \qquad n \geq 1,$$

where $\mathrm{GL}(n+1,\mathbf{C})$ is the Lie group of complex $(n+1)\times(n+1)$ matrices with center $\mathrm{GL}(1,\mathbf{C})$ consisting of scalar matrices. The maximal torus normalizer for $\mathrm{PGL}(n+1,\mathbf{C})$ is

$$N(\mathrm{PGL}(n+1,\mathbf{C})) = \frac{\mathrm{GL}(1,\mathbf{C})^{n+1}}{\mathrm{GL}(1,\mathbf{C})} \rtimes \Sigma_{n+1}$$

where $\Sigma_{n+1} = W(\mathrm{PGL}(n+1,\mathbf{C})) \subset \mathrm{PGL}(n+1,\mathbf{C})$ is the Weyl group of permutation matrices. It is known [**24**, **35**] that

eq:afamHOH1

$$H^0(W;\check{T}) = \begin{cases} \mathbf{Z}/2 & n=1 \\ 0 & n>1 \end{cases}, \qquad H^1(W;\check{T}) = \begin{cases} \mathbf{Z}/2 & n=3 \\ 0 & n\neq 3 \end{cases} \tag{3.1}$$

for $\mathrm{PGL}(n+1,\mathbf{C})$. For all n, $\mathrm{PGL}(n+1,\mathbf{C}) = \mathrm{PSL}(n+1,\mathbf{C})$. When $n+1$ is odd, $\mathrm{PGL}(n+1,\mathbf{C}) = \mathrm{PSL}(n+1,\mathbf{C}) = \mathrm{SL}(n+1,\mathbf{C})$ as 2-compact groups.

sec:pglnc

1. The structure of $\mathrm{PGL}(n+1,\mathbf{C})$

In this and the following section we use the results of Chapter 2 to show that the 2-compact groups $\mathrm{PGL}(n+1,\mathbf{C})$, $n \geq 1$, are uniquely N-determined. This section provides the information about the Quillen category needed for the calculation (3.18) of the higher limit obstruction groups from 2.48 and 2.51.

sec:Apglnct

3.2. The toral subcategory of $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$. We consider the full subcategory of $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$ generated by the toral nontrivial elementary abelian 2-groups in $\mathrm{PGL}(n+1,\mathbf{C})$, $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\leq t}$ (2.50) .

tpglnC

3.3. Lemma. *The monomorphism $\nu\colon V\to \mathrm{PGL}(n+1,\mathbf{C})$ is toral if and only if it lifts to a morphism $V \to \mathrm{GL}(n+1,\mathbf{C})$. If $n+1$ is odd, all objects of $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$ are toral.*

Proof. Any monomorphism $V \to \mathrm{GL}(n+1,\mathbf{C}) \to \mathrm{PGL}(n+1,\mathbf{C})$ is toral since it is toral already in $\mathrm{GL}(n+1,\mathbf{C})$ by complex representation theory. Conversely, any toral monomorphism $V \to \mathrm{GL}(1,\mathbf{C})^{n+1}/\mathrm{GL}(1,\mathbf{C}) \subset \mathrm{PGL}(n+1,\mathbf{C})$ lifts to $\mathrm{GL}(1,\mathbf{C})$ since $\mathrm{GL}(1,\mathbf{C})$ is divisible. When $n+1$ is odd, $\mathrm{PGL}(n+1,\mathbf{C}) = \mathrm{SL}(n+1,\mathbf{C}) \subset \mathrm{GL}(n+1,\mathbf{C})$ as 2-compact groups so all monomorphisms $V \to \mathrm{PGL}(n+1,\mathbf{C})$ are toral. □

Let

$$e_i = \mathrm{diag}(+1,\dots,+1,-1,+1,\dots,+1) \in \mathrm{GL}(n+1,\mathbf{C}), \qquad 1 \leq i \leq n+1,$$

be the diagonal matrix with -1 in position i and $+1$ at all other positions. The maximal toral elementary abelian 2-groups

$$\Delta_{n+1} = \langle e_1,\dots,e_{n+1}\rangle = \langle \mathrm{diag}(\pm1,\dots,\pm1)\rangle \cong (\mathbf{Z}/2)^{n+1} \subset \mathrm{GL}(n+1,\mathbf{C}),$$
$$P\Delta_{n+1} = \langle e_1,\dots,e_{n+1}\rangle \,/\, \langle e_1\cdots e_{n+1}\rangle \cong (\mathbf{Z}/2)^{n} \subset \mathrm{PGL}(n+1,\mathbf{C}),$$

have Quillen automorphism groups $\Sigma_{n+1} \cong \mathbf{A}(\mathrm{GL}(n+1,\mathbf{C}))(\Delta_{n+1}) \cong \mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))(P\Delta_{n+1})$.

ma:afamtoralcat

3.4. Lemma. *The inclusion functors*

$$\mathbf{A}(\Sigma_{n+1},\Delta_{n+1}) \to \mathbf{A}(\mathrm{GL}(n+1,\mathbf{C})), \qquad \mathbf{A}(\Sigma_{n+1},P\Delta_{n+1}) \to \mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\leq t}$$

are equivalence of categories.

PROOF. This is a general fact; the first part of [47, 2.8] also holds for the case $p = 2$. However, it may be more illustrative to prove the lemma directly in this special case.

By complex represention theory, any nontrivial elementary abelian 2-group in $\mathrm{GL}(n+1, \mathbf{C})$ is conjugate to a subgroup of Δ_{n+1} and $\mathbf{A}(\mathrm{GL}(n+1,\mathbf{C}))(\Delta_{n+1}) = \Sigma_{n+1}$. Thus there is a faithful inclusion functor $\mathbf{A}(\Sigma_{n+1}, \Delta_{n+1}) \to \mathbf{A}(\mathrm{GL}(n+1,\mathbf{C}))$ which is surjective on the sets of isomorphism classes of objects. It remains to show that this functor is full. Since any morphism in the category $\mathbf{A}(\mathrm{GL}(n+1,\mathbf{C}))$ is an isomorphism followed by an inclusion, it is enough to show that any conjugation induced isomorphism $V_1 \to V_2$ between nontrivial subgroups $V_1, V_2 \subset \Delta_{n+1}$ is actually induced from conjugation by an element of $N(\mathrm{GL}(n+1,\mathbf{C}))$. But this is well-known fact from Lie group theory easily derived from eg [7, IV.2.5].

Any toral nontrivial elementary abelian 2-group in $\mathrm{PGL}(n+1, \mathbf{C})$ is the image of a elementary abelian 2-group in $\mathrm{GL}(n+1,\mathbf{C})$ and hence conjugate to subgroup of $P\Delta_{n+1}$. Since any $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$-morphism between subgroups of $P\Delta_{n+1}$ are induced from conjugation with an element of $N(\mathrm{PGL}(n+1,C))$, it follows that $\mathbf{A}(\Sigma_{n+1}, P\Delta_{n+1}) \to \mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\leq t}$ is an equivalence of categories. □

For any partition $n+1 = i_0 + i_1 + \cdots + i_r$ of $n+1$ into a sum of r positive integers, let $(\pm 1)^{i_0}(\pm 1)^{i_1}\cdots(\pm 1)^{i_r}$ denote the diagonal matrix

$$\mathrm{diag}(\overbrace{\pm 1, \dots \pm 1}^{i_0}, \overbrace{\pm 1, \dots \pm 1}^{i_1}, \dots, \overbrace{\pm 1, \dots \pm 1}^{i_r})$$

in $\mathrm{GL}(n+1,\mathbf{C})$.

For any partition (i_0, i_1) of $n+1 = i_0 + i_1$ into a sum of two positive integers $i_0 \geq i_1 \geq 1$, let $L[i_0, i_1] \subset \mathrm{PGL}(n+1,\mathbf{C})$ be the image in $\mathrm{PGL}(n+1,\mathbf{C})$ of the elementary abelian 2-group

$$L[i_0,i_1]^* = \langle (+1)^{i_0}(-1)^{i_1}, (-1)^{n+1} \rangle$$

in $\mathrm{GL}(n+1,\mathbf{C})$. The centralizer of $L[i_0, i_1]$ is

eq:afamCL

$$C_{\mathrm{PGL}(n+1,\mathbf{C})}L[i_0,i_1] = \begin{cases} \frac{\mathrm{GL}(i_0,\mathbf{C})^2}{\mathrm{GL}(1,\mathbf{C})} \rtimes C_2 & i_0 = i_1 \\ \frac{\mathrm{GL}(i_0,\mathbf{C})\times \mathrm{GL}(i_1,\mathbf{C})}{\mathrm{GL}(1,\mathbf{C})} & i_0 > i_1 \end{cases} \tag{3.5}$$

where the action of

$$C_2 = \left\langle \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \right\rangle$$

interchanges the two $\mathrm{GL}(i_0, \mathbf{C})$-factors. The center of the centralizer of $L[i_0, i_1]$ is

eq:afamZCL

$$ZC_{\mathrm{PGL}(n+1,\mathbf{C})}L[i_0,i_1] = \begin{cases} L[i_0,i_1] & i_0 = i_1 \\ \frac{\mathrm{GL}(1,\mathbf{C})\times \mathrm{GL}(1,\mathbf{C})}{\mathrm{GL}(1,\mathbf{C})} & i_0 > i_1 \end{cases} \tag{3.6}$$

For any partition (i_0, i_1, i_2) of $n+1 = i_0 + i_1 + i_2$ into a sum of three positive integers $i_0 \geq i_1 \geq i_2 \geq 1$ let $P[i_0, i_1, i_2] \subset \mathrm{PGL}(n+1,\mathbf{C})$ be the image in $\mathrm{PGL}(n+1,\mathbf{C})$ of the elementary abelian 2-group

$$P[i_0,i_1,i_2]^* = \langle (+1)^{i_0}(-1)^{i_1}(+1)^{i_2}, (+1)^{i_0}(+1)^{i_1}(-1)^{i_2}, (-1)^{n+1} \rangle$$

in $\mathrm{GL}(n+1,\mathbf{C})$. The centralizer of $P[i_0, i_1, i_2]$ is

eq:afamCP3

$$C_{\mathrm{PGL}(n+1,\mathbf{C})}P[i_0,i_1,i_2] = \frac{\mathrm{GL}(i_0,\mathbf{C}) \times \mathrm{GL}(i_1,\mathbf{C}) \times \mathrm{GL}(i_2,\mathbf{C})}{\mathrm{GL}(1,\mathbf{C})} \tag{3.7}$$

so that the center of the centralizer is

eq:afamZCP3

$$ZC_{\mathrm{PGL}(n+1,\mathbf{C})}P[i_0,i_1,i_2] = \frac{\mathrm{GL}(1,\mathbf{C}) \times \mathrm{GL}(1,\mathbf{C}) \times \mathrm{GL}(1,\mathbf{C})}{\mathrm{GL}(1,\mathbf{C})} \tag{3.8}$$

connected.

For any partition (i_0, i_1, i_2, i_3) of $n+1$ into a sum $n+1 = i_0 + i_1 + i_2 + i_3$ of $n+1$ into a sum of four positive integers $i_0 \geq i_1 \geq i_2 \geq i_3 \geq 1$ let $P[i_0, i_1, i_2, i_3] \subset \mathrm{PGL}(n+1,\mathbf{C})$ be the image in $\mathrm{PGL}(n+1,\mathbf{C})$ of the elementary abelian 2-group

$$P[i_0,i_1,i_2,i_3]^* = \langle (+1)^{i_0}(-1)^{i_1}(+1)^{i_2}(-1)^{i_3}, (+1)^{i_0}(+1)^{i_1}(-1)^{i_2}(-1)^{i_3}, (-1)^{n+1} \rangle$$

The centralizer of $P[i_0,i_1,i_2,i_3]$ is

eq:afamCP

$$C_{\mathrm{PGL}(n+1,\mathbf{C})}P[i_0,i_1,i_2,i_3]=\begin{cases}\frac{\mathrm{GL}(i_0,\mathbf{C})^4}{\mathrm{GL}(1,\mathbf{C})}\rtimes(C_2\times C_2) & i_0=i_1=i_2=i_3\\ \frac{\mathrm{GL}(i_0,\mathbf{C})^2\times\mathrm{GL}(i_2,\mathbf{C})^2}{\mathrm{GL}(1,\mathbf{C})}\rtimes C_2 & i_0=i_1>i_2=i_3\\ \frac{\mathrm{GL}(i_0,\mathbf{C})\times\mathrm{GL}(i_1,\mathbf{C})\times\mathrm{GL}(i_2,\mathbf{C})\times\mathrm{GL}(i_3,\mathbf{C})}{\mathrm{GL}(1,\mathbf{C})} & \text{otherwise}\end{cases}\tag{3.9}$$

where

$$C_2\times C_2=\left\langle\begin{pmatrix}0&E&0&0\\E&0&0&0\\0&0&0&E\\0&0&E&0\end{pmatrix},\begin{pmatrix}0&0&E&0\\0&0&0&E\\E&0&0&0\\0&E&0&0\end{pmatrix}\right\rangle,\qquad C_2=\left\langle\begin{pmatrix}0&E&0&0\\E&0&0&0\\0&0&0&E\\0&0&E&0\end{pmatrix}\right\rangle$$

The center of the centralizer of $P[i_0,i_1,i_2,i_3]$ is

eq:afamZCP

$$ZC_{\mathrm{PGL}(n+1,\mathbf{C})}P[i_0,i_1,i_2,i_3]=\tag{3.10}$$
$$\begin{cases}P[i_0,i_1,i_2,i_3] & i_0=i_1=i_2=i_3\\ \frac{\mathrm{GL}(1,\mathbf{C})\times\mathrm{GL}(1,\mathbf{C})}{\mathrm{GL}(1,\mathbf{C})}\times\langle(+1)^{i_0}(-1)^{i_1}(+1)^{i_2}(-1)^{i_3}\rangle & i_0=i_1>i_2=i_3\\ \frac{\mathrm{GL}(1,\mathbf{C})\times\mathrm{GL}(1,\mathbf{C})\times\mathrm{GL}(1,\mathbf{C})\times\mathrm{GL}(1,\mathbf{C})}{\mathrm{GL}(1,\mathbf{C})} & \text{otherwise}\end{cases}$$

We collect the information about the toral subcategory that we shall need later on in the folllowing proposition. Let $P(m,k)$ denote the number of partitions of m into sums of k natural integers.

prop:afamtoral

3.9. PROPOSITION. *The category $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$ contains precisely*

- $P(n+1,2)$ *isomorphism classes of toral rank one objects represented by the lines $L[i_0,i_1]$.*
- $P(n+1,3)+P(n+1,4)$ *isomorphism classes of toral rank two objects represented by the planes $P[i_0,i_1,i_2]$ and $P[i_0,i_1,i_2,i_3]$.*

The centralizers of these objects are listed in (3.5), (3.7), and (3.9).

The automorphism groups are easily computed using complex representation theory because

$$\mathbf{A}(\mathrm{GL}(n+1,\mathbf{C}))(P[i_0,i_1,i_2,i_3]^*)\to\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))(P[i_0,i_1,i_2,i_3])$$

is surjective (as in 3.16). One finds that

$$\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))P[i_0,i_1,i_2,i_3]=\begin{cases}\mathrm{GL}(2,\mathbf{F}_2) & \ell(i_0,i_1,i_2,i_3)\ge 3\\ C_2 & \ell(i_0,i_1,i_2,i_3)=2\\ \{1\} & \ell(i_0,i_1,i_2,i_3)=1\end{cases}$$

where $\ell(i_0,i_1,i_2,i_3)=\max_{1\le j\le 4}\#\{k\mid i_k=i_j\}$ is the maximal number of repetitions in the sequence (i_0,i_1,i_2,i_3). This formula also holds for the objects $P[i_0,i_1,i_2]$ when interpreted as $P[i_0,i_1,i_2,0]$.

sec:afamlhs

2. Centralizers of objects of $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\le t}_{\le 2}$ are LHS

In this section we check that all toral objects of rank ≤ 2 have LHS (2.§2.26) centralizers.

ex:LHS

3.10. LEMMA. *The centralizers of the objects of $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\le t}_{\le 2}$,*

(1) $\frac{\mathrm{GL}(i,\mathbf{C})^2}{\mathrm{GL}(1,\mathbf{C})}\rtimes C_2$ *(3.5),*
(2) $\frac{\mathrm{GL}(i,\mathbf{C})^4}{\mathrm{GL}(1,\mathbf{C})}\rtimes(C_2\times C_2)$ *(3.9),*
(3) $\frac{\mathrm{GL}(i_0,\mathbf{C})^2\times\mathrm{GL}(i_2,\mathbf{C})^2}{\mathrm{GL}(1,\mathbf{C})}\rtimes C_2$ *(3.9)*

are LHS.

PROOF. (1) Let $X=\frac{\mathrm{GL}(i,\mathbf{C})^2}{\mathrm{GL}(1,\mathbf{C})}\rtimes C_2$, $i\ge 1$, where the C_2-action switches the two $\mathrm{GL}(i,\mathbf{C})$-factors. For $i=1$, X is a 2-compact toral group, hence LHS. For $i=2$ explicit computer computation yields

$\frac{\mathrm{GL}(i,\mathbf{C})^2}{\mathrm{GL}(1,\mathbf{C})}\rtimes C_2$	$H^1(\pi;\check{T}^{W_0})$	$H^1(W;\check{T})$	$H^1(W_0;\check{T})^{\pi}$	$H^1(W_0;\check{T})$
$i=2$	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$

so X is manifestly LHS in this case (even though X_0 is not regular). For $i > 2$, $\theta(X_0)$ is bijective and thus X is LHS by 2.28. ($\theta(X_0)$ is injective by 2.21.(1) and surjective by 2.21.(2) for $i \neq 4$ and for $i = 4$ by inspection or by 2.23 and 2.24 for all $i > 2$.)

(2) Let $X = \frac{\mathrm{GL}(i,\mathbf{C})^4}{\mathrm{GL}(1,\mathbf{C})} \rtimes (C_2 \times C_2)$, $i \geq 1$, where $C_2 \times C_2 = \langle (12)(34), (13)(24) \rangle$ permutes the four $\mathrm{GL}(i,\mathbf{C})$-factors. For $i = 1$, X is a 2-compact toral group, hence LHS. For $i = 2$ explicit computer computation yields

$\frac{\mathrm{GL}(i,\mathbf{C})^4}{\mathrm{GL}(1,\mathbf{C})} \rtimes (C_2 \times C_2)$	$H^1(\pi; \check{T}^{W_0})$	$H^1(W; \check{T})$	$H^1(W_0; \check{T})^\pi$	$H^1(W_0; \check{T})$
$i = 2$	$\mathbf{Z}/2$	$(\mathbf{Z}/2)^3$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^8$

so X is manifestly LHS in this case. (Alternatively, observe that X_0 is regular (2.24, 2.23), the kernel of $\theta(X_0)$ is $(\mathbf{Z}/2)^4$, and $\theta(X_0)^\pi$ is surjective because $H^1(C_2 \times C_2; (\mathbf{Z}/2)^4) = 0$ for the regular representation.) For $i > 2$, we see as in 3.10.(1) above that $\theta(X_0)$ is bijective and hence X is LHS by 2.28.

(3) Let $X = \frac{(\mathrm{GL}(i_0,\mathbf{C}) \times \mathrm{GL}(i_2,\mathbf{C}))^2}{\mathrm{GL}(1,\mathbf{C})} \rtimes C_2$, $1 \leq i_0 < i_2$, where C_2 switches the two identical factors. Using 2.23 and 2.24 we see (details omitted) that X_0 is regular. By 2.21.(1), $\theta(X_0)$ is in fact bijective except when i_0 or i_2 is 2. In those cases, the kernel of $\theta(X_0)$ is $(\mathbf{Z}/2)^2$ and $\theta(X_0)^{C_2}$ is surjective as $H^1(C_2; (\mathbf{Z}/2)^2) = 0$ for the regular representation. Therefore X is LHS by 2.28. □

sec:afamlim

3. Limits over the Quillen category of $\mathrm{PGL}(n+1,\mathbf{C})$

In this section we show that the problem of computing the higher limits of the functors $\pi_i(BZC_{\mathrm{PGL}(n+1,\mathbf{C})})$, $i = 1, 2$, (2.47) is concentrated on the nontoral objects of the Quillen category.

lemma:pijB

3.11. Lemma. [**47**, 2.8] *Let $V \subset P\Delta_{n+1}$ be a nontrivial subgroup representing an object of $\mathbf{A}(\Sigma_{n+1}, P\Delta_{n+1}) = \mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\leq t}$ (3.4). Then*

$$\check{Z}C_{\mathrm{PGL}(n+1,\mathbf{C})}(V) = \check{T}^{\Sigma_{n+1}(V)}$$

where $\check{T} = \check{T}(\mathrm{PGL}(n,\mathbf{C}))$ is the discrete approximation [**18**, §3] *to the maximal torus of* $\mathrm{PGL}(n+1,\mathbf{C})$ *and $\Sigma_{n+1}(V)$ is the point-wise stabilizer subgroup (2.68).*

Proof. Let $\nu^* \colon V \to T(\mathrm{GL}(n+1,\mathbf{C}))$ be a lift to $\mathrm{GL}(n+1,\mathbf{C})$ of the inclusion homomorphism of V into $T(\mathrm{PGL}(n+1,\mathbf{C}))$. Then

$$C_{\mathrm{GL}(n+1,\mathbf{C})}(\nu^* V) = \prod_{\rho \in V^\vee} \mathrm{GL}(i_\rho, \mathbf{C}), \qquad \Sigma_{n+1}(\nu^* V) = \prod_{\rho \in V^\vee} \Sigma_{i_\rho}$$

where $i \colon V^\vee \to \mathbf{Z}$ records the multiplicity of each linear character $\rho \in V^\vee$ in the representation ν^*. Using [**47**, 5.11] and 9.20 we get that

$$C_{\mathrm{PGL}(n+1,\mathbf{C})}(V) = \frac{C_{\mathrm{GL}(n+1,\mathbf{C})}(\nu^* V)}{\mathrm{GL}(1,\mathbf{C})} \rtimes V^\vee_{\nu^*}, \qquad \Sigma_{n+1}(V) = \Sigma_{n+1}(\nu^* V) \rtimes V^\vee_{\nu^*}$$

where $V^\vee_{\nu^*} = \{\zeta \in V^\vee = \mathrm{Hom}(V, \mathrm{GL}(1,\mathbf{C})) \mid \forall \rho \in V^\vee \colon i_{\zeta\rho} = i_\rho\}$. The semi-direct products are obtained because the elements of $V^\vee_{\nu^*}$ can be effectuated by permutations from Σ_{n+1} that fix $V \subset \mathrm{PGL}(n+1,\mathbf{C})$ point-wise. The discrete approximation [**18**, §3] to the center of the centralizer is therefore

$$\begin{aligned}
\check{Z}C_{\mathrm{PGL}(n+1,\mathbf{C})}(V) &= \check{Z}\left(\frac{\prod \mathrm{GL}(i_\rho, \mathbf{C})}{\mathrm{GL}(1,\mathbf{C})} \rtimes V^\vee_{\nu^*}\right) \overset{(9.14)}{=} \check{Z}\left(\frac{\prod \mathrm{GL}(i_\rho, \mathbf{C})}{\mathrm{GL}(1,\mathbf{C})}\right)^{V^\vee_{\nu^*}} \\
&= \left(\frac{\prod \check{Z}\mathrm{GL}(i_\rho, \mathbf{C})}{\mathrm{GL}(1,\mathbf{C})}\right)^{V^\vee_{\nu^*}} = \left(\frac{\check{T}(\mathrm{GL}(n+1,\mathbf{C}))^{\Sigma_{n+1}(\nu^* V)}}{\mathrm{GL}(1,\mathbf{C})}\right)^{V^\vee_{\nu^*}} \\
&= \left(\check{T}(\mathrm{PGL}(n+1,\mathbf{C}))^{\Sigma_{n+1}(\nu^* V)}\right)^{V^\vee_{\nu^*}} = \check{T}(\mathrm{PGL}(n+1,\mathbf{C}))^{\Sigma_{n+1}(V)}
\end{aligned}$$

where the penultimate equality sign is justified by the fact that $H^1(\Sigma_{n+1}(\nu^* V); \mathrm{GL}(1,\mathbf{C})) \to H^1(\Sigma_{n+1}(\nu^* V); \check{T}(\mathrm{GL}(n+1,\mathbf{C})))$ is injective. □

Define $\pi_i(\mathrm{B}ZC_{\mathrm{PGL}(n+1,\mathbf{C})})_{\not\leq t}$ to be the subfunctor of $\pi_i(\mathrm{B}ZC_{\mathrm{PGL}(n+1,\mathbf{C})})$ (2.47) that vanishes on all toral objects and is unchanged on all nontoral objects of the Quillen category. This means that

lefn:nontoralsub

$$\pi_i(\mathrm{B}ZC_{\mathrm{PGL}(n+1,\mathbf{C})})_{\not\leq t}(V,\nu) = \begin{cases} 0 & (V,\nu) \text{ is toral} \\ \pi_i(\mathrm{B}ZC_{\mathrm{PGL}(n+1,\mathbf{C})})(V,\nu) & (V,\nu) \text{ is nontoral} \end{cases} \tag{3.12}$$

for all objects (V,ν) of $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$. The reason for introducing this subfunctor is that in the computation of the higher limits, we can ignore the toral objects.

orallimforpglnC

3.13. Corollary. *When* $n > 1$,

$$\lim{}^*(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C})); \pi_i(BZC)_{\not\leq t}) \cong \lim{}^*(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C})); \pi_i(BZC)), \quad i = 1,2,$$

where $\pi_i(BZC) = \pi_i(\mathrm{B}ZC_{\mathrm{PGL}(n+1,\mathbf{C})})$ *(2.47).*

Proof. The result of Lemma 3.11 is (2.31) equivalent to

$$\pi_i(BZC)(V) = H^{2-i}(\Sigma_{n+1}(V); L), \qquad V \subset P\Delta_{n+1},$$

where L is the $\mathbf{Z}_2\Sigma_{n+1}$-module $\pi_2 BT(\mathrm{PGL}(n+1,\mathbf{C}))$ and therefore (2.69)

$$\lim{}^j(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\leq t}, \pi_i(\mathrm{B}ZC)) = \begin{cases} H^{2-i}(\Sigma_{n+1}, L) & j = 0 \\ 0 & j > 0 \end{cases}$$

where the cohomology groups $H^{2-i}(\Sigma_{n+1}; L)$, $i = 1,2$, are trivial for $n > 1$ (3.1).

Since the quotient functor $\pi_i(\mathrm{B}ZC)/\pi_i(\mathrm{B}ZC)_{\not\leq t}$ vanishes on all nontoral objects

$$\lim{}^j(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C})), \pi_i(\mathrm{B}ZC)/\pi_i(\mathrm{B}ZC)_{\not\leq t}) \overset{[47,\ 13.12]}{\cong} \lim{}^j(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\leq t}, \pi_i(\mathrm{B}ZC))$$

We conclude that $\lim^*(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C})), \pi_i(\mathrm{B}ZC)/\pi_i(\mathrm{B}ZC)_{\not\leq t}) = 0$. The long exact coefficient functor sequence for higher limits now shows that $\lim^*(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C})), \pi_i(\mathrm{B}ZC)_{\not\leq t})$ and $\lim^*(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C})), \pi_i(\mathrm{B}ZC))$ are isomorphic. □

4. The category $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{[\ ,\ ]\neq 0}$

:nontoralapglnc

For any nontrivial elementary abelian 2-group V in $\mathrm{PGL}(n+1,\mathbf{C})$, let $[\ ,\ ]\colon V \times V \to \mathbf{F}_2$ be the symplectic bilinear form [28, II.9.1] given by $[u\mathbf{C}^\times, v\mathbf{C}^\times] = r$ if $[u,v] = (-E)^r$ where $u, v \in \mathrm{GL}(n+1,\mathbf{C})$ are such that $u\mathbf{C}^\times$, $v\mathbf{C}^\times \in V$. (The elements $[u,v]$ and u^2 lie in the center $\mathbf{C}^\times$ of $\mathrm{GL}(n+1,\mathbf{C})$ so that $E = [u^2,v] = [u,v]^u[u,v] = [u,v]^2$ and thus $[u,v] \in \mathbf{C}^\times$ has order 2. Therefore $[u,v] = [u,v]^{-1} = [v,u]$.)

altform

3.14. Lemma. V *in* $\mathrm{PGL}(n+1,\mathbf{C})$ *is toral* $\iff$ $[V,V] = 0$

Proof. Let $e_i\mathbf{C}^\times$, $1 \leq i \leq d$, be a basis for V. Since $\mathbf{C}^\times$ is divisible, we can assume that each $e_i \in \mathrm{GL}(n+1,\mathbf{C})$ has order 2. If $[V,V] = 0$, these e_is commute and span a lift to $\mathrm{GL}(n+1,\mathbf{C})$ of $V \subseteq \mathrm{PGL}(n+1,\mathbf{C})$. □

An extra special 2-group is of *positive type* if it is isomorphic to a central product of dihedral groups D_8 of order 8 [56, p 145–146].

griesslemma

3.15. Lemma. [23, 3.1] [47, 5.4] *Let* $\nu\colon V \to \mathrm{PGL}(n,\mathbf{C})$ *be a nontoral monomorphism of a nontrivial elementary abelian* 2*-group* V *into* $\mathrm{PGL}(n+1,\mathbf{C})$*. Then there exists a morphism of short exact sequences of groups*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z(P) & \longrightarrow & PE & \longrightarrow & V & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \nu} & & \\ 1 & \longrightarrow & \mathbf{C}^\times & \longrightarrow & \mathrm{GL}(n+1,\mathbf{C}) & \longrightarrow & \mathrm{PGL}(n+1,\mathbf{C}) & \longrightarrow & 1 \end{array}$$

where PE *is the direct product of an extra special* 2*-group* $P \subseteq \mathrm{GL}(n+1,\mathbf{C})$ *of positive type and an elementary abelian* 2*-group* $E \subseteq \mathrm{GL}(n+1,\mathbf{C})$ *with* $P \cap E = \{1\} = [P,E]$.

Let $G = \langle P, E, i\rangle = P \circ C_4 \times E$ be the group generated by E and the central product $P \circ C_4$ of P and the cyclic group $C_4 = \langle i\rangle \subseteq \mathbf{C}^\times$ with $C_2 = \langle -E\rangle$ amalgamated. The image of G in $\mathrm{PGL}(n+1,\mathbf{C})$ is V.

Let $\mathbf{A}(\mathrm{GL}(n,\mathbf{C}))(G)$ be the subgroup, isomorphic to $N_{\mathrm{GL}(n+1,\mathbf{C})}(G)/G \cdot C_{\mathrm{GL}(n+1,\mathbf{C})}(G)$, of $\mathrm{Out}(G)$ consisting of all outer automorphisms of G induced from conjugation in $\mathrm{GL}(n+1,\mathbf{C})$ [47, 5.8]. In other words, $\mathbf{A}(\mathrm{GL}(n,\mathbf{C}))(G) = \mathrm{Out}_{\mathrm{tr}}(G)$ is the group of trace preserving outer automorphisms of G.

Qauto

3.16. LEMMA. $\mathbf{A}(\mathrm{GL}(n+1,\mathbf{C}))(G) \to \mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))(V)$ *is surjective.*

PROOF. Suppose that $B \in \mathrm{GL}(n+1,\mathbf{C})$ is such that $V^{B\mathbf{C}^\times} = V$. Then $G^B \subseteq G\cdot\mathbf{C}^\times$: For any $g \in G$ there exist $h \in G$ and $z \in \mathbf{C}^\times$ such that $g^B = hz$. But since G has exponent 4, $z^4 = 1$ so $z \in C_4$ and $g^B \in G$. □

A monomorphic conjugacy class $\nu\colon V \to \mathrm{PGL}(n+1,\mathbf{C})$ is said to be a $(2d+r,r)$ object of $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$ if the underlying symplectic vector space of (V,ν) is isomorphic to $V = H^d \times V^\perp$ where H denotes the symplectic plane over $\mathbf{F}_2$ and $\dim_{\mathbf{F}_2} V^\perp = r$ [28, II.9.6] (so that $\dim_{\mathbf{F}_2} V = 2d+r$). An (r,r) object is the same thing as an r-dimensional toral object. We write $\mathrm{Sp}(V)$ or $\mathrm{Sp}(2d+r,r)$ (abbreviated to $\mathrm{Sp}(2d)$ if $r=0$) for the group of linear automorphisms of V that preserve the symplectic form.

nonTpglnC

3.17. COROLLARY. *Suppose that $n+1 = 2^d m$ for some natural numbers $d \geq 1$ and $m \geq 1$.*

(1) *There is up to isomorphism a unique $(2d,0)$ object H^d of $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$, and*

$$\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))(H^d) = \mathrm{Sp}(2d), \quad C_{\mathrm{PGL}(n+1,\mathbf{C})}(H^d) = H^d \times \mathrm{PGL}(m,\mathbf{C})$$

for this object.

nonTpglnC2

(2) *Isomorphism classes of $(2d+r,r)$, $r>0$, objects V of $\mathbf{A}(\mathrm{PGL}(2^d m,\mathbf{C}))$ correspond bijectively to isomorphism classes of (r,r) objects $V^\perp$ of $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))$, and*

$$\mathbf{A}(\mathrm{PGL}(2^d m,\mathbf{C}))(V) = \begin{pmatrix} \mathrm{Sp}(2d) & 0 \\ * & \mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(V^\perp)\end{pmatrix}$$
$$C_{\mathrm{PGL}(2^d m,\mathbf{C})}(V) = V/V^\perp \times C_{\mathrm{PGL}(m,\mathbf{C})}(V^\perp)$$

for these objects.

PROOF. 1. The group $2_+^{1+2d} \circ 4$ has [29, 7.5] 2^{1+2d} characters of degree 1 and 2 irreducible characters of degree 2^d (interchanged by the action of $\mathrm{Out}(2_+^{1+2d} \circ 4) \cong \mathrm{Sp}(2d) \times \mathrm{Aut}(C_4)$ [22, pp. 403–404]) given by

$$\chi_\lambda(g) = \begin{cases} 2^d\lambda(g) & g \in C_4 \\ 0 & g \notin C_4 \end{cases}$$

where $\lambda\colon C_4 \to \mathbf{C}^\times$ is an injective group homomorphism ($\lambda(i) = \pm i$). The linear characters vanish on the derived group $2 = [2_+^{1+2d} \circ 4, 2_+^{1+2d} \circ 4]$ but the irreducible characters of degree 2^d do not. Thus the only faithful representations of $2_+^{1+2d} \circ 4$ with central centers are multiples $m\chi_\lambda$ of χ_λ for a fixed λ. Phrased slightly differently, $\mathrm{GL}(m2^d,\mathbf{C})$ contains up to conjugacy a unique subgroup with central center isomorphic to $2_+^{1+2d} \circ 4$. For this group and its image H^d in $\mathrm{PGL}(2^d m,\mathbf{C})$ we have

$$\mathbf{A}(\mathrm{GL}(m2^d,\mathbf{C}))(2_+^{1+2d} \circ 4, 2_+^{1+2d} \circ 4) \cong \mathrm{Sp}(2d) \cong \mathbf{A}(\mathrm{PGL}(m2^d,\mathbf{C}))(H^d, H^d)$$
$$C_{\mathrm{GL}(m2^d,\mathbf{C})}(2_+^{1+2d} \circ 4) \cong \mathrm{GL}(m,\mathbf{C}), \quad C_{\mathrm{PGL}(m2^d,\mathbf{C})}(H^d) \cong H^d \times \mathrm{PGL}(m,\mathbf{C})$$

where the last isomorphism is a consequence of [47, 5.9].

2. The $(2d+r,r)$ object (V,ν) of $\mathbf{A}(\mathrm{PGL}(2^d m,\mathbf{C}))$ and the $(r,0)$ object $(V^\perp,\nu^\perp)$ of $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))$ correspond to each other iff there is an m-dimensional representation $\mu\colon V^\perp \to \mathrm{GL}(m,\mathbf{C})$ such that $\mathbf{C}^{2^d} \otimes \mu$ is a lift of $\nu|V^\perp$ and μ a lift of $\nu^\perp$. According to 3.15 any lift of $\nu|V^\perp$ has this form for some μ uniquely determined up to the action of $(V^\perp)^\vee$.

We use 3.16 to calculate the Quillen automorphism group of a $(2d+r,r)$ object $H^d \times V^\perp$ of $\mathbf{A}(\mathrm{PGL}(2^d m,\mathbf{C}))$. Let $H^d \times V^\perp$ be covered by the group $P \circ C_4 \times V^\perp$ as in 3.15. Let α be an automorphism of $P \circ C_4$, let β be any homomorphism of the form $P \circ C_4 \to H^d \to V^\perp$, and let γ be any Quillen automorphism of $(V^\perp, \nu^\perp)$. Choose a homomorphism $\zeta_1 \colon P \circ C_4 \to H^d \times C_4/C_2 \to C_4$ such that $\lambda(\zeta_1(x)\alpha(x)) = \lambda(x)$ for all $x \in C_4$ and a homomorphism $\zeta_2 \colon V^\perp \to C_4$ such that $\lambda(\zeta_2(v))\mu(\gamma(v)) = \mu(v)$ for all $v \in V^\perp$. Then the automorphism of $P \circ C_4$ that takes (x, v) to $(\zeta_1(x)\zeta_2(v)\alpha(x), \beta(x) + \gamma(v))$ preserves the trace of $\chi_\lambda \# \mu$ and therefore the automorphism induced on the quotient is a Quillen automorphism of $H^d \times V^\perp$. Conversely, any automorphism of $P \circ C_4 \times V^\perp$ takes the center $C_4 \times V^\perp$ isomorphically to itself and hence it is of the form $(x, v) \to (\zeta(x, v)\alpha(x), \beta(x) + \gamma(v))$ for some automorphism α of $P \circ C_4$, some homomorphism $\beta \colon P \circ C_4 \to V^\perp$ vanishing on C_4, and some homomorphism $\zeta \colon P \circ C_4 \times V^\perp \to C_4$. Such an automorphism preserves the trace of $\chi_\lambda \# \mu$ iff $\lambda(\zeta(x, v)\alpha(x)) = \mu(\gamma(v))$ for all $(x, v) \in Z(P \circ C_4 \times V^\perp) = C_4 \times V^\perp$. But this means that the induced automorphism of $H^d \times V^\perp$ is of the stated form. □

We conclude that the nontoral objects of $\mathbf{A}(\mathrm{PGL}(2m,\mathbf{C}))$ of rank ≤ 4 are

- One $(2,0)$ object H, $\mathbf{A}(\mathrm{PGL}(2m,\mathbf{C}))(V) = \mathrm{Sp}(2)$,
- $P(m,2)$ $(3,1)$ objects V, $\mathbf{A}(\mathrm{PGL}(2m,\mathbf{C}))(V) = \mathrm{Sp}(3,1)$,
- $P(m,3)+P(m,4)$ $(4,2)$ objects E, $\mathbf{A}(\mathrm{PGL}(2m,\mathbf{C}))(E) = \begin{pmatrix} \mathrm{Sp}(2) & 0 \\ * & \mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(E^\perp) \end{pmatrix}$ where $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(E^\perp) = 1$, C_2, or $\mathrm{GL}(E^\perp)$,
- One $(4,0)$ object H^2 if m is even, $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(H^2) = \mathrm{Sp}(4)$.

This information will be needed in the next section as input for Oliver's cochain complex [53] for computing higher limits of the functors $\pi_i(\mathrm{B}ZC_{\mathrm{PGL}(n+1,\mathbf{C})})_{\not\leq t}$.

sec:Afam

5. Higher limits of the functor $\pi_i BZC_{\mathrm{PGL}(n+1,\mathbf{C})}$ on $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{[\ ,\ ]\neq 0}$

We compute the higher limits from 2.48.(2) and 2.51.(4) by means of 3.13 and Oliver's cochain complex [53].

lim=0 3.18. Lemma. *The low degree higher limits of the functors* $\pi_i BZC_{\mathrm{PGL}(n+1,\mathbf{C})}$, $i = 1, 2$, *are:*

(1) $\lim^j(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C})), \pi_1 BZC_{\mathrm{PGL}(n+1,\mathbf{C})}) = 0$ *for* $j = 1, 2$,
(2) $\lim^j(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C})), \pi_2 BZC_{\mathrm{PGL}(n+1,\mathbf{C})}) = 0$ *for* $j = 2, 3$,

for all $n \geq 1$.

For any elementary abelian 2-group E in $\mathrm{PGL}(n+1,\mathbf{C})$ we shall write

$$[E] = \mathrm{Hom}_{\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))(E)}(\mathrm{St}(E), \pi_1(BZC_{\mathrm{PGL}(n+1,\mathbf{C})}(E)))$$

for the $\mathbf{F}_2$-vector space of $\mathbf{F}_2\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))(E)$-module homomorphisms from the Steinberg representation $\mathrm{St}(E)$ over $\mathbf{F}_2$ of $\mathrm{GL}(E)$ to $\pi_1(BZC_{\mathrm{PGL}(n+1,\mathbf{C})}(E))$.

Oliver's cochain complex for computing the first limits of the functor $\pi_1(BZC_{\mathrm{PGL}(n+1,\mathbf{C})})_{\not\leq t}$ has the form

bobccc
$$0 \to [H] \xrightarrow{d^1} \prod_{1\leq i\leq [m/2]} [H\#L[m-i,i]] \xrightarrow{d^2} [H\#P[1,1,m-2]] \times \prod_{2<i<[m/2]} [H\#P[1,i-1,m-i]] \tag{3.19}$$

where we only list some of the nontoral rank four objects. Here,

$$[H] = \mathrm{Hom}_{\mathrm{Sp}(2)}(\mathrm{St}(H), H) \cong \mathbf{F}_2$$
$$[H\#L[m-i,i]] = \mathrm{Hom}_{\mathrm{Sp}(3,1)}(\mathrm{St}(V), V) \cong \mathbf{F}_2, \quad V = H\#L[m-i,i],$$
$$[H\#P[1,1,m-2]] = \mathrm{Hom}_{\begin{pmatrix}\mathrm{Sp}(2) & 0\\ * & C_2\end{pmatrix}}(\mathrm{St}(E_2), E_2/E_2^{\perp}) \cong \mathbf{F}_2, \quad E_2 = H\#P[1,1,m-2],$$
$$[H\#P[1,i-1,m-i]] = \mathrm{Hom}_{\begin{pmatrix}\mathrm{Sp}(2) & 0\\ * & 1\end{pmatrix}}(\mathrm{St}(E_i), E_i) \cong \mathbf{F}_2 \times \mathbf{F}_2, \quad E_i = H\#P[1,i-1,m-i]$$

where $2 < i \leq [m/2]$ in the last line. The dimensions of these spaces were found using the computer algebra program *magma*. It suffices to show that the first differential d^1 is injective and that the second differential d^2 has rank $[m/2] - 1$.

Let $H = \mathbf{F}_2 e_1 + \mathbf{F}_2 e_2$ be a 2-dimensional vector space over $\mathbf{F}_2$ with basis $\{e_1, e_2\}$ and symplectic inner product matrix

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

Let $\mathbf{F}_2[1]$ be the 3-dimensional $\mathbf{F}_2$-vector space on all length zero flags $[L]$ of nontrivial proper subspaces $L \subset H$. The Steinberg module $\mathrm{St}(H)$ for H is the 2-dimensional $\mathbf{F}_2\mathrm{GL}(H)$-module that is the kernel for the augmentation $d\colon \mathbf{F}_2[1] \to \mathbf{F}_2$ given by $d[L] = 1$ for all L. Let $f\colon \mathrm{St}(H) \to H$ be the restriction to $\mathrm{St}(H)$ of the $\mathbf{F}_2\mathrm{GL}(H)$-module homomorphism $\overline{f}\colon \mathbf{F}_2[1] \to H$ given by $\overline{f}[L] = L$.

Let $V = \mathbf{F}_2 e_1 + \mathbf{F}_2 e_2 + \mathbf{F}_2 e_3$ be a 3-dimensional vector space over $\mathbf{F}_2$ with basis $\{e_1, e_2, e_3\}$ and (degenerate) symplectic inner product matrix

$$\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

Let $\mathbf{F}_2[1]$ be the 21-dimensional $\mathbf{F}_2$-vector space on all length one flags $[P > L]$ and $\mathbf{F}_2[0]$ the 14-dimensional $\mathbf{F}_2$-vector space on all length zero flags, $[P]$ or $[L]$, of non-trivial and proper subspaces of V. The Steinberg module $\mathrm{St}(V)$ over $\mathbf{F}_2$ for V is the $2^3 = 8$-dimensional kernel of the linear map $d\colon \mathbf{F}_2[1] \to \mathbf{F}_2[0]$ given by $d[P > L] = [P] + [L]$. Define $df\colon \mathrm{St}(V) \to V$ to be the restriction to $\mathrm{St}(V)$ of the linear map $\overline{df}\colon \mathbf{F}_2[1] \to V$ given by

eq:defdfafam
$$\overline{df}[P > L] = \begin{cases} L & P \cap P^{\perp} = \{0\} \\ 0 & \text{otherwise} \end{cases} \tag{3.20}$$

on the basis vectors.

Let $E = \mathbf{F}_2e_1 + \mathbf{F}_2e_2 + \mathbf{F}_2e_3 + \mathbf{F}_2e_4$ be a 4-dimensional vector space over $\mathbf{F}_2$ with basis $\{e_1, e_2, e_3, e_4\}$ and (degenerate) symplectic inner product matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Let $\mathbf{F}_2[2]$ be the 315-dimensional $\mathbf{F}_2$-vector space on all length two flags $[V > P > L]$ and $\mathbf{F}_2[1]$ the also 315-dimensional $\mathbf{F}_2$-vector space on all length one flags, $[P > L]$ or $[V > L]$ or $[V > P]$, of non-trivial, proper subspaces of E. The Steinberg module $\mathrm{St}(E)$ over $\mathbf{F}_2$ for E is the $2^6 = 64$-dimensional kernel of the linear map $d\colon \mathbf{F}_2[2] \to \mathbf{F}_2[1]$ given by $d[V > P > L] = [P > L] + [V > L] + [V > P]$. Define $F_1 = \overline{F}_1|\,\mathrm{St}(E)\colon \mathrm{St}(E) \to E$ as the restriction to $\mathrm{St}(E)$ of the linear map $\overline{F}_1\colon \mathbf{F}_2[2] \to E$ with values

defF1 (3.21)
$$\overline{F}_1[V > P > L] = \begin{cases} L & P \cap P^\perp = 0, V \cap V^\perp = \mathbf{F}_2e_3 \\ 0 & \text{otherwise} \end{cases}$$

on the basis elements. Define $F_2 = \overline{F}_2|\,\mathrm{St}(E)\colon \mathrm{St}(E) \to E$ similarly but replace the condition $V \cap V^\perp = \mathbf{F}_2e_3$ by $V \cap V^\perp = \mathbf{F}_2e_4$. The linear maps F_1 and F_2 are $\begin{pmatrix} \mathrm{Sp}(2) & 0 \\ * & 1 \end{pmatrix}$-equivariant because this group preserves the symplectic inner product on E and preserves $V^\perp = \mathbf{F}_2\langle e_3, e_4\rangle$ pointwise.

lemma:fF1F2basis 3.22. Lemma. *Let f and F_1, F_2 be the linear maps defined above.*

(1) *The vector f is a basis for $[H]$.*
(2) *The vector df is a basis for $[H\#L[m-i,i]]$, $1 \leq i \leq [m/2]$.*
(3) *The vector F_2 is a basis for $[H\#P[1,1,m-2]]$.*
(4) *The set $\{F_1, F_2\}$ is a basis for $[H\#P[1,i-1,m-i]]$, $2 < i \leq [m/2]$. The sum $F_1 + F_2$ is the linear map defined as in (3.21) but with condition $V \cap V^\perp = \mathbf{F}_2e_3$ replaced by $V \cap V^\perp = \mathbf{F}_2(e_3 + e_4)$.*

Proof. This can be directly verified by machine computation. □

Proof of Lemma 3.18. Since we already know that these higher limits vanish when $n+1$ is odd (3.3, 3.13) we can assume that $n + 1 = 2m$ is even.
(1) See 8.2 for the case $m = 1$ and assume now that $m \geq 2$. The image in $[H\#L[m-i,i]]$ of $f \in [H]$ is

$$df_{L[m-i,i]}[P > L] = \begin{cases} L & P = H \\ 0 & \text{otherwise} \end{cases}$$

which equals df (3.20). For $1 < i \leq [m/2]$, let

$$ddf_{L[m-i,i]}[V > P > L] = \begin{cases} L & V = H\#L[m-i,i],\ P = H \\ 0 & \text{otherwise} \end{cases}$$

The object $H\#P[1,1,m-2]$ receives morphisms from $H\#L[m-1,1]$ and (when $m > 2$) $H\#L[m-2,2]$. Using a computer program one easily checks that $ddf_{L[m-1,1]} = F_2 = ddf_{L[m-2,2]}$ in $[H\#P[1,1,m-2]]$. The object $H\#P[1,i-1,m-i]$ receives morphisms from $H\#L[m-1,1]$, $H\#L[m-i+1,i-1]$, and $H\#L[m-i,i]$. Using a computer program one easily checks that $ddf_{L[m-i+1,i-1]} = F_1$, $ddf_{L[m-i,i]} = F_1$, and $ddf_{L[m-1,1]} = F_1 + F_2$ in $[H\#P[1,i-1,m-i]$. For $m = 2$ or $m = 3$, the cochain complexes (3.19) take the form

$$0 \to [H] \xrightarrow{d^1} [H\#L[1,1]] \xrightarrow{d^2} 0$$
$$0 \to [H] \xrightarrow{d^1} [H\#L[2,1]] \xrightarrow{d^2} [H\#P[1,1,1]]$$

where d^1 is an isomorphism. For $m \geq 4$, and with our choice of basis (3.22), the matrix for the differential d^1 is the injective $(1 \times [m/2])$-matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}$$

and the matrix for d^2 (or rather, the components of d^2 shown in 3.19) is the $([m/2] \times (2[m/2]-3))$-matrix

	$[H\#P[1,1,8]]$	$[H\#P[1,2,7]]$	$[H\#P[1,3,6]]$	$[H\#P[1,4,5]]$
$[H\#L[9,1]]$	(1)	$(1\ \ 1)$	$(1\ \ 1)$	$(1\ \ 1)$
$[H\#L[8,2]]$	(1)	$(1\ \ 0)$		
$[H\#L[7,3]]$		$(0\ \ 1)$	$(1\ \ 0)$	
$[H\#L[6,4]]$			$(0\ \ 1)$	$(1\ \ 0)$
$[H\#L[5,5]]$				$(0\ \ 1)$

(shown here for $m = 10$) of rank $[m/2] - 1$.
(2) Oliver's cochain complex for computing these higher limits over $\mathbf{A}(\mathrm{PGL}(2m, \mathbf{C}))$ involve the $\mathbf{Z}_2$-modules (3.17.(2))

$$\mathrm{Hom}_{\begin{pmatrix} \mathrm{Sp}(2) & 0 \\ * & \mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(E^{\perp}) \end{pmatrix}}(\mathrm{St}(E), \pi_2(BZC_{\mathrm{PGL}(2m,\mathbf{C})}(E^{\perp}))), \quad \dim_{\mathbf{F}_2} E = 3, 4,$$

that are submodules of finite products of $\mathbf{Z}_2$-modules of the form

$$\mathrm{Hom}_{\begin{pmatrix} \mathrm{Sp}(2) & 0 \\ * & 1 \end{pmatrix}}(\mathrm{St}(E), \mathbf{Z}_2), \quad \dim_{\mathbf{F}_2} E = 3, 4,$$

where the action on $\mathbf{Z}_2$ is trivial. According to the computer program *magma*, these latter modules are trivial. □

CHAPTER 4

The D-family

sec:dfam

Let $\mathrm{GL}(2n,\mathbf{R})$, $n \geq 1$, be the matrix group of $2n \times 2n$ real matrices and $\mathrm{SL}(2n,\mathbf{R})$ the closed subgroup of matrices with determinant 1. The D-family is the infinite family of matrix groups

$$\mathrm{PSL}(2n,\mathbf{R}) = \frac{\mathrm{SL}(2n,\mathbf{R})}{\langle -E \rangle}, \quad n \geq 4,$$

with trivial center. These groups also exist for $n = 1, 2, 3$; however, $\mathrm{PSL}(2,\mathbf{R}) = \{1\}$ is the trivial group, and $\mathrm{PSL}(4,\mathbf{R}) = \mathrm{PGL}(2,\mathbf{C})^2$, $\mathrm{PSL}(6,\mathbf{R}) = \mathrm{PGL}(4,\mathbf{C})$ are already known to be uniquely N-determined (1.2).

The maximal torus, the maximal torus normalizer of $\mathrm{GL}(2n,\mathbf{R})$, $\mathrm{SL}(2n,\mathbf{R})$, and $\mathrm{PSL}(2n,\mathbf{R})$ are

eq:dfamTNW

$$(4.1) \qquad \begin{aligned} &T(\mathrm{GL}(2n,\mathbf{R})) = \mathrm{SL}(2,\mathbf{R})^n, && N(\mathrm{GL}(2n,\mathbf{R})) = \mathrm{GL}(2,\mathbf{R}) \wr \Sigma_n \\ &T(\mathrm{SL}(2n,\mathbf{R})) = \mathrm{SL}(2,\mathbf{R})^n, && N(\mathrm{SL}(2n,\mathbf{R})) = \mathrm{SL}(2n,\mathbf{R}) \cap N(\mathrm{GL}(2n,\mathbf{R})) \\ &T(\mathrm{PSL}(2n,\mathbf{R})) = \frac{\mathrm{SL}(2,\mathbf{R})^n}{\langle -E \rangle}, && N(\mathrm{PSL}(2n,\mathbf{R})) = \frac{N(\mathrm{SL}(2n,\mathbf{R}))}{\langle -E \rangle} \end{aligned}$$

In all three cases, the maximal torus normalizer is the semi-direct product for the action of the Weyl group

eq:dfamW

$$(4.2) \qquad \begin{aligned} &W(\mathrm{GL}(2n,\mathbf{R})) = \Sigma_2 \wr \Sigma_n, \quad \Sigma_2 = W(\mathrm{GL}(2,\mathbf{R})) = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle, \\ &W(\mathrm{SL}(2n,\mathbf{R})) = A_{2n} \cap (\Sigma_2 \wr \Sigma_n) = W(\mathrm{PSL}(2n,\mathbf{R})) \end{aligned}$$

on the maximal torus. It is known that for $n \geq 3$ [**5**, **24**, **34**, **35**]

eq:dfamH01WT

$$(4.3) \qquad H^0(W;\check{T})(\mathrm{PSL}(2n,\mathbf{R})) = 0, \qquad H^1(W;\check{T})(\mathrm{PSL}(2n,\mathbf{R})) = \begin{cases} \mathbf{Z}/2 & n = 3 \\ \mathbf{Z}/2 \times \mathbf{Z}/2 & n = 4 \\ 0 & n > 4 \end{cases}$$

for these projective groups. (The group of outer Lie automorphisms of the Lie group $\mathrm{PSL}(8,\mathbf{R})$, isomorphic to Σ_3, is faithfully represented in $H^1(W;\check{T})(\mathrm{PSL}(8,\mathbf{R}))$.)

The Lie groups

$$\mathrm{GL}(2n,\mathbf{R}) = \mathrm{SL}(2n,\mathbf{R}) \rtimes \langle D \rangle, \quad \mathrm{PGL}(2n,\mathbf{R}) = \mathrm{PSL}(2n,\mathbf{R}) \rtimes \langle D \langle -E \rangle \rangle$$

are the semi-direct products of their identity components with the order two subgroup generated by the matrix $D = \mathrm{diag}(-1, 1, \dots, 1)$ (or any other order two matrix with negative determinant) and conjugation with D induces an outer automorphism of the Lie groups $\mathrm{SL}(2n,\mathbf{R})$ and $\mathrm{PSL}(2n,\mathbf{R})$.

1. The structure of $\mathrm{PSL}(2n,\mathbf{R})$

sec:structure

In this section we investigate the Quillen category $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))$ (2.45) for $\mathrm{PSL}(2n,\mathbf{R})$ (and related 2-compact groups $\mathrm{SL}(2n,\mathbf{R})$, $\mathrm{GL}(2n,\mathbf{R})$, $\mathrm{PGL}(2n,\mathbf{R})$).

Consider the elementary abelian 2-groups

eq:dfamt

$$
\begin{aligned}
&t(\mathrm{SL}(2n,\mathbf{R})) = t(\mathrm{GL}(2n,\mathbf{R})) = \langle e_1,\dots,e_n\rangle \subset \mathrm{SL}(2n,\mathbf{R}) \subset \mathrm{GL}(2n,\mathbf{R})\\
&\Delta_{2n} = \langle e_1,\dots,e_n,c_1,\dots,c_n\rangle = \langle \mathrm{diag}(\pm1,\dots,\pm1)\rangle \cong (\mathbf{Z}/2)^{2n} \subset \mathrm{GL}(2n,\mathbf{R})\\
&P\Delta_{2n} = \Delta_{2n}/\langle e_1\cdots e_n\rangle \cong (\mathbf{Z}/2)^{2n-1} \subset \mathrm{PGL}(2n,\mathbf{R}) \qquad (e_1\cdots e_n = -E)\\
(4.4)\quad &S\Delta_{2n} = \langle e_1,\dots,e_n,c_1c_2,\dots,c_1c_n\rangle = \mathrm{SL}(2n,\mathbf{R})\cap\Delta_{2n} \cong (\mathbf{Z}/2)^{2n-1} \subset \mathrm{SL}(2n,\mathbf{R})\\
&PS\Delta_{2n} = S\Delta_{2n}/\langle e_1\cdots e_n\rangle \cong (\mathbf{Z}/2)^{2n-2} \subset \mathrm{PSL}(2n,\mathbf{R}) \qquad (e_1\cdots e_n = -E)\\
&t(\mathrm{PSL}(2n,\mathbf{R})) = t(\mathrm{PGL}(2n,\mathbf{R})) = \langle I,e_1,\dots,e_n\rangle/\langle e_1\dots e_n\rangle \subset \mathrm{PSL}(2n,\mathbf{R}) \subset \mathrm{PGL}(2n,\mathbf{R})\\
&Pt(\mathrm{SL}(2n,\mathbf{R})) = Pt(\mathrm{GL}(2n,\mathbf{R})) = \langle e_1,\dots,e_n\rangle/\langle e_1\cdots e_n\rangle \subset \mathrm{SL}(2n,\mathbf{R}) \subset \mathrm{GL}(2n,\mathbf{R})
\end{aligned}
$$

where

eq:dfamIc

$$
\begin{aligned}
e_j &= \mathrm{diag}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix},\dots,\begin{pmatrix}-1&0\\0&-1\end{pmatrix},\dots,\begin{pmatrix}1&0\\0&1\end{pmatrix}\right) \in \mathrm{SL}(2n,\mathbf{R}), \quad 1\le j\le n\\
(4.5)\qquad I &= \mathrm{diag}\left(\begin{pmatrix}0&-1\\1&0\end{pmatrix},\dots,\begin{pmatrix}0&-1\\1&0\end{pmatrix}\right) \in \mathrm{SL}(2n,\mathbf{R}),\\
c_j &= \mathrm{diag}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix},\dots,\begin{pmatrix}-1&0\\0&+1\end{pmatrix},\dots,\begin{pmatrix}1&0\\0&1\end{pmatrix}\right) \in \mathrm{GL}(2n,\mathbf{R}), \quad 1\le j\le n
\end{aligned}
$$

The matrices e_j and c_j have order two and commute with each other while $Ie_j = e_jI$, $Ic_j = e_jc_jI$, and $I^2 = e_1\cdots e_n = -E$.

The representation of the Weyl groups

WgroupsGL

$$W(\mathrm{GL}(2n,\mathbf{R})) = \langle c_1,\dots,c_n\rangle \rtimes \Sigma_n = \Sigma_2 \wr \Sigma_n, \tag{4.6}$$

WgroupsSL

$$W(\mathrm{SL}(2n,\mathbf{R})) = \langle c_1c_2,\dots,c_1c_n\rangle \rtimes \Sigma_n = A_{2n} \cap (\Sigma_2 \wr \Sigma_n) \tag{4.7}$$

on the maximal toral elementary abelian 2-group $t(\mathrm{SL}(2n,\mathbf{R})) = t(\mathrm{GL}(2n,\mathbf{R}))$ is trivial on the subgroup $\langle c_1,\dots,c_n\rangle = \Sigma_2^n$ while $\Sigma_n \subset \mathrm{GL}(n,\mathbf{C}) \subset \mathrm{SL}(2n,\mathbf{R})$ permutes the n basis vectors $e_1,\dots,e_n$ of $t(\mathrm{SL}(2n,\mathbf{R})) = t(\mathrm{GL}(2n,\mathbf{R}))$.

Let V be a nontrivial elementary abelian 2-group in $\mathrm{PGL}(2n,\mathbf{R})$ and V^* its inverse image in $\mathrm{GL}(2n,\mathbf{R})$. Let $q\colon V\to\mathbf{F}_2=\{0,1\}$ be the function and $[\ ,\ ]\colon V\times V\to\mathbf{F}_2=\{0,1\}$ the bilinear map given by ${v^*}^2 = (-E)^{q(v)}$ and $[v_1^*,v_2^*] = (-E)^{[v_1,v_2]}$ where $v^*, v_1^*, v_2^* \in \mathrm{SL}(2n,\mathbf{R})$ are preimages of $v, v_1, v_2 \in \mathrm{PSL}(2n,\mathbf{R})$, respectively. The equations

$$[v_1,v_2] = [v_2,v_1], \quad [v,v] = 0, \quad q(v_1+v_2) = q(v_1)+q(v_2)+[v_1,v_2]$$

show that q is the quadratic function associated to the symplectic bilinear form $[\ ,\ ]$ [28, p. 356]. The bilinear form is the deviation from linearity of the quadratic function. Define $V^\perp \supset R(V)$ to be the subgroups

$$V^\perp = \{v\in V \mid [v,V]=0\} \supset \{v\in V^\perp \mid q(v)=0\} = R(V)$$

of V. Since q is a group homomorphism on $V^\perp$, the subgroup $R(V)$ is either all of $V^\perp$ or a subgroup of index 2.

In the following we write $G\circ H$ for the product of the groups G and H with a common central subgroup amalgamated. The subgroup $\mho_1(V^*)$ is generated by all squares of elements of V^* [28, III.10.4].

lemma:Vast

4.8. Lemma. *Let V be a nontrivial elementary abelian 2-group in* $\mathrm{PGL}(2n,\mathbf{R})$. *The preimage V^* in* $\mathrm{GL}(2n,\mathbf{R})$ *is*

$$V^* = \begin{cases} C_2\times V & q(V)=0\\ C_4\circ V & [V,V]=0,\ q(V)\neq 0\\ P\times R(V) & [V,V]\neq 0,\ q(V^\perp)=0\\ (C_4\circ P)\times R(V) & [V,V]\neq 0,\ q(V^\perp)\neq 0\end{cases}$$

where $C_2 = \langle -E\rangle \subset C_4 \subset \mathrm{SL}(2n,\mathbf{R})$, $P = 2_\pm^{1+2d}$ is extraspecial, $C_4\circ P$ is generalized extraspecial with center of order 4, and $\mho_1(V^*) \subset \langle -E\rangle$.

PROOF. As long as the bilinear form is trivial, $[V, V] = 0$, V^* is abelian and the structure theorem for finitely generated abelian groups applies. Assume that the bilinear form does not completely vanish, $[V, V] \neq 0$. Then V^* is nonabelian with commutator subgroup $[V^*, V^*] = C_2$. Write $V = U \times R(V)$ for some nontrivial subgroup U complementary to $R(V)$. Then $V^\perp = V^\perp \cap (U \times R(V)) = (V^\perp \cap U) \times R(V)$ and $q(V^\perp) = q(V^\perp \cap U)$. If U^* denotes the preimage of U, we have $V^* = U^*(C_2 \times R(V)) = U^* \times R(V)$ as the preimage of $R(V)$, $C_2 \times R(V)$, is central in V^*. The commutator subgroup $[U^*, U^*] = [U^*R(V), U^*R(V)] = [V^*, V^*] = C_2$ and the center $Z(U^*)$ is the preimage of $V^\perp \cap U$. If $q(V^\perp) = 0$, $R(V) = V^\perp$ and $V^\perp \cap U = R(V) \cap U$ is trivial so $Z(U^*) = C_2$ and $U^* = P$ is extraspecial. If $q(V^\perp) \neq 0$, $R(V)$ has index 2 in $V^\perp$, $V^\perp \cap U$ has order 2, and $q(V^\perp \cap U) \neq 0$ so that $Z(U^*)$ contains an element of order 4. Therefore $Z(U^*) = C_4$ and U^* is generalized extraspecial. There are two isomorphism classes of such groups but only $U^* = C_4 \circ D_8 \circ \cdots D_8 = C_4 \circ P$ has elementary abelian abelianization [**56**, Ex. 8, p. 146]. □

For instance, the preimage of the maximal toral elementary abelian 2-group $t(\mathrm{PSL}(2n, \mathbf{R})$ of $\mathrm{PSL}(2n, \mathbf{R})$ is the abelian group

eq:tPSL2nR
$$t(\mathrm{PSL}(2n, \mathbf{R}))^* = \langle I, e_1, \ldots, e_n \rangle \,, \tag{4.9}$$

generated by I and $t(\mathrm{SL}(2n, \mathbf{R}))$.

r:toralnontoral
4.10. COROLLARY. *Let V be a nontrivial elementary abelian 2-group in $\mathrm{PSL}(2n, \mathbf{R})$. If*

$q(V) = 0$, $[V, V] = 0$: *V is toral in $\mathrm{PSL}(2n, \mathbf{R})$ $\iff$ $V^* = C_2 \times V$ is toral in $\mathrm{SL}(2n, \mathbf{R})$*
$q(V) \neq 0$, $[V, V] = 0$: *V is toral*
$q(V) \neq 0$, $[V, V] \neq 0$: *V is nontoral*

PROOF. We have

$$V \text{ is toral} \iff V \subset t(\mathrm{PSL}(2n, \mathbf{R})) \iff V^* \subset t(\mathrm{PSL}(2n, \mathbf{R}))^*$$

where the symbol '$\subset$' reads 'is subconjugate to'. In the first case of the corollary, the preimage V^* contains no elements of order 4 so that

$$V^* \subset t(\mathrm{PSL}(2n, \mathbf{R}))^* \iff V^* \subset t(\mathrm{SL}(2n, \mathbf{R}))$$

as $t(\mathrm{SL}(2n, \mathbf{R}))$ consists of the elements of order ≤ 2 in $t(\mathrm{PSL}(2n, \mathbf{R}))^*$. In the second case, $V^* = C_4 \times R(V)$ so that $R(V) \subset C_{\mathrm{SL}(2n,\mathbf{R})}(I) = \mathrm{GL}(n, \mathbf{C})$. But any complex representation of the elementary abelian 2-group $R(V)$ is toral, so $R(V) \subset t(\mathrm{GL}(n, \mathbf{C})) = t(\mathrm{SL}(2n, \mathbf{R}))$ and $V^* \subset \langle C_4, t(\mathrm{SL}(2n, \mathbf{R}))\rangle = t(\mathrm{PSL}(2n, \mathbf{R}))^*$. In the third case, the nonabelian group V^* can not be a subgroup of the abelian group $t(\mathrm{PSL}(2n, \mathbf{R}))^*$. □

lemma:Vconj
4.11. LEMMA. *Let V_1 and V_2 be elementary abelian 2-groups in $\mathrm{PSL}(2n, \mathbf{R})$. Then*

$$V_1 \text{ and } V_2 \text{ are conjugate in } \mathrm{PSL}(2n, \mathbf{R}) \iff V_1^* \text{ and } V_2^* \text{ are conjugate in } \mathrm{SL}(2n, \mathbf{R})$$

where $V_1^*, V_2^* \subset \mathrm{SL}(2n, \mathbf{R})$ are the preimages.

PROOF. This is clear. □

Write $\mathbf{A}(\mathrm{PGL}(2n, \mathbf{R}))^{q=0}$ and $\mathbf{A}(\mathrm{PGL}(2n, \mathbf{R}))^{\leq t, q=0}$ for the full subcategories of $\mathbf{A}(\mathrm{PGL}(2n, \mathbf{R}))$ generated by all elementary abelian 2-groups $V \subset \mathrm{PGL}(2n, \mathbf{R})$ with trivial quadratic function q, respectively, all toral elementary abelian 2-groups $V \subset \mathrm{PGL}(2n, \mathbf{R})$ with trivial quadratic function q. Define $\mathbf{A}(\mathrm{PSL}(2n, \mathbf{R}))^{q=0}$ and $\mathbf{A}(\mathrm{PSL}(2n, \mathbf{R}))^{\leq t, q=0}$ similarly as full subcategories of $\mathbf{A}(\mathrm{PSL}(2n, \mathbf{R}))$.

cor:equivcat
4.12. LEMMA. *Write* GL *for* $\mathrm{GL}(2n, \mathbf{R})$, SL *for* $\mathrm{SL}(2n, \mathbf{R})$, *and* PSL *for* $\mathrm{PSL}(2n, \mathbf{R})$. *The inclusion functors*

$$\begin{array}{ll}
\mathbf{A}(\Sigma_{2n}, \Delta_{2n}) \to \mathbf{A}(\mathrm{GL}) & \mathbf{A}(\Sigma_{2n}, S\Delta_{2n}) \to \mathbf{A}(\mathrm{SL}) \\
 & \mathbf{A}(W(\mathrm{SL}), t(\mathrm{SL})) \to \mathbf{A}(\mathrm{SL})^{\leq t} \\
\mathbf{A}(\Sigma_{2n}, P\Delta_{2n}) \to \mathbf{A}(\mathrm{PGL})^{q=0} & \mathbf{A}(\Sigma_{2n}, PS\Delta_{2n}) \to \mathbf{A}(\mathrm{PSL})^{q=0} \\
 & \mathbf{A}(W(\mathrm{PSL}), t(\mathrm{PSL})) \to \mathbf{A}(\mathrm{PSL})^{\leq t} \\
 & \mathbf{A}(W(\mathrm{PSL}), Pt(\mathrm{SL})) \to \mathbf{A}(\mathrm{PSL})^{\leq t, q=0}
\end{array}$$

are equivalences of categories. In particular, $\mathbf{A}(\mathrm{SL})$ *and* $\mathbf{A}(\mathrm{PSL})$ *are full subcategories of* $\mathbf{A}(\mathrm{GL})$ *and* $\mathbf{A}(\mathrm{PGL})$*, respectively. (See 2.68 for the meaning of* $\mathbf{A}(\Sigma_{2n},\Delta_{2n})$*.)*

PROOF. By real representation theory any nontrivial elementary abelian 2-group of $\mathrm{GL}(2n,\mathbf{R})$ is conjugate to a subgroup V of Δ_{2n} and

$$C_{\mathrm{GL}(2n,\mathbf{R})}(V)=\prod_{\rho\in V^\vee}\mathrm{GL}(i_\rho,\mathbf{R})$$

where $i\colon V^\vee\to\mathbf{Z}$ records the multiplicity of $\rho\in V^\vee$ in the representation $V\subset\Delta_{2n}\subset\mathrm{GL}(2n,\mathbf{R})$. Observe that Δ_{2n} is the maximal elementary abelian 2-group in $C_{\mathrm{GL}(2n,\mathbf{R})}(V)$. (For any $i\geq 1$, $\mathrm{GL}(i,\mathbf{R})$ contains the subgroup Δ_i, consisting of diagonal matrices with ±1 in the diagonal, as a maximal elementary abelian 2-group.) Therefore, by the standard argument from [7, IV.2.5], used also in 3.4, any group homomorphism between two nontrivial subgroups of Δ_{2n} induced by conjugation with a matrix from $\mathrm{GL}(2n,\mathbf{R})$, is in fact induced by conjugation with a matrix from $N_{\mathrm{GL}(2n,\mathbf{R})}(\Delta_{2n})=\Delta_{2n}\rtimes\Sigma_{2n}$ [54, Lemmma 3]. Thus the inclusion functor $\mathbf{A}(\Sigma_{2n},\Delta_{2n})\to\mathbf{A}(\mathrm{GL}(2n,\mathbf{R}))$ is a category equivalence.

Any nontrivial elementary abelian 2-group $V\subset\mathrm{PGL}(2n,\mathbf{R})$ with $q(V)=0$ is conjugate to a subgroup of $P\Delta_{2n}$ since V^*, the preimage in $\mathrm{GL}(2n,\mathbf{R})$, is conjugate to subgroup of Δ_{2n}. Let V_1,V_2 be two nontrivial subgroups of $P\Delta_{2n}$. From the commutative diagram of morphism sets

$$\begin{array}{ccc}\mathbf{A}(\Sigma_{2n},\Delta_{2n})(V_1^*,V_2^*) & = & \mathbf{A}(\mathrm{GL}(2n,\mathbf{R}))(V_1^*,V_2^*)\\ \downarrow & & \downarrow\\ \mathbf{A}(\Sigma_{2n},P\Delta_{2n})(V_1,V_2) & \hookrightarrow & \mathbf{A}(\mathrm{PGL}(2n,\mathbf{R}))^{q=0}(V_1,V_2)\end{array}$$

we see that the the bottom horizontal arrow is a bijection. This implies that $\mathbf{A}(\Sigma_{2n},P\Delta_{2n})\to\mathbf{A}(\mathrm{PGL}(2n,\mathbf{R}))^{q=0}$ is an equivalence of categories.

Any nontrivial elementary abelian 2-group in $\mathrm{SL}(2n,\mathbf{R})$ is conjugate in $\mathrm{GL}(2n,\mathbf{R})$ to a subgroup of $\mathrm{SL}(2n,\mathbf{R})\cap\Delta_{2n}=S\Delta_{2n}$ (4.4). The Quillen category of $\mathrm{SL}(2n,\mathbf{R})$ is a full subcategory of the Quillen catgory of $\mathrm{GL}(2n,\mathbf{R})$ since $C_{\mathrm{GL}(2n,\mathbf{R})}(V)\not\subset\mathrm{SL}(2n,\mathbf{R})$ for all objects V of $\mathbf{A}(\mathrm{SL}(2n,\mathbf{R}))$. Thus the inclusion functor $\mathbf{A}(\Sigma_{2n},S\Delta_{2n})\to\mathbf{A}(\mathrm{SL}(2n,\mathbf{R}))$ is an equivalence of categories.

Any toral elementary abelian 2-group in $\mathrm{SL}(2n,\mathbf{R})$ is conjugate to a subgroup of $t(\mathrm{SL}(2n,\mathbf{R}))$ by its very definition (2.50). Any morphism between two nontrivial subgroups of $t(\mathrm{SL}(2n,\mathbf{R}))$ induced by conjugation with a matrix from $\mathrm{SL}(2n,\mathbf{R})$, is in fact induced by conjugation with a matrix from $N(\mathrm{SL}(2n,\mathbf{R}))$ and hence from $W(\mathrm{SL}(2n,\mathbf{R}))$ [7, IV.2.5]. Thus $\mathbf{A}(W(\mathrm{SL}),t(\mathrm{SL}))\to\mathbf{A}(\mathrm{SL}(2n,\mathbf{R}))^{\leq t}$ is a category equivalence. The same argument can be used to identify the toral subcategory for $\mathrm{PSL}(2n,\mathbf{R})$ (and it is actually a general fact that the inclusion functor $\mathbf{A}(W(X),t(X))\to\mathbf{A}(X)^{\leq t}$ is an equivalence of categories where $t(X)\to X$ is the maximal toral elementary abelian p-group in the connected p-compact group X [47, 2.8]).

Any nontrivial toral elementary abelian 2-group $V\subset\mathrm{PSL}(2n,\mathbf{R})$ with $q(V)=0$ is conjugate to a subgroup of $Pt(\mathrm{SL})$ (4.4) since V^*, the preimage (4.8) in $\mathrm{GL}(2n,\mathbf{R})$, is conjugate to subgroup of $t(\mathrm{SL})\subset t(\mathrm{PSL})^*$ (4.9). As $\mathbf{A}(\mathrm{PSL})^{\leq t,q=0}$ is a full subcategory of $\mathbf{A}(\mathrm{PSL})^{\leq t}=\mathbf{A}(W(\mathrm{PSL}),t(\mathrm{PSL}))$, this means that $\mathbf{A}(W(\mathrm{PSL}),Pt(\mathrm{SL}))\to\mathbf{A}(\mathrm{PSL})^{\leq t,q=0}$ is a category equivalence.

□

We now specialize to full subcategory $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))^{\leq t}_{\leq 2}$ (2.50).

prop:Apslr2nleqt 4.13. PROPOSITION. *The chart*

$\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))^{\leq t}_{\leq 2}$	*Lines*		*Planes*	
	$q=0$	$q\neq 0$	$q=0$	$q\neq 0$
n *even*	$n/2$	2	$P(n,3)+P(n,4)$	$n/2+[n/4]$
n *odd*	$[n/2]$	1	$P(n,3)+P(n,4)$	$[n/2]$

gives the number of isomorphism classes of toral objects of rank 1 *and* 2 *in* $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))$*.*

When n is even, the $\frac{n}{2}$ toral lines with $q=0$ are $L(2i,2n-2i)$, $1\leq i\leq\frac{n}{2}$, and the two toral lines with $q\neq0$ are I and I^D. The toral planes with $q=0$ are the planes $P(2i_0,2i_1,2i_2,0)$ where

(i_0, i_1, i_2) is a partition of n into three natural numbers, $P(2i_0, 2i_1, 2i_2, 2i_3)$ where (i_0, i_1, i_2, i_3) is a partition of n into four natural numbers, and the toral planes with $q \neq 0$ are $I\#L(i, n-i)$, $1 \leq i \leq \frac{n}{2}$, and $I\#L(i, n-i)^D$ for even i.

When n is odd, the $\left[\frac{n}{2}\right]$ toral lines with $q = 0$ are $L(2i, 2n-2i)$, $1 \leq i \leq \left[\frac{n}{2}\right]$, and the toral line with $q \neq 0$ is I. The toral planes with $q = 0$ are the planes $P(2i_0, 2i_1, 2i_2, 0)$ where (i_0, i_1, i_2) is a partition of n into three natural numbers, $P(2i_0, 2i_1, 2i_2, 2i_3)$ where (i_0, i_1, i_2, i_3) is a partition of n into four natural numbers, and the toral planes with $q \neq 0$ are $I\#L(i, n-i)$, $1 \leq i \leq \left[\frac{n}{2}\right]$.

In (4.14) and (4.15) we list the centralizers of the rank one objects and in (4.16) and (4.17) the centralizers of the rank two objects.

Proposition 4.13 is the conclusion of the following considerations.

For any partition $i = (i_0, i_1)$ of $n = i_0 + i_1$ into a sum of two positive integers $i_0 \geq i_1 \geq 1$ let $L[i] = L[2i_0, 2i_1] \subset t(\mathrm{SL}(2n, \mathbf{R})) \subset \mathrm{SL}(2n, \mathbf{R})$ be the toral subgroup generated by

$$\mathrm{diag}(\overbrace{+E, \ldots, +E}^{i_0}, \overbrace{-E, \ldots, -E}^{i_1})$$

Then the centralizer (of the image in PSL($2n, \mathbf{R}$)) of this subgroup is

CLqeq0

$$C_{\mathrm{PSL}(2n,\mathbf{R})} L[2i_0, 2i_1] = \begin{cases} \frac{\mathrm{SL}(2i_0,\mathbf{R}) \times \mathrm{SL}(2i_i,\mathbf{R})}{\langle -E \rangle} \rtimes \langle \mathrm{diag}(D_1, D_2) \rangle & i_0 \neq i_1 \\ \frac{\mathrm{SL}(2i_0,\mathbf{R})^2}{\langle -E \rangle} \rtimes \left\langle \mathrm{diag}(D_1, D_2), \begin{pmatrix} O & E \\ E & 0 \end{pmatrix} \right\rangle & i_0 = i_1 \end{cases} \tag{4.14}$$

where $D_j = \mathrm{diag}(-1, 1, \ldots, 1) \in \mathrm{GL}(2i_j, \mathbf{R})$ are matrices of determinant -1. The diagonal matrix $\mathrm{diag}(D_1, D_2)$ acts on the identity component of the centralizer by the outer action on both factors. In the second case, which only occurs when $n = 2i_0$ is even, the matrix $\begin{pmatrix} O & E \\ E & 0 \end{pmatrix}$ acts by permuting the factors.

The element $I \in t(\mathrm{PSL}(2n, \mathbf{R}))^* \subset \mathrm{SL}(2n, \mathbf{R})$ of order four generates an order two toral subgroup of PSL($2n, \mathbf{R}$) with centralizer [47, 5.11]

CLneq0

$$C_{\mathrm{PSL}(2n,\mathbf{R})}(I) = \begin{cases} \mathrm{GL}(n, \mathbf{C})/\langle -E \rangle & n \text{ odd} \\ \mathrm{GL}(n, \mathbf{C})/\langle -E \rangle \rtimes \langle c_1 \cdots c_n \rangle & n \text{ even} \end{cases} \tag{4.15}$$

where, in the even case, the component group acts on the identity component through the unstable Adams operation ψ^{-1}. The nontrivial outer automorphism of PSL($2n, \mathbf{R}$) takes I to I^D where $I \neq I^D$ if and only if n is even (9.4.(4)).

For any partition $i = (i_0, i_1, i_2, 0)$ of $n = i_0 + i_1 + i_2$ into a sum of three positive integers $i_0 \geq i_1 \geq i_2 > 0$ or any partition $i = (i_0, i_1, i_2, i_3)$ of $n = i_0 + i_1 + i_2 + i_3$ into a sum of four positive integers $i_0 \geq i_1 \geq i_2 \geq i_3 > 0$ let $P[i] = P[2i_0, 2i_1, 2i_2, 2i_3] \subset t(\mathrm{SL}(2n, \mathbf{R})) \subset \mathrm{SL}(2n, \mathbf{R})$ be the subgroup generated by the two elements

$$\mathrm{diag}(\overbrace{+E, \ldots, +E}^{i_0}, \overbrace{-E, \ldots, -E}^{i_1}, \overbrace{+E, \ldots, +E}^{i_2}, \overbrace{-E, \ldots, -E}^{i_3})$$
$$\mathrm{diag}(\overbrace{+E, \ldots, +E}^{i_0}, \overbrace{+E, \ldots, +E}^{i_1}, \overbrace{-E, \ldots, -E}^{i_2}, \overbrace{-E, \ldots, -E}^{i_3})$$

The centralizers in PSL($2n, \mathbf{R}$) are

CVqeq0

$$C_{\mathrm{PSL}(2n,\mathbf{R})} P(i) = \begin{cases} \frac{\mathrm{SL}(2i_0,\mathbf{R})^2 \times \mathrm{SL}(2i_2,\mathbf{R})^2}{\langle -E,-E,-E,-E \rangle} \rtimes \left(\ker \left(C_2^{S(i)} \to C_2 \right) \rtimes \mathbf{Z}/2 \right) & i = (2i_0, 2i_0, 2i_2, 2i_2) \\ \frac{\mathrm{SL}(2i_0,\mathbf{R})^4}{\langle -E,-E,-E,-E \rangle} \rtimes \left(\ker \left(C_2^{S(i)} \to C_2 \right) \rtimes (\mathbf{Z}/2 \times \mathbf{Z}/2) \right) & i = (2i_0, 2i_0, 2i_0, 2i_0) \\ \frac{\prod_{S(i)} \mathrm{SL}(2i_j,\mathbf{R})}{\langle -E \rangle} \rtimes \ker \left(C_2^{S(i)} \to C_2 \right) & \text{otherwise} \end{cases} \tag{4.16}$$

where $\ker\left(C_2^{S(i)} \to C_2\right) = \langle \mathrm{diag}(D_1, D_2, E, E), \mathrm{diag}(D_1, E, D_3, E), \mathrm{diag}(D_1, E, E, D_4)\rangle$ (when $\#S(i) = 4$) is generated by diagonal matrices, $D_j = \mathrm{diag}(-1, 1, \ldots, 1) \in \mathrm{GL}(2i_j, \mathbf{R})$, and

$$\mathbf{Z}/2 = \left\langle \begin{pmatrix} 0 & E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & 0 & E \\ 0 & 0 & E & 0 \end{pmatrix} \right\rangle, \quad \mathbf{Z}/2 \times \mathbf{Z}/2 = \left\langle \begin{pmatrix} 0 & E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & 0 & E \\ 0 & 0 & E & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & E & 0 \\ 0 & 0 & 0 & E \\ E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \end{pmatrix} \right\rangle$$

are generated by block permutation matrices. (The component group of the first line is $C_2 \times D_8$; the component group of second line is extra special of order 32 isomorphic to $D_8 \circ D_8$.)

For any partition $i = (i_0, i_1)$ of $n = i_0 + i_1$ into a sum of two positive integers $i_0 \geq i_1 > 0$ let $I\#L[i_0, i_1] \subset \mathrm{PSL}(2n, \mathbf{R})$ be the elementary abelian 2-group that is the quotient of

$$(I\#L[i_0, i_1])^* = \langle I, \mathrm{diag}(\overbrace{+E, \ldots, +E}^{i_0}, \overbrace{-E, \ldots, -E}^{i_1})\rangle \subset t(\mathrm{PSL}(2n, \mathbf{R}))^*$$

It follows that

CVqneq0 (4.17)
$$C_{\mathrm{PSL}(2n,\mathbf{R})} I\#L(i_0, i_1) = \begin{cases} \frac{\mathrm{GL}(i_0,\mathbf{C}) \times \mathrm{GL}(i_1,\mathbf{C})}{\langle -E, -E\rangle} & n \text{ odd} \\ \frac{\mathrm{GL}(i_0,\mathbf{C}) \times \mathrm{GL}(i_1,\mathbf{C})}{\langle -E, -E\rangle} \rtimes \langle c_1 \cdots c_n\rangle & n \text{ even}, i_0 \neq i_1 \\ \frac{\mathrm{GL}(i,\mathbf{C}) \times \mathrm{GL}(i,\mathbf{C})}{\langle -E, -E\rangle} \rtimes \langle c_1 \cdots c_n, P\rangle & n \text{ even}, i_0 = i_1 \end{cases}$$

where $P = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$ permutes the two identical factors.

prop:IL 4.18. PROPOSITION. *$I\#L(i, n-i) \neq I\#L(i, n-i)^D$ if and only if n and i are even.*

PROOF. The automorphism group of $\langle i\rangle \times \langle \varepsilon\rangle = C_4 \times C_2 = I\#L(i, n-i)^*$ is the dihedral group of order eight

$$\mathrm{Aut}(C_4 \times C_2) = \langle a, b \mid a^4, b^2, bab = a^3\rangle$$

generated by the two automorphisms given by $a(i) = i\varepsilon$, $a(\varepsilon) = i^2\varepsilon$ and $b(i) = i$, $b(\varepsilon) = i^2\varepsilon$. The automorphism $a^2 \in Aut(C_4) \subset \mathrm{Aut}(C_4 \times C_2)$ is induced by conjugation with the matrix

$$\mathrm{diag}(P, \ldots, P), \qquad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

of determinant $(-1)^n$. Thus $A(\mathrm{SL}(2n, \mathbf{R}))(I\#L(i, n-i)^*) \neq A(\mathrm{GL}(2n, \mathbf{R}))(I\#L(i, n-i)^*)$ and $I\#L(i, n-i) = I\#L(i, n-i)^D$ when n is odd (9.2).

Assume now that n is even. The group of trace preserving automorphisms

$$\mathbf{A}(\mathrm{GL}(2n, \mathbf{R}))(C_4 \times C_2) = \begin{cases} \langle a^2, ba\rangle & 2i < n \\ \mathrm{Aut}(C_4 \times C_2) & 2i = n \end{cases}$$

has index 2 in general but is actually equal to the full automorphism group in case $i = n/2$. The conjugating matrix for ba is

$$\mathrm{diag}(\overbrace{P, \ldots, P}^{i}, \overbrace{E, \ldots, E}^{n-i})$$

of determinant $(-1)^i$. Thus $I\#L(i, n-i) = I\#L(i, n-i)^D$ when i is odd. If $n = 2i$ then the conjugating matrices for the automorphisms a and b are

$$\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \mathrm{diag}(\overbrace{P, \ldots, P}^{i}, \overbrace{E, \ldots, E}^{i}) \qquad \text{and} \qquad \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$$

The permutation matrix for b has positive determinant and the matrix for a has determinant $(-1)^i$. Thus $I\#L(i, n-i) = I\#L(i, n-i)^D$ if and only if i is odd. □

sec:lhs

2. Centralizers of objects of $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))^{\leq t}_{\leq 2}$ are LHS

In this section we check that all toral objects of rank ≤ 2 have LHS (2.26) centralizers.

lemma:lhs

4.19. LEMMA. *The centralizers of the objects of* $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))^{\leq t}_{\leq 2}$,

lhs1 (1) $\mathrm{GL}(i,\mathbf{C})/\langle -E\rangle \rtimes C_2$, $1 \leq i$ (4.15)
lhs2 (2) $\mathrm{SL}(2i_0,\mathbf{R}) \circ \mathrm{SL}(2i_1,\mathbf{R}) \rtimes C_2$, $1 \leq i_0 < i_1$ (4.14)
lhs3 (3) $(\mathrm{SL}(2i,\mathbf{R}) \circ \mathrm{SL}(2i,\mathbf{R})) \rtimes (C_2 \times C_2)$, $1 \leq i$ (4.14)
lhs4 (4) $C_{\mathrm{PSL}(2n,\mathbf{R})}(V)$, $q(V) = 0$ (4.16)
lhs5 (5) $C_{\mathrm{PSL}(2n,\mathbf{R})}(V)$, $q(V) \neq 0$ (4.17)

are LHS.

The cases of interest here are summarized in the following charts, obtained by use of a computer, for rank one centralizers with quadratic form $q = 0$ (4.14)

$\mathrm{SL}(2i_0,\mathbf{R}) \circ \mathrm{SL}(2i_1,\mathbf{R})$	$\ker\theta$	$\mathrm{Hom}(W,\check{T}^W)$	$H^1(W;\check{T})$	θ
$1 = i_0, 2 = i_1$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^2$	$\mathbf{Z}/2$	0
$1 = i_0, 3 = i_1$	0	$\mathbf{Z}/2$	$(\mathbf{Z}/2)^2$	mono
$1 = i_0, 4 \leq i_1$	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$	iso
$2 = i_0 < i_1$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^3$	$\mathbf{Z}/2$	epi
$3 \leq i_0 < i_1$	0	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^2$	iso

$\mathrm{SL}(2i,\mathbf{R}) \circ \mathrm{SL}(2i,\mathbf{R})$	$\ker\theta$	$\mathrm{Hom}(W,\check{T}^W)$	$H^1(W;\check{T})$	θ
$i = 2$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^3$	0
$i \geq 3$	0	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^2$	iso

and $q \neq 0$ (4.15)

$\mathrm{GL}(i,\mathbf{C})/\langle -E\rangle$	$\ker\theta$	$\mathrm{Hom}(W,\check{T}^W)$	$H^1(W;\check{T})$	θ
$i = 2$	$\mathbf{Z}/2$	$(\mathbf{Z}/2)^2$	$\mathbf{Z}/2$	epi
$i = 3$	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$	iso
$i = 4$	0	$\mathbf{Z}/2$	$(\mathbf{Z}/2)^2$	mono
$i > 4$	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$	iso

and for rank two centralizers with quadratic form $q = 0$ (4.16)

$\mathrm{SL}(2i_0,\mathbf{R})^2 \circ \mathrm{SL}(2i_1,\mathbf{R})^2$	$\ker\theta$	$\mathrm{Hom}(W,\check{T}^W)$	$H^1(W;\check{T})$	θ
$1 = i_0, 2 = i_1$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^{12}$	$(\mathbf{Z}/2)^8$	epi
$1 = i_0, 2 < i_1$	0	$(\mathbf{Z}/2)^6$	$(\mathbf{Z}/2)^6$	iso
$2 = i_0 < i_1$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^{18}$	$(\mathbf{Z}/2)^{14}$	epi
$2 < i_0 < i_1$	0	$(\mathbf{Z}/2)^{12}$	$(\mathbf{Z}/2)^{12}$	iso

$\prod_{j=0}^2 \mathrm{SL}(2i_j,\mathbf{R})/\langle -E\rangle$	$\ker\theta$	$\mathrm{Hom}(W;\check{T}^W)$	$H^1(W;\check{T})$	θ
$1 = i_0, 2 = i_1 < i_2$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^6$	$(\mathbf{Z}/2)^4$	epi
$1 = i_0, 2 < i_1 < i_2$	0	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^4$	iso
$2 = i_0 < i_1 < i_2$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^8$	$(\mathbf{Z}/2)^6$	epi
$2 < i_0 < i_1 < i_2$	0	$(\mathbf{Z}/2)^6$	$(\mathbf{Z}/2)^6$	iso

$\prod_{j=0}^3 \mathrm{SL}(2i_j,\mathbf{R})/\langle -E\rangle$	$\ker\theta$	$\mathrm{Hom}(W;\check{T}^W)$	$H^1(W;\check{T})$	θ
$1 = i_0, 2 = i_1 < i_2 < i_3$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^{12}$	$(\mathbf{Z}/2)^{10}$	epi
$1 = i_0, 2 < i_1 < i_2 < i_3$	0	$(\mathbf{Z}/2)^9$	$(\mathbf{Z}/2)^9$	iso
$2 = i_0 < i_1 < i_2 < i_3$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^{15}$	$(\mathbf{Z}/2)^{13}$	epi
$2 < i_0 < i_1 < i_2 < i_3$	0	$(\mathbf{Z}/2)^{12}$	$(\mathbf{Z}/2)^{12}$	iso

$\mathrm{SL}(2i,\mathbf{R})^4/\langle -E\rangle$	$\ker\theta$	$\mathrm{Hom}(W;\check{T}^W)$	$H^1(W;\check{T})$	θ
$2 = i$	$(\mathbf{Z}/2)^8$	$(\mathbf{Z}/2)^{24}$	$(\mathbf{Z}/2)^{16}$	epi
$3 \leq i$	0	$(\mathbf{Z}/2)^{12}$	$(\mathbf{Z}/2)^{12}$	iso

and with quadratic form $q \neq 0$ (4.17)

$\mathrm{GL}(i_0,\mathbf{C})\circ\mathrm{GL}(i_1,\mathbf{C})$	$\ker\theta$	$\mathrm{Hom}(W;\check{T}^W)$	$H^1(W;\check{T})$	θ
$1=i_0, 2=i_1$	$\mathbf{Z}/2$	$(\mathbf{Z}/2)^2$	$\mathbf{Z}/2$	epi
$1=i_0, 2<i_1$	0	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^2$	iso
$2=i_0<i_1$	$\mathbf{Z}/2$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^3$	epi
$2<i_0<i_1$	0	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^4$	iso

$\mathrm{GL}(i,\mathbf{C})\circ\mathrm{GL}(i,\mathbf{C})$	$\ker\theta$	$\mathrm{Hom}(W;\check{T}^W)$	$H^1(W;\check{T})$	θ
$2=i$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^3$	
$3\leq i$	0	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^4$	iso

Proof of Lemma 4.19. (1) Let $X=\mathrm{GL}(i,\mathbf{C})/\langle -E\rangle\rtimes C_2$ for $i\geq 1$. Since the Weyl group for X is a direct product $W=W_0\times C_2$, X is LHS.
(2) Let $X=(\mathrm{SL}(2i_0,\mathbf{R})\circ\mathrm{SL}(2i_1,\mathbf{R}))\rtimes C_2$ for $1\leq i_0<i_1$. The first problematic case is when $i_0=1$ and $i_1=2$ or 3. In this case, explicit computer computation results in the chart

$\mathrm{SL}(2,\mathbf{R})\circ\mathrm{SL}(2i_1,\mathbf{R}))\rtimes C_2$	$H^1(\pi;\check{T}^{W_0})$	$H^1(W;\check{T})$	$H^1(W_0;\check{T})$	$H^1(W_0;\check{T})^\pi$
$i_1=2$	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$
$i_1=3$	0	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^2$

showing that X is LHS. The second problematic case is $2=i_0<i_1$ where $\theta(X_0)$ is epimorphic. Since $H^1(W_0;\check{T})=\mathbf{Z}/2$, also $\theta(X_0)^\pi$ is epimorphic so that X is LHS (2.28).
(3) Let $X=(\mathrm{SL}(2i,\mathbf{R})\circ\mathrm{SL}(2i,\mathbf{R}))\rtimes(C_2\times C_2)$ for $i\geq 1$. X is a 2-compact toral group when $i=1$ and hence obviously LHS. For $i\geq 2$ explicit computer computation gives

$(\mathrm{SL}(2i,\mathbf{R})\circ\mathrm{SL}(2i,\mathbf{R}))\rtimes(C_2\times C_2)$	$H^1(\pi;\check{T}^{W_0})$	$H^1(W;\check{T})$	$H^1(W_0;\check{T})$	$H^1(W_0;\check{T})^\pi$
$i=2$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^3$	$(\mathbf{Z}/2)^2$
$i\geq 3$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^3$	$(\mathbf{Z}/2)^2$	$\mathbf{Z}/2$

so X is manifestly LHS for $i=2$. For $i>2$, $\theta(X_0)$ is bijective so X is LHS (2.28).
(4) Let $X=(\mathrm{SL}(2i,\mathbf{R})^4/\langle -E\rangle)\rtimes(D_8\circ D_8)$ for $i\geq 1$. When $i=1$, X is a 2-compact toral group which are all LHS. When $i=2$ explicit computer computation gives

$(\mathrm{SL}(2i,\mathbf{R})^4/\langle -E\rangle)\rtimes(D_8\circ D_8)$	$H^1(\pi;\check{T}^{W_0})$	$H^1(W;\check{T})$	$H^1(W_0;\check{T})$	$H^1(W_0;\check{T})^\pi$
$i=2$	$(\mathbf{Z}/2)^7$	$(\mathbf{Z}/2)^9$	$(\mathbf{Z}/2)^{16}$	$(\mathbf{Z}/2)^2$

so X is LHS by definition. For $i>2$, $\theta(X_0)$ is bijective.
(5) Let $X=(\mathrm{SL}(2i_0,\mathbf{R})^2\circ\mathrm{SL}(2i_1,\mathbf{R})^2)\rtimes(C_2\times D_8)$ for $1\leq i_0<i_1$. The problematic cases are $i_0=1, i_1=2$ and $2=i_0<i_1$ where $\theta(X_0)$ is surjective but not bijective. With the help of computer computations we obtain the table

$(\mathrm{SL}(2i_0,\mathbf{R})^2\circ\mathrm{SL}(2i_1,\mathbf{R})^2)\rtimes(D_8\times C_2)$	$H^1(\pi;\check{T}^{W_0})$	$H^1(W;\check{T})$	$H^1(W_0;\check{T})$	$H^1(W_0;\check{T})^\pi$
$i_0=1, i_1=2$	$(\mathbf{Z}/2)^5$	$(\mathbf{Z}/2)^7$	$(\mathbf{Z}/2)^8$	$(\mathbf{Z}/2)^2$
$i_0=2, 3\leq i_1$	$(\mathbf{Z}/2)^6$	$(\mathbf{Z}/2)^{11}$	$(\mathbf{Z}/2)^{14}$	$(\mathbf{Z}/2)^5$

showing that X is LHS in these cases also.
(6) Let $X=\frac{\prod_{j=0}^2\mathrm{SL}(2i_j,\mathbf{R})}{\langle -E\rangle}\rtimes C_2^2$. The problematic cases are $1=i_0, 2=i_1<i_2$ and $2=i_0<i_1<i_2$. With the help of computer computations we obtain the table

$\frac{\prod_{j=0}^3\mathrm{SL}(2i_j,\mathbf{R})}{\langle -E\rangle}\rtimes C_2^2$	$H^1(\pi;\check{T}^{W_0})$	$H^1(W;\check{T})$	$H^1(W_0;\check{T})$	$H^1(W_0;\check{T})^\pi$
$i_0=1, 2=i_1<i_2$	$(\mathbf{Z}/2)^3$	$(\mathbf{Z}/2)^6$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^3$
$2=i_0<i_1<i_2$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^9$	$(\mathbf{Z}/2)^6$	$(\mathbf{Z}/2)^5$

showing that X is LHS in these cases also.
(7) Let $X=\frac{\prod_{j=0}^3\mathrm{SL}(2i_j,\mathbf{R})}{\langle -E\rangle}\rtimes C_2^3$. The problematic cases are $1=i_0, 2=i_1<i_2<i_3$ and $2=i_0<i_1<i_2<i_3$. With the help of computer computations we obtain the table

$\frac{\prod_{j=0}^{3}\mathrm{SL}(2i_j,\mathbf{R})}{\langle -E\rangle}\rtimes C_2^3$	$H^1(\pi;\check{T}^{W_0})$	$H^1(W;\check{T})$	$H^1(W_0;\check{T})$	$H^1(W_0;\check{T})^{\pi}$
$i_0=1, 2=i_1<i_2<i_3$	$(\mathbf{Z}/2)^8$	$(\mathbf{Z}/2)^{16}$	$(\mathbf{Z}/2)^{10}$	$(\mathbf{Z}/2)^8$
$2=i_0<i_1<i_2<i_3$	$(\mathbf{Z}/2)^9$	$(\mathbf{Z}/2)^{20}$	$(\mathbf{Z}/2)^{13}$	$(\mathbf{Z}/2)^{11}$

showing that X is LHS in these cases also.
(8) The 2-compact group $(\mathrm{GL}(i,\mathbf{C})\circ\mathrm{GL}(i,\mathbf{C}))\rtimes(\mathbf{Z}/2\times\mathbf{Z}/2)$ is LHS for $i>2$ where θ is bijective. When $i=2$ we find

$(\mathrm{GL}(i,\mathbf{C})\circ\mathrm{GL}(i,\mathbf{C}))\rtimes(C_2\times C_2)$	$H^1(\pi;\check{T}^{W_0})$	$H^1(W;\check{T})$	$H^1(W_0;\check{T})$	$H^1(W_0;\check{T})^{\pi}$
$i=2$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^3$	$(\mathbf{Z}/2)^2$

so X is also LHS in this case.
(9) Let $X=\mathrm{GL}(i_0,\mathbf{C})\circ\mathrm{GL}(i_1,\mathbf{C})\rtimes C_2$, $1\le i_0<i_1$. Since the identity component has surjective θ-homomorphism and the component group $\pi=C_2$ acts trivially on $H^1(W_0;\check{T})$, X is LHS by 2.28. The values of the relevant cohomology groups are

$(\mathrm{GL}(i_0,\mathbf{C})\circ\mathrm{GL}(i_1,\mathbf{C}))\rtimes C_2$	$H^1(\pi;\check{T}^{W_0})$	$H^1(W;\check{T})$	$H^1(W_0;\check{T})$	$H^1(W_0;\check{T})^{\pi}$
$1=i_0, 2=i_1$	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$
$1=i_0, 2<i_1$	0	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^2$
$2=i_0<i_1$	0	$(\mathbf{Z}/2)^3$	$(\mathbf{Z}/2)^3$	$(\mathbf{Z}/2)^3$
$2<i_0<i_1$	0	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^4$

according to computer computations. □

3. The limit of the functor $H^1(W_0;\check{T})^{W/W_0}$ on $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))^{\le t}_{\le 2}$

sec:lim0

Let $H^1(W_0;\check{T})\colon \mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))^{\le t}_{\le t}\to\mathbf{Ab}$ be the functor that takes the toral elementary abelian 2-group $V\subset t(\mathrm{PSL}(2n,\mathbf{R}))$ to the abelian group $H^1(W_0(C_{\mathrm{PSL}(2n,\mathbf{R})}(V);\check{T}))$, and $H^1(W_0;\check{T})^{W/W_0}$ the functor that takes V to the the invariants for the action of the component group $\pi_0 C_{\mathrm{PSL}(2n,\mathbf{R})}(V)$ on this first cohomology group (2.53).

prop:lim0H1

4.20. Proposition. *The restriction map*

$$H^1(W(\mathrm{PSL}(2n,\mathbf{R});\check{T})\to\lim{}^0(\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))^{\le t}_{\le 2},H^1(W_0;\check{T})^{W/W_0})$$

is an isomorphism for all $n\ge 4$.

Proof. Consider first that case where $n=4$. The 2-compact group $X=\mathrm{PSL}(8,\mathbf{R})$ contains (4.13) the four rank one elementary abelian 2-groups $L(2,6), L(4,4), I, I^D$ with centralizers $\mathrm{SL}(2,\mathbf{R})\circ\mathrm{SL}(6,\mathbf{R})\rtimes C_2$, $\mathrm{SL}(4,\mathbf{R})\circ\mathrm{SL}(4,\mathbf{R})\rtimes(C_2\times C_2)$, $\mathrm{GL}(4,\mathbf{C})/\langle -E\rangle$ (twice). The claim of the proposition follows from the fact, verifiable by computer computations, that in all four cases, the restriction $H^1(W;\check{T}(X))\to H^1(W_0(C_X(L));\check{T})^{W/W_0}$ happens to be isomorphism.

For $n>4$, the claim is that the limit of the functor $H^1(W_0;\check{T})^{W/W_0}$ is trivial. In fact, even the limit of the functor $H^1(W_0;\check{T})$ is trivial. To see this, recall (4.13) that $\mathrm{PSL}(2n,\mathbf{R})$ contains the toral lines $L(2i,2n-2i)$, $1\le i\le[n/2]$, I and also I^D when n is even. Computer computations show that the morphisms

$$H^1(W_0;\check{T})(L(2,2n-2))\hookrightarrow H^1(W_0;\check{T})(I\#L(1,n-1))\hookleftarrow H^1(W_0;\check{T})(I)$$

are injective with and that their images intersect trivially. When $n\ge 6$ is even, also the images of the injective morphisms

$$H^1(W_0;\check{T})(L(4,2n-4))\hookrightarrow H^1(W_0;\check{T})(I\#L(2,n-2)^D)\hookleftarrow H^1(W_0;\check{T})(I^D)$$

intersect trivially. More computer computations show that, similarly, the morphisms

$$H^1(W_0;\check{T})(L(2i,2n-2i))\hookrightarrow H^1(W_0;\check{T})(I\#L(i,n-i))\hookleftarrow H^1(W_0;\check{T})(I),\qquad 1\le i\le[n/2],$$

are injective with and that their images intersect trivially. □

4. Rank two nontoral objects of $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))$

sec:NdetD

In this section we take a closer look at the nontoral rank two objects of $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))$ in order to verify the conditions of 2.63.

Nontoral rank two objects P of $\mathrm{PSL}(2n,\mathbf{R})$ satisfy either $q(P)=0$ or $[P,P]\neq 0$ (4.10) and the latter case only occurs if n is even.

$q(P)=0$: For any partition $i_0\geq i_i\geq i_2\geq i_3\geq 1$, let

$$P[i_0,i_1,i_2,i_3]^* = \left\langle (+1)^{i_0}(-1)^{i_1}(+1)^{i_2}(-1)^{i_3}, (+1)^{i_0}(+1)^{i_1}(-1)^{i_2}(-1)^{i_3}, -E\right\rangle \subset \Delta_{2n},$$
$$P[i_0,i_1,i_2,i_3] = P[i_0,i_1,i_2,i_3]^*/\langle -E\rangle \subset P\Delta_{2n}$$

where we apply the notation from notation from 3.§1. Then $P[i_0,i_1,i_2,i_3]^*\subset S\Delta_{2n}$ if and only if i_0, i_1, i_2, and i_3 all have the same parity and $P[i_0,i_1,i_2,i_3]^*$ is nontoral iff this parity is odd. It follows that the set of isomorphism classes of nontoral rank two objects of $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))^{q=0}$ corresponds bijectively to the $P(n+2,4)$ partitions $i=[i_0,i_1,i_2,i_3]$ of $n+2$ into sums of 4 natural numbers, $n+2=i_0+i_1+i_2+i_3$, $i_0\geq i_1\geq i_2\geq i_3\geq 1$. The correspondence is via the map

$$i=[i_0,i_1,i_2,i_3]\to P[i]=P[2i_0-1,2i_1-1,2i_2-1,2i_3-1]$$

that to the partition $i=[i_0,i_1,i_2,i_3]$ associates the quotient $P[i]\subset PS\Delta_{2n}$ of $P[i]^*\subset S\Delta_{2n}$ generated by the three elements

$$v_1=\mathrm{diag}(\overbrace{+1,\dots,+1}^{2i_0-1},\overbrace{-1,\dots,-1}^{2i_1-1},\overbrace{+1,\dots,+1}^{2i_2-1},\overbrace{-1,\dots,-1}^{2i_3-1})$$
$$v_2=\mathrm{diag}(\overbrace{+1,\dots,+1}^{2i_0-1},\overbrace{+1,\dots,+1}^{2i_1-1},\overbrace{-1,\dots,-1}^{2i_2-1},\overbrace{-1,\dots,-1}^{2i_3-1}),$$
$$v_3=\mathrm{diag}(-1,\dots,-1)$$

The centralizer of $P[i]^*$ in $\mathrm{SL}(2n,\mathbf{R})$ is

$$C_{\mathrm{SL}(2n,\mathbf{R})}(P[i]^*) = \mathrm{SL}(2n,\mathbf{R})\cap C_{\mathrm{SL}(2n,\mathbf{R})}(P[i]^*)$$
$$= \mathrm{SL}(2n,\mathbf{R})\cap\Big(\prod_{j=0}^{3}\mathrm{GL}(2i_j-1,\mathbf{R})\Big) = P[i]^*\times\prod_{j=0}^{3}\mathrm{SL}(2i_j-1,\mathbf{R})$$

and centralizer of $P[i]$ in $\mathrm{PSL}(2n,\mathbf{R})$ is therefore [47, 5.11]

eq:CPi1

$$C_{\mathrm{PSL}(2n,\mathbf{R})}(P[i]) = P[i]\times\Big(\prod_{j=0}^{3}\mathrm{SL}(2i_j-1,\mathbf{R})\Big)\rtimes P[i]_i^\vee \tag{4.21}$$

where $P[i]_i^\vee$ is a group of permutation matrices isomorphic to C_2 if $i=[i_0,i_0,i_2,i_2]$, to $C_2\times C_2$ if $i=[i_0,i_0,i_0,i_0]$, and trivial in all other cases. Note that $P[i]^*$ is contained in $N(\mathrm{SL}(2n,\mathbf{R}))=\mathrm{SL}(2n,\mathbf{R})\cap\mathrm{GL}(2,\mathbf{R})\wr\Sigma_n$ because we may write

eq:e1

$$v_1=\mathrm{diag}(\overbrace{E,\dots,E}^{i_0-1},R,\overbrace{-E,\dots,-E}^{i_1-1},\overbrace{E,\dots,E}^{i_2-1},R,\overbrace{-E,\dots,-E}^{i_3-1}),\qquad R=\begin{pmatrix}1&0\\0&-1\end{pmatrix} \tag{4.22}$$

eq:e2

$$v_2=\mathrm{diag}(\overbrace{E,\dots,E}^{i_0-1},E,\overbrace{E,\dots,E}^{i_1-1},\overbrace{-E,\dots,-E}^{i_2-1},-E,\overbrace{-E,\dots,-E}^{i_3-1}) \tag{4.23}$$

and that the centralizer of $P[i]^*$ there is

$$C_{N(\mathrm{SL}(2n,\mathbf{R}))}(P[i]^*) = \mathrm{SL}(2n,\mathbf{R})\cap C_{\mathrm{GL}(2,\mathbf{R})\wr\Sigma_n}(v_1)\cap C_{\mathrm{GL}(2,\mathbf{R})\wr\Sigma_n}(v_2)$$
$$\overset{(9.10)}{=} \mathrm{SL}(2n,\mathbf{R})\cap C_{\mathrm{GL}(2,\mathbf{R})\wr(\Sigma_{i_0+i_1-1}\times\Sigma_{i_2+i_3-1})}(v_1) \overset{(9.10)}{=} P[i]^*\times\Big(\prod_{j=0}^{3}\mathrm{GL}(2,\mathbf{R})\wr\Sigma_{i_j-1}\Big)$$

It follows that the centralizer of $P[i]$ in $N(\mathrm{PSL}(2n,\mathbf{R}))$ is

$$C_{N(\mathrm{PSL}(2n,\mathbf{R}))}(P[i]) = P[i]\times\Big(\prod_{j=0}^{3}\mathrm{GL}(2,\mathbf{R})\wr\Sigma_{i_j-1}\Big)\rtimes P[i]_i^\vee = N(C_{\mathrm{PSL}(2n,\mathbf{R})}(P[i]))$$

For instance, if $i=[i_0,i_0,i_2,i_2]$, then $P[i]_i^\vee$ is the group of order two generated by $\mathrm{diag}(C_0,C_2)\in N(\mathrm{PSL}(2n,\mathbf{R}))$ where C_0 is the $(4i_0-2)\times(4i_0-2)$ matrix

$$\begin{pmatrix} 0 & 0 & E \\ 0 & T & 0 \\ E & 0 & 0 \end{pmatrix}, \qquad T=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and C_2 is a similar $(4i_2-2)\times(4i_2-2)$ matrix. Thus $P[i]\subset N(\mathrm{PSL}(2n,\mathbf{R}))$ is a preferred lift [**45**] of $P[i]\subset\mathrm{PSL}(2n,\mathbf{R})$. The two other preferred lifts [**44**, 6.2] of $P[i]\subset\mathrm{PSL}(2n,\mathbf{R})$ are obtained by composing the inclusion $P[i]\subset N(\mathrm{PSL}(2n,\mathbf{R}))$ with the inner automorphism given by the permutation matrices $(1,2)(2i_0,2n-2i_3+1)$

$$(\overbrace{+1,+1,\ldots,+1}^{2i_0-1},\overbrace{-1,\ldots,-1}^{2i_1-1},\overbrace{+1,\ldots,+1}^{2i_2-1},\overbrace{-1,\ldots,-1}^{2i_3-1})$$

$$(\overbrace{+1,+1,\ldots,+1}^{2i_0-1},\overbrace{+1,\ldots,+1}^{2i_1-1},\overbrace{-1,\ldots,-1}^{2i_2-1},\overbrace{-1,\ldots,-1}^{2i_3-1})$$

and $(1,2)(2i_0,2n-2i_3+2)$

$$(\overbrace{+1,+1,\ldots,+1}^{2i_0-1},\overbrace{-1,\ldots,-1}^{2i_1-1},\overbrace{+1,\ldots,+1}^{2i_2-1},\overbrace{-1,\ldots,-1}^{2i_3-1})$$

$$(\overbrace{+1,+1,\ldots,+1}^{2i_0-1},\overbrace{+1,\ldots,+1}^{2i_1-1},\overbrace{-1,\ldots,-1}^{2i_2-1},\overbrace{-1,\ldots,-1}^{2i_3-1})$$

taking v_1 and v_2 as in (4.22, 4.23) to

$$v_1=\mathrm{diag}(\overbrace{E,\ldots,E}^{i_0-1},E,\overbrace{-E,\ldots,-E}^{i_1-1},\overbrace{E,\ldots,E}^{i_2-1},-E,\overbrace{-E,\ldots,-E}^{i_3-1}),$$
$$v_2=\mathrm{diag}(\overbrace{E,\ldots,E}^{i_0-1},R,\overbrace{E,\ldots,E}^{i_1-1},\overbrace{-E,\ldots,-E}^{i_2-1},R,\overbrace{-E,\ldots,-E}^{i_3-1})$$

respectively to

$$v_1=\mathrm{diag}(\overbrace{E,\ldots,E}^{i_0-1},R,\overbrace{-E,\ldots,-E}^{i_1-1},\overbrace{E,\ldots,E}^{i_2-1},R,\overbrace{-E,\ldots,-E}^{i_3-1}),$$
$$v_2=\mathrm{diag}(\overbrace{E,\ldots,E}^{i_0-1},R,\overbrace{E,\ldots,E}^{i_1-1},\overbrace{-E,\ldots,-E}^{i_2-1},R,\overbrace{-E,\ldots,-E}^{i_3-1})$$

In the same way as above, we see that these are really preferred lifts of $P[i]$. The three lifts are not conjugate in $N(\mathrm{PSL}(2n,\mathbf{R}))$ because the intersection with the maximal torus is v_2 in case (4.22, 4.23) but v_1 and v_1+v_2 in the two other cases. Observe that all three preferred lifts of $P[i]$ have the same image in $W(\mathrm{PSL}(2n,\mathbf{R}))=\pi_0N(\mathrm{PSL}(2n,\mathbf{R}))\subset\pi_0\mathrm{GL}(2,\mathbf{R})\wr\Sigma_n$. Observe also that the inclusion $P[i]\times P[i]_i^\vee\to C_{\mathrm{PSL}(2n,\mathbf{R})}(P[i])$ induces an isomorphism on component groups and that the centralizer

$$C_{\mathrm{PSL}(2n,\mathbf{R})}(P[i]\times P[i]_i^\vee)=C_{C_{\mathrm{PSL}(2n,\mathbf{R})}(P[i])}(P[i]_i^\vee)$$
$$=\begin{cases} P[i]\times\mathrm{SL}(2i_0-1,\mathbf{R}) & i=[i_0,i_0,i_0,i_0] \\ P[i]\times\mathrm{SL}(2i_0-1,\mathbf{R})\times\mathrm{SL}(2i_2-1,\mathbf{R}) & i=[i_0,i_0,i_2,i_2] \\ C_{\mathrm{PSL}(2n,\mathbf{R})}(P[i]) & \text{otherwise} \end{cases}$$

has nontrivial identity component when $n>2$.

$[P,P]\neq 0$: $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))$ contains (up to isomorphism) four rank two objects with nontrivial inner product, namely H_+, H_+^D, H_-, and H_-^D where $H_\pm$ is the image of $2_\pm^{1+2}\subset\mathrm{SL}(2n,\mathbf{R})$ (4.50).

The extraspecial 2-group $2_+^{1+2} \subset \mathrm{SL}(4n,\mathbf{R})$ is described in 9.4.(6) or, alternatively, in 9.7 as

$$2_+^{1+2} = \Big\langle \operatorname{diag}\Big(\overbrace{\begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}, \dots, \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}}^{n}\Big), \operatorname{diag}\Big(\overbrace{\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}}^{n}\Big)\Big\rangle = \langle g_1, g_2 \rangle$$

Note that 2_+^{1+2} is contained in $N(\mathrm{SL}(4n,\mathbf{R})) = \mathrm{SL}(4n,\mathbf{R}) \cap (\mathrm{GL}(2,\mathbf{R}) \wr \Sigma_{2n})$ and that its centralizer there is

$$\begin{aligned} C_{N(\mathrm{SL}(4n,\mathbf{R}))}(2_+^{1+2}) &= \mathrm{SL}(4n,\mathbf{R}) \cap C_{\mathrm{GL}(4n,\mathbf{R})}(v_1) \cap C_{\mathrm{GL}(2,\mathbf{R})\wr\Sigma_{2n}}(v_2) \\ &\overset{(9.10)}{=} \mathrm{SL}(4n,\mathbf{R}) \cap C_{\mathrm{GL}(2,\mathbf{R})^n \rtimes (C_2 \wr \Sigma_n)}(v_1) = \mathrm{GL}(2,\mathbf{R}) \wr \Sigma_n = N(\mathrm{GL}(2n,\mathbf{R})) \end{aligned}$$

It follows, as in 4.50, that the centralizer of H_+ in $N(\mathrm{PSL}(4n,\mathbf{R}))$ is

$$C_{N(\mathrm{SL}(4n,\mathbf{R}))}(H_+) = H_+ \times C_{N(\mathrm{SL}(4n,\mathbf{R}))}(2_+^{1+2}) / \langle -E \rangle = N(H_+ \times \mathrm{PGL}(2n,\mathbf{R}))$$

which means that $H_+ \subset N(\mathrm{PSL}(4n,\mathbf{R}))$ is a preferred lift of $H_+ \subset \mathrm{PSL}(4n,\mathbf{R})$. Another preferred lift can be obtained by pre-composing the inclusion $H_+ \subset N(\mathrm{PSL}(4n,\mathbf{R}))$ with the nontrivial automorphism in $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(H_+) = O^+(2,\mathbf{F}_2)$. The final preferred lift is

$$\begin{aligned} (2_+^{1+2})^{\operatorname{diag}(\overbrace{B,\dots,B}^{n})} &= \Big\langle -(g_1g_2)^{\operatorname{diag}(B,\dots,B)}, g_2^{\operatorname{diag}(B,\dots,B)} \Big\rangle, \qquad B = \frac{1}{\sqrt{2}}\begin{pmatrix} E & I \\ I & E \end{pmatrix}, \\ -(g_1g_2)^{\operatorname{diag}(B,\dots,B)} &= \operatorname{diag}\Big(\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \dots, \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\Big), \qquad g_2^B = g_2 \end{aligned}$$

Also this subgroup is actually contained in the maximal torus normalizer with centralizer

$$\begin{aligned} C_{N(\mathrm{SL}(4n,\mathbf{R}))}(2_+^{1+2})^{\operatorname{diag}(B,\dots,B)} &= \mathrm{SL}(4n,\mathbf{R}) \cap C_{\mathrm{GL}(2,\mathbf{R})\wr\Sigma_{2n}}\big(-(g_1g_2)^{\operatorname{diag}(B,\dots,B)}\big) \cap C_{\mathrm{GL}(4n,\mathbf{R})}(g_2) \\ &\overset{(9.10)}{=} (\mathrm{GL}(1,\mathbf{C})^2 \rtimes C_2) \wr \Sigma_n \cap C_{\mathrm{GL}(4n,\mathbf{R})}(g_2) = (\mathrm{GL}(1,\mathbf{C}) \rtimes C_2) \wr \Sigma_n = N(\mathrm{GL}(2n,\mathbf{R})) \end{aligned}$$

where

$$C_2 = \Big\langle \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix} \Big\rangle, \qquad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Observe that, for all three preferred lifts of H_+, the image in the Weyl group $W(\mathrm{PSL}(4n,\mathbf{R})) = \pi_0 N(\mathrm{PSL}(4n,\mathbf{R})) \subset \pi_0 \mathrm{GL}(2,\mathbf{R}) \wr \Sigma_{2n}$ is the order 2 subgroup of Σ_{2n} generated by the permutation $(1,2)(3,4)\cdots(2n-1,2n)$. Observe also the inclusion $H_+ \# L(1,2n-1) \to C_{\mathrm{PSL}(4n,\mathbf{R})}(H_+)$ (4.39) induces an isomorphism on component groups and that the centralizer $C_{\mathrm{PSL}(4n,\mathbf{R})}(H_+ \# L(1,2n-1))$ has nontrivial identity component (according to the proof of 4.54) when $n \geq 2$.

The extraspecial 2-group $2_-^{1+2} \subset \mathrm{SL}(4n,\mathbf{R})$ is described in 9.4.(7) or, alternatively, in 9.7 as

$$2_-^{1+2} = \Big\langle \operatorname{diag}\Big(\overbrace{\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \dots, \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}}^{n}\Big), \operatorname{diag}\Big(\overbrace{\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}}^{n}\Big)\Big\rangle = \langle g_1, g_2 \rangle$$

Note that 2_-^{1+2} is contained in $N(\mathrm{SL}(4n,\mathbf{R})) = \mathrm{SL}(4n,\mathbf{R}) \cap (\mathrm{GL}(2,\mathbf{R}) \wr \Sigma_{2n})$ and that its centralizer there is

$$\begin{aligned} C_{N(\mathrm{SL}(4n,\mathbf{R}))}(2_-^{1+2}) &= \mathrm{SL}(4n,\mathbf{R}) \cap C_{\mathrm{GL}(2,\mathbf{R})\wr\Sigma_{2n}}(g_1) \cap C_{\mathrm{GL}(4n,\mathbf{R})}(g_2) \\ &\overset{(9.10)}{=} (\mathrm{GL}(1,\mathbf{C})^2 \rtimes C_2) \wr \Sigma_n \cap C_{\mathrm{GL}(4n,\mathbf{R})}(g_2) = \Big\langle \mathrm{GL}(1,\mathbf{C}), \begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix} \Big\rangle \wr \Sigma_n \overset{(1)}{=} N(\mathrm{GL}(n,\mathbf{H})) \end{aligned}$$

It follows, as in 4.50, that the centralizer of H_- in $N(\mathrm{PSL}(4n,\mathbf{R}))$ is

$$C_{N(\mathrm{SL}(4n,\mathbf{R}))}(H_-) = H_- \times C_{N(\mathrm{SL}(4n,\mathbf{R}))}(2_+^{1+2}) / \langle -E \rangle = N(H_- \times \mathrm{PGL}(n,\mathbf{H}))$$

which means that $H_- \subset N(\mathrm{PSL}(4n,\mathbf{R}))$ is a preferred lift of $H_- \subset \mathrm{PSL}(4n,\mathbf{R})$. The two other preferred lifts can be obtained by pre-composing the inclusion $H_- \subset N(\mathrm{PSL}(4n,\mathbf{R}))$ with the nontrivial automorphisms in $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(H_-) = O^-(2,\mathbf{F}_2) = \mathrm{Sp}(2,\mathbf{F}_2) = \mathrm{GL}(2,\mathbf{F}_2)$. Observe that, for all three preferred lifts of H_-, the image in the Weyl group $W(\mathrm{PSL}(4n,\mathbf{R})) = \pi_0 N(\mathrm{PSL}(4n,\mathbf{R})) \subset \pi_0 \mathrm{GL}(2,\mathbf{R}) \wr \Sigma_{2n}$ is the order 2 subgroup of Σ_{2n} generated by $(1,2)(3,4)\cdots(2n-$

$1, 2n)$. Observe also that H_- is contained in the rank three subgroup $H_+\#L(1,n-1)$ (4.41) whose centralizer has a nontrivial identity component when $n \geq 2$ (according to the proof of 4.54).

We conclude that for every nontoral rank two object P of $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))$ the identity component $C_{\mathrm{PSL}(2n,\mathbf{R})}(P)_0$ of the centralizer is center-less. By (part of) [42, 5.2], the homomorphism

$$\mathrm{Aut}(C_{\mathrm{PSL}(2n,\mathbf{R})}(P)) \to \mathrm{Aut}(\pi_0 C_{\mathrm{PSL}(2n,\mathbf{R})}(P)) \times \mathrm{Aut}(C_{\mathrm{PSL}(2n,\mathbf{R})}(P)_0),$$

obtained by applying the functors π_0 and $(\ )_0$, is injective. Under the inductive assumption that $C_{\mathrm{PSL}(2n,\mathbf{R})}(P)_0$ (see (4.21) and (4.50)) has $\pi_*(N)$-determined automorphisms it then follows from Lemma 2.63 and diagram (2.64) that condition (3) of Theorem 2.51 is satisfied.

5. Limits over the Quillen category of $\mathrm{PSL}(2n,\mathbf{R})$

sec:quillen

In this section we show that the problem of computing the higher limits of the functors $\pi_i(BZC_{\mathrm{PSL}(2n,\mathbf{R})})$, $i=1,2$, (2.47) is concentrated on objects of the Quillen category with $q \neq 0$.

lemma:HOV

4.24. Lemma. *Let $V \subset PS\Delta_{2n}$ (4.4) be a nontrivial subgroup representing an object of the category $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))^{q=0} = \mathbf{A}(\Sigma_{2n}, PS\Delta_{2n})$ (4.12). Then*

$$ZC_{\mathrm{PSL}(2n,\mathbf{R})}(V) = PS\Delta_{2n}^{\Sigma_{2n}(V)}$$

where $\Sigma_{2n}(V) \subset \Sigma_{2n}$ is the point-wise stabilizer subgroup (2.68).

Proof. Let $\nu^*\colon V \to S\Delta_{2n}$ be a lift to $\mathrm{SL}(2n,\mathbf{R})$ of the inclusion homomorphism of V into $\mathrm{PSL}(2n,\mathbf{R})$. Then

$$C_{\mathrm{SL}(2n,\mathbf{R})}(\nu^*V) = \mathrm{SL}(2n,\mathbf{R}) \cap \prod_{\rho\in V^\vee} \mathrm{GL}(i_\rho,\mathbf{R}), \qquad \Sigma_{2n}(\nu^*V) = \prod_{\rho\in V^\vee} \Sigma_{i_\rho}$$

where $i\colon V^\vee \to \mathbf{Z}$ records the multiplicity of each $\rho \in V^\vee$ in the representation ν^*. Write

$$\nu^*(v) = \mathrm{diag}(\overbrace{\rho_1(v),\dots,\rho_1(v)}^{i_1},\dots,\overbrace{\rho_m(v),\dots,\rho_m(v)}^{i_m})$$

where $\rho_1,\dots,\rho_m \in V^\vee = \mathrm{Hom}(V, C_2)$ are pairwise distinct homomorphisms $V \to C_2 = \langle \pm 1\rangle$ and $i_1 + \cdots + i_m = 2n$. There is a corresponding decomposition $\{1,\dots,2n\} = I_1 \cup \cdots \cup I_m$ of the set $\{1,\dots,2n\}$ into k disjoint subsets I_j containing i_j elements.

Using [47, 5.11] and 9.20 we get that

$$C_{\mathrm{PSL}(2n,\mathbf{R})}(V) = \frac{C_{\mathrm{SL}(2n,\mathbf{R})}(\nu^*V)}{\langle -E\rangle} \rtimes V_{\nu^*}^\vee, \qquad \Sigma_{2n}(V) = \Sigma_{2n}(\nu^*V) \rtimes V_{\nu^*}^\vee$$

where $V_{\nu^*}^\vee = \{\zeta \in \mathrm{Hom}(V, \mathrm{GL}(1,\mathbf{R})) \mid \forall \rho \in V^\vee\colon i_{\zeta\rho} = i_\rho\}$. Suppose that ζ is a nontrivial element of $V_{\nu^*}^\vee$. Choose a vector $v \in V$ such that $\zeta(v) = -1$. Then the determinant of $\nu^*(v)$ is $(-1)^n$ for $\nu^*(v)$ consists of an equal number of $+1$ and -1. Thus n is even. Let σ be the permutation associated to ζ that moves the subset I_j monotonically to I_k where $\zeta\rho_j = \rho_k$. Then σ is even for it is a product of n transpositions. In this way we imbed $V_{\nu^*}^\vee$ as a subgroup of the alternating group $A_{2n} \subset \mathrm{PSL}(2n,\mathbf{R})$ to obtain the semi-direct products.

The center of the centralizer is

$$\begin{aligned} ZC_{\mathrm{PSL}(2n,\mathbf{R})}(V) &= Z\left(\frac{\prod \mathrm{SL}(2n,\mathbf{R}) \cap \mathrm{GL}(i_\rho,\mathbf{R})}{\langle -E\rangle} \rtimes V_{\nu^*}^\vee\right) \overset{(9.14)}{=} Z\left(\frac{\prod \mathrm{SL}(2n,\mathbf{R}) \cap \mathrm{GL}(i_\rho,\mathbf{R})}{\langle -E\rangle}\right)^{V_{\nu^*}^\vee} \\ &\overset{(9.18)}{=} \left(\frac{\mathrm{SL}(2n,\mathbf{R}) \cap \prod Z\mathrm{GL}(i_\rho,\mathbf{R})}{\langle -E\rangle}\right)^{V_{\nu^*}^\vee} = \left(\frac{S\Delta_{2n}^{\Sigma_{2n}(\nu^*V)}}{\langle -E\rangle}\right)^{V_{\nu^*}^\vee} \\ &= \left(PS\Delta_{2n}^{\Sigma_{2n}(\nu^*V)}\right)^{V_{\nu^*}^\vee} = PS\Delta_{2n}^{\Sigma_{2n}(V)} \end{aligned}$$

where the penultimate equlity sign is justified by observing that the coefficient group homomorphism $H^1(\Sigma_{2n}(\nu^*V); \langle -E\rangle) \to H^1(\Sigma_{2n}(\nu^*V); S\Delta_{2n}) \to H^1(\Sigma_{2n}(\nu^*V); \Delta_{2n})$ is injective. □

Let $\pi_i(BZC) = \pi_i(BZC_{\mathrm{PSL}(2n,\mathbf{R})})$ (2.47).

cor:dfamlimqeq0

4.25. Corollary. $\lim^*(\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))^{q=0}, \pi_i(BZC)) = 0$ *for $n \geq 2$ and $i = 1, 2$.*

PROOF. This is obvious for $i = 2$ as $\pi_2(BZC) = 0$. For $i = 1$, use 2.69 to compute the limits of the functor $\pi_1(BZC)(V) = PS\Delta_{2n}^{\Sigma_{2n}(V)}$ (4.24). The fixed-point group $PSD_{2n}^{\Sigma_{2n}} = 0$ since PSD_{2n} is an irreducible $\mathbf{F}_2\Sigma_{2n}$-module of dimension $2n-2$ for $n \geq 2$. $\square$

dfamZCtoralqeq0

4.26. LEMMA. *Let $V \subset Pt(\mathrm{SL}) = Pt(\mathrm{SL}(2n,\mathbf{R}))$ (4.4) be a nontrivial subgroup representing an object of the category $\mathbf{A}(A_{2n} \cap (\Sigma_2 \wr \Sigma_n), Pt(\mathrm{SL})) = \mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))^{\leq t, q=0}$ (4.12). Then*

$$ZC_{\mathrm{PSL}(2n,\mathbf{R})}(V) = Pt(\mathrm{SL})^{(A_{2n}\cap(\Sigma_2\wr\Sigma_n))(V)}$$

where $(A_{2n} \cap (\Sigma_2 \wr \Sigma_n))(V)$ is the point-wise stabilizer subgroup (2.68).

PROOF. The point-wise stabilizer subgroups are

:dfamstabgroups

$$(A_{2n} \cap (\Sigma_2 \wr \Sigma_n))(V) = A_{2n} \cap \Sigma_{2n}(V), \qquad \Sigma_{2n}(V) = \Sigma_2^n \rtimes \Sigma_n(V) \tag{4.27}$$

Because these stabilizer subgroup have these particular forms and $PS\Delta_{2n}^{\Sigma_2^n} = Pt(\mathrm{SL})$, we get

$$ZC_{\mathrm{PSL}(2n,\mathbf{R})}(V) = PS\Delta_{2n}^{\Sigma_{2n}(V)} = PS\Delta_{2n}^{\Sigma_2^n \rtimes \Sigma_n(V)} = Pt(\mathrm{SL})^{\Sigma_n(V)} = Pt(\mathrm{SL})^{A_{2n}\cap(\Sigma_2^n \cap \Sigma_n(V))}$$

from 4.24 $\square$

lfamlimtoralqeq0

4.28. COROLLARY. $\lim^*(\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))^{\leq t,q=0}, \pi_i(BZC)) = 0$ *for $n \geq 2$ and $i = 1, 2$.*

PROOF. Similar to 4.25 but using $H^0(A_{2n} \cap (\Sigma_2 \wr \Sigma_n); Pt(\mathrm{SL})) = H^0(\Sigma_n; Pt(\mathrm{SL})) = 0$. $\square$

mma:dfamZCtoral

4.29. LEMMA. *Let $V \subset t(\mathrm{PSL}) = t(\mathrm{PSL}(2n,\mathbf{R}))$ (4.4) be a nontrivial subgroup representing an object of the category $\mathbf{A}(A_{2n} \cap (\Sigma_2 \wr \Sigma_n), t(\mathrm{PSL})) = \mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))^{\leq t}$ (4.12) where $n \geq 32$. Then*

$$\check{Z}C_{\mathrm{PSL}(2n,\mathbf{R})}(V) = \check{T}^{(A_{2n}\cap(\Sigma_2\wr\Sigma_n))(V)}$$

where $\check{T} = \check{T}(\mathrm{PSL}(2n,\mathbf{R}))$ is the discrete approximation [18, §3] to the maximal torus of $\mathrm{PSL}(2n,\mathbf{R})$ *and $(A_{2n} \cap (\Sigma_2 \wr \Sigma_n))(V)$ is the point-wise stabilizer subgroup of V (2.68).*

PROOF. Consider first the case where $V \subset Pt(\mathrm{SL}) \subset t(\mathrm{PSL})$. One checks that $\check{T}^{A_{2n}\cap\Sigma_2^n} = Pt(\mathrm{SL})$ for $n > 2$. Since $(A_{2n} \cap (\Sigma_2 \wr \Sigma_n))(V) \supset A_{2n} \cap \Sigma_2^n$ we get

$$ZC_{\mathrm{PSL}(2n,\mathbf{R})}(V) \overset{4.26}{=} Pt(\mathrm{SL})^{(A_{2n}\cap(\Sigma_2\wr\Sigma_n))(V)} = \check{T}^{(A_{2n}\cap(\Sigma_2\wr\Sigma_n))(V)}$$

in this case.

Consider next the case where V^*, the preimage of V in $\mathrm{SL}(2n,\mathbf{R})$, contains I (4.5) so that $V^* = \langle I, U\rangle$ (4.8) for some (possibly trivial) elementary abelian 2-group $U \subset t(\mathrm{SL}) \subset C_{\mathrm{SL}(2n,\mathbf{R})}(C_4) = \mathrm{GL}(n,\mathbf{C})$. Then

$$C_{\mathrm{SL}(2n,\mathbf{R})}(V^*) = \prod_{\rho\in U^\vee} \mathrm{GL}(i_\rho,\mathbf{C}), \qquad (\Sigma_2 \wr \Sigma_n)(V^*) = \Sigma_n(U) \subset A_{2n}$$

where $i\colon U^\vee \to \mathbf{Z}$ records the multiplicity of the linear character $\rho \in U^\vee$ in the representation $\nu^*\colon U \to \mathrm{GL}(n,\mathbf{C})$ and $\Sigma_n(U)$ is point-wise stabilizer subgroup for the action of $\Sigma_n = W(\mathrm{GL}(n,\mathbf{C}))$ on $t(\mathrm{SL}) = t(\mathrm{GL}(n,\mathbf{C})) = \langle e_1, \dots e_n\rangle$. It now follows [47, 5.11] and 9.20, as in (4.15) and (4.17), that

$$C_{\mathrm{PSL}(2n,\mathbf{R})}(V) = \begin{cases} \frac{C_{\mathrm{SL}(2n,\mathbf{R})}(V^*)}{\langle -E\rangle} & n \text{ odd} \\ \frac{C_{\mathrm{SL}(2n,\mathbf{R})}(V^*)}{\langle -E\rangle} \rtimes (U_{\nu^*}^\vee \times \langle c_1\cdots c_n\rangle) & n \text{ even}\end{cases}$$

$$A_{2n} \supset (\Sigma_2 \wr \Sigma_n)(V) = \begin{cases} \Sigma_n(U) & n \text{ odd} \\ \Sigma_n(U) \rtimes (U_{\nu^*}^\vee \times \langle c_1\cdots c_n\rangle) & n \text{ even}\end{cases}$$

where $U_{\nu^*}^\vee = \{\zeta \in U^\vee = \mathrm{Hom}(U, \langle -E\rangle) \mid \forall \rho \in U^\vee \colon i_{\zeta\rho} = i_\rho\}$ can be realized as a subgroup of Σ_n and $\langle c_1\cdots c_n\rangle$ is the diagonal order two subgroup of Σ_2^n. Consequently, if n is odd,

$$\check{Z}C_{\mathrm{PSL}(2n,\mathbf{R})}(V) = \check{Z}\left(\frac{\prod \mathrm{GL}(i_\rho,\mathbf{C})}{\langle -E\rangle}\right) = \frac{\prod \check{Z}\mathrm{GL}(i_\rho,\mathbf{C})}{\langle -E\rangle} = \frac{\check{T}(\mathrm{SL}(2n,\mathbf{R}))^{\Sigma_n(U)}}{\langle -E\rangle} = \check{T}^{\Sigma_n(U)} = \check{T}^{(A_{2n}\cap(\Sigma_2\wr\Sigma_n))(V)}$$

and if n is even,

$$\begin{aligned}\check{Z}C_{\mathrm{PSL}(2n,\mathbf{R})}(V) &= \check{Z}\left(\frac{\prod \mathrm{GL}(i_\rho, \mathbf{C})}{\langle -E\rangle} \rtimes (U_{\nu^*} \times \langle c_1 \cdots c_n\rangle)\right) \overset{9.14}{=} \left(\frac{\prod \check{Z}\mathrm{GL}(i_\rho, \mathbf{C})}{\langle -E\rangle}\right)^{U_{\nu^*} \times \langle c_1 \cdots c_n\rangle} \\ &= \left(\frac{\check{T}(\mathrm{SL}(2n, \mathbf{R}))^{\Sigma_n(U)}}{\langle -E\rangle}\right)^{U_{\nu^*} \times \langle c_1 \cdots c_n\rangle} = \left(\check{T}^{\Sigma_n(U)}\right)^{U_{\nu^*} \times \langle c_1 \cdots c_n\rangle} \\ &= \check{T}^{U_{\nu^*} \times \langle c_1 \cdots c_n\rangle} = \check{T}^{(A_{2n} \cap (\Sigma_2 \wr \Sigma_n))(V)}\end{aligned}$$

where we use that $H^1(\Sigma_n(U); \langle -E\rangle) \to H^1(\Sigma_n(U); \check{T}(\mathrm{SL}(2n, \mathbf{R})))$ is injective. (In fact, the center of the centralizer, $\check{Z}C_{\mathrm{PSL}(2n,\mathbf{R})}(V)$, is a product, $\check{T}^{\Sigma_n(U)}$, of 2-compact tori when n is odd, and a finite abelian group, $\check{T}^{\Sigma_n(U) \rtimes (U^\vee_{\nu^*} \rtimes \langle c_1 \cdots c_n\rangle)} = (\check{T}^{\langle c_1 \cdots c_n\rangle})^{\Sigma_n(U) \rtimes U^\vee_{\nu^*}} = t(\mathrm{PSL})^{\Sigma_n(U) \rtimes U^\vee_{\nu^*}}$ when n is even.) □

Lemma 4.26 can also be proved along the lines of [**47**, 2.8] using 2.33.

or:dfamtorallim 4.30. COROLLARY. $\lim^*(\mathbf{A}(\mathrm{PSL}(2n, \mathbf{R}))^{\leq t}, \pi_i(BZC)) = 0$ *for* $n \geq 3$ *and* $i = 1, 2$.

PROOF. Similar to 4.25 but using that that $H^0(W; \check{T})(\mathrm{PSL}(2n, \mathbf{R})) = 0$ for $n \geq 3$ (4.3). □

As we shall next see, Corollaries 4.25, 4.28 and 4.30 reduce the problem of computing the graded abelian group $\lim^*(\mathbf{A}(\mathrm{PSL}(2n, \mathbf{R})), \pi_i(BZC))$ considerably.

Let $\mathbf{A}$ be a category containing two full subcategories, $\mathbf{A}_j$, $j = 1, 2$, such that any object of $\mathbf{A}$ with a morphism to an object of $\mathbf{A}_j$ is an object of $\mathbf{A}_j$. Write $\mathbf{A}_1 \cap \mathbf{A}_2$ for the full subcategory with objects $\mathrm{Ob}(\mathbf{A}_1 \cap \mathbf{A}_2) = \mathrm{Ob}(\mathbf{A}_1) \cap \mathrm{Ob}(\mathbf{A}_2)$ and $\mathbf{A}_1 \cup \mathbf{A}_2$ for the full subcategory with objects $\mathrm{Ob}(\mathbf{A}_1 \cup \mathbf{A}_2) = \mathrm{Ob}(\mathbf{A}_1) \cup \mathrm{Ob}(\mathbf{A}_2)$. Let $M \colon \mathbf{A} \to \mathbf{Ab}$ be a functor taking values in abelian groups. Consider the subfunctor M_{12} of M given by

$$M_{12}(a) = \begin{cases} 0 & a \in \mathrm{Ob}(\mathbf{A}_1 \cup \mathbf{A}_2) \\ M(a) & a \notin \mathrm{Ob}(\mathbf{A}_1 \cup \mathbf{A}_2) \end{cases}$$

We now state a kind of Mayer–Vietoris sequence argument for cohomology of categories.

4.31. LEMMA. *If the graded abelian groups* $\lim^*(\mathbf{A}_1, M)$, $\lim^*(\mathbf{A}_2, M)$, *and* $\lim^*(\mathbf{A}_1 \cap \mathbf{A}_2, M)$ *are trivial, then* $\lim^*(\mathbf{A}, M_{12}) \cong \lim^*(\mathbf{A}; M)$.

PROOF. Consider also the subfunctor M_1 of M given by

$$M_1(a) = \begin{cases} 0 & a \in \mathrm{Ob}(\mathbf{A}_1) \\ M(a) & a \notin \mathrm{Ob}(\mathbf{A}_1) \end{cases}$$

Then there are natural transformations $M_{12} \to M_1 \to M$ of functors. The induced long exact sequences imply that it suffices to show $\lim^*(\mathbf{A}; M/M_1) = 0 = \lim^*(\mathbf{A}; M_1/M_{12})$.

The quotient functor M/M_1 vanishes outside $\mathbf{A}_1$ where it agrees with M and therefore [**47**, 13.12] $\lim^*(\mathbf{A}; M/M_1) \cong \lim(\mathbf{A}_1; M)$ which is trivial by assumption.

The same argument applied to $\mathbf{A}_2$ instead of $\mathbf{A}$ gives that $\lim^*(\mathbf{A}_2; M/M_1) \cong \lim(\mathbf{A}_1 \cap \mathbf{A}_2; M)$. Since this abelian group is trivial by assumption, we have that $\lim^*(\mathbf{A}_2; M_1) \cong \lim^*(\mathbf{A}_2; M)$. Also this abelian group is trivial by assumption.

The quotient functor M_1/M_{12} vanishes outside $\mathbf{A}_1 \cup \mathbf{A}_2$ where it agrees with M_1 and therefore $\lim^*(\mathbf{A}; M_1/M_{12}) \cong \lim(\mathbf{A}_1 \cup \mathbf{A}_2; M_1)$. Here, the functor M_1 vanishes outside $\mathbf{A}_2$ and hence $\lim(\mathbf{A}_1 \cup \mathbf{A}_2; M_1) \cong \lim^*(\mathbf{A}_2; M_1)$. Since we just showed that this abelian group is trivial, we have that $\lim^*(\mathbf{A}; M_1/M_{12}) = 0$. □

We conclude that

$$\lim{}^* \left(\mathbf{A}(\mathrm{PSL}(2n, \mathbf{R})), \pi_j(BZC_{\mathrm{PSL}(2n,\mathbf{R})})_{12}\right) = \lim{}^* \left(\mathbf{A}(\mathrm{PSL}(2n, \mathbf{R})), \pi_j(BZC_{\mathrm{PSL}(2n,\mathbf{R})})\right)$$

where $\pi_j(BZC_{\mathrm{PSL}(2n,\mathbf{R})})_{12}$ is the subfunctor of $\pi_j(BZC_{\mathrm{PSL}(2n,\mathbf{R})})$ given by

$$\pi_j(BZC_{\mathrm{PSL}(2n,\mathbf{R})})_{12}(V) = \begin{cases} 0 & V \text{ is toral or } q(V) = 0 \\ \pi_j(BZC_{\mathrm{PSL}(2n,\mathbf{R})}(V)) & V \text{ is nontoral and } q(V) \neq 0 \end{cases}$$

According to 4.10 we have

$$V \text{ is nontoral and } q(V) \neq 0 \iff [V,V] \neq 0$$

for all elementary abelian 2-groups V in $\mathrm{PSL}(2n,\mathbf{R})$. Thus the problem of computing the higher limits of the functors $\pi_i(BZC_{\mathrm{PSL}(2n,\mathbf{R})})$ is concentrated on the full subcategory $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))^{[\ ,\ ]\neq 0}$ of $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))$ generated by all elementary abelian 2-groups $V \subset \mathrm{PSL}(2n,\mathbf{R})$ with nontrivial inner product. Note that if $\mathrm{PSL}(2n,\mathbf{R})$ contains an elementary abelian 2-group V with $[V,V] \neq 0$ then $\mathrm{PSL}(2n,\mathbf{R})$ in particular contains such a subgroup of rank two. The preimage in $\mathrm{SL}(2n,\mathbf{R})$ of rank two $V \subset \mathrm{PSL}(2n,\mathbf{R})$ with nontrivial inner product is an extraspecial 2-group $2^{1+2}_{\pm}$ with central $\mho_1$ (4.8) so that, by real representation theory (9.5), n must be even.

sec:lim

6. Higher limits of the functors $\pi_i BZC_{\mathrm{PSL}(4n,\mathbf{R})}$ on $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))^{[\,,\,]\neq 0}$

In this section we compute the first higher limits of the functors $\pi_i BZC_{\mathrm{PSL}(4n,\mathbf{R})}$, $i=1,2$, by means of Oliver's cochain complex [53].

lemma:lim=0

4.32. LEMMA. *$\lim^1 \pi_1 BZC_{\mathrm{PSL}(4n,\mathbf{R})} = 0 = \lim^2 \pi_1 BZC_{\mathrm{PSL}(4n,\mathbf{R})}$ and $\lim^2 \pi_2 BZC_{\mathrm{PSL}(4n,\mathbf{R})} = 0 = \lim^3 \pi_2 BZC_{\mathrm{PSL}(4n,\mathbf{R})}$.*

The case $i=2$ is easy. Since $\pi_2 BZC_{\mathrm{PSL}(4n,\mathbf{R})}$ has value 0 on all objects of $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))^{[\,,\,]\neq 0}$ of rank ≤ 4 (4.54) it is immediate from Oliver's cochain complex that $\lim^2$ and $\lim^3$ of this functor are trivial.

We shall therefore now concentrate on the case $i=1$. The claim of the above lemma is that Oliver's cochain complex [53]

eq:Olccc

$$0 \to \prod_{|P|=2^2} [P] \xrightarrow{d^1} \prod_{|V|=2^3} [V] \xrightarrow{d^2} \prod_{|E|=2^4} [E] \xrightarrow{d^3} \cdots \tag{4.33}$$

is exact at objects of rank ≤ 3. Here, as a matter of notational convention,

$$[E] = \mathrm{Hom}_{\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(E)}(\mathrm{St}(E), E)$$

stands for the $\mathbf{F}_2$-vector space of $\mathbf{F}_2\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(E)$-module homomorphisms from the Steinberg module $\mathrm{St}(E)$ to E. The Steinberg module is the $\mathbf{F}_2\mathrm{GL}(E)$-module obtained in the following way.

Let $P = \mathbf{F}_2 e_1 + \mathbf{F}_2 e_2$ be a 2-dimensional vector space over $\mathbf{F}_2$ with basis vectors e_1, e_2. Let $\mathbf{F}_2[0]$ be the 3-dimensional $\mathbf{F}_2$-vector space on length zero flags, $[L]$, of nontrivial and proper subspaces L of P. The Steinberg module $\mathrm{St}(P)$ is the 2-dimensional kernel of the augmentation map $d\colon \mathbf{F}_2[0] \to \mathbf{F}_2$ given by $d[L]=1$.

Let $V = \mathbf{F}_2 e_1 + \mathbf{F}_2 e_2 + \mathbf{F}_2 e_3$ be a 3-dimensional vector space over $\mathbf{F}_2$ with basis vectors e_1, e_2, e_3. Let $\mathbf{F}_2[1]$ be the 21-dimensional $\mathbf{F}_2$-vector space on length one flags $[P > L]$ of nontrivial and proper subspaces of V and $\mathbf{F}_2[0]$ the 14-dimensional $\mathbf{F}_2$-vector space on all length 0 flags, $[P]$ or $[L]$, of nontrivial and proper subspaces of V. The Steinberg module $\mathrm{St}(V)$ over $\mathbf{F}_2$ for V is the 2^3-dimensional kernel of the linear map $d\colon \mathbf{F}_2[1] \to \mathbf{F}_2[0]$ given by $d[P>L] = [P]+[L]$.

prop:H+H-

4.34. PROPOSITION. *$H_+ \neq H_+^D$ and $H_- \neq H_-^D$ in $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))$. The automorphism groups of the objects H_+ and H_- (4.50) are*

$$\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(H_+) = O^+(2,\mathbf{F}_2) \cong C_2, \qquad \mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(H_-) = O^-(2,\mathbf{F}_2) = \mathrm{GL}(2,\mathbf{F}_2),$$

and the dimensions of the spaces of equivariant maps are

$$\dim[H_+] = 2, \qquad \dim[H_-] = 1$$

PROOF. The first part was proved in 4.50. The Quillen automorphismgroup $\mathbf{A}(\mathrm{SL}(4n,\mathbf{R}))(2_{\pm}^{1+2}) = \mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(2_{\pm}^{1+2}) = \mathrm{Out}(2_{\pm}^{1+2}) \cong O^{\pm}(2,\mathbf{F}_2)$ where the isomorphism is induced by abelianization $2_{\pm}^{1+2} \to H_{\pm}$ (9.4.(2), 9.4.(3), 9.5). According to *magma*, $\dim[H_+]=2$ and $\dim[H_-]=1$. $\square$

The $\mathbf{F}_2\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(H_+)$-equivariant maps given by

eq:deff+

$$f_+[L] = L, \qquad f_0[L] = \begin{cases} H_+^{\mathbf{A}(H_+)} & q(L)=0 \\ 0 & \text{otherwise} \end{cases} \tag{4.35}$$

form a basis for the 2-dimensional space $[H_+]$. The $\mathbf{F}_2\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(H_-)$-equivariant map given by

eq:deff-

$$f_-[L] = L \tag{4.36}$$

is a basis for the 1-dimensional space $[H_-]$.

The quadratic function (9.5) $q(v_1,v_2,v_3) = v_1^2 + v_2 v_3$ on V_0 (4.51) has automorphism group

$$O(q) \cong \mathrm{Sp}(2,\mathbf{F}_2) = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \subset \mathrm{GL}(3,\mathbf{F}_2)$$

of order 6.

prop:V0

4.37. PROPOSITION. *$V_0 \neq V_0^D$ in $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))$. The automorphism group $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(V_0) = O(q)$ and $\dim[V_0] = 4$.*

PROOF. See 9.4.(5) for the first part. According to *magma*, $\dim[V_0] = 4$. □

The four $\mathbf{F}_2\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(V_0)$-module homomorphisms

eq:basisV0

$$\{df_+, df_0, df_-, f_0\} \tag{4.38}$$

given by

$$df_+[P > L] = \begin{cases} L & P = H_+ \\ 0 & \text{otherwise} \end{cases} \qquad df_0[P > L] = \begin{cases} P^{\mathbf{A}(P)} & P = H_+, q(L) = 0 \\ 0 & \text{otherwise} \end{cases}$$
$$df_-[P > L] = \begin{cases} L & P = H_- \\ 0 & \text{otherwise} \end{cases} \qquad f_0[P > L] = \begin{cases} V_0^{\mathbf{A}(V_0)} & [P,P] = 0, q(L) = 0 \\ 0 & \text{otherwise} \end{cases}$$

is a basis for $[V_0]$.

The quadratic function on $H_+\#L(i,2n-i) \in \mathrm{Ob}(\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R})))$, $0 \leq i \leq n$, $q(v_1,v_2,v_3) = v_1v_2$, has automorphism group

$$O(q) = \begin{pmatrix} O^+(2,\mathbf{F}_2) & 0 \\ * & 1 \end{pmatrix} = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \right\rangle$$

of order $|O^+(2,\mathbf{F}_2)| \cdot 2^2 = 8$.

prop:H+L

4.39. PROPOSITION. *$H_+\#L(i,2n-i) \neq (H_+\#L(i,2n-i))^D \iff i$ is even. The Quillen automorphism group is*

$$\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(H_+\#L(i,2n-i)) = \begin{cases} \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \right\rangle & i \text{ odd} \\ O(q) & i \text{ even} \end{cases}$$

and the dimension of the space of equivariant maps is

$$\dim[H_+\#L(i,2n-i)] = \begin{cases} 6 & i \text{ odd} \\ 3 & i \text{ even} \end{cases}$$

PROOF. $H_+\#L(i,2n-i) \subset \mathrm{PSL}(4n,\mathbf{R})$ is (4.51) the quotient of

$$G = \langle \mathrm{diag}(R,\ldots,R), \mathrm{diag}(T,\ldots,T), \mathrm{diag}(\overbrace{-E,\ldots,-E}^{i}, \overbrace{E,\ldots,E}^{2n-i})\rangle = \langle g_1,g_2,g_3\rangle \subset \mathrm{SL}(4n,\mathbf{R})$$

The centralizer of G is $\mathrm{GL}(4n,\mathbf{R})$ is contained in the centralizer of its subgroup 2_+^{1+2} which is contained in $\mathrm{SL}(4n,\mathbf{R})$ (9.4). Observe that

- R and T are conjugate in $\mathrm{GL}(2,\mathbf{R})$
- Conjugation with $\mathrm{diag}(\overbrace{T,\ldots,T}^{i}, \overbrace{E,\ldots,E}^{2n-i})$ induces $(g_1,g_2,g_3) \xrightarrow{\phi_1} (g_1g_3,g_2,g_3)$
- Conjugation with $\mathrm{diag}(\overbrace{R,\ldots,R}^{i}, \overbrace{E,\ldots,E}^{2n-i})$ induces $(g_1,g_2,g_3) \xrightarrow{\phi_2} (g_1,g_2g_3,g_3)$
- When $i = n$, conjugation with $\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$ induces $(g_1,g_2,g_3) \xrightarrow{\phi} (g_1,g_2,-g_3)$

Consider the automorphism groups

$$\mathbf{A}(\mathrm{SL}(4n,\mathbf{R}))(G) \subset \mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G) \subset \mathrm{Out}(G) \to O(q) \subset \mathrm{Aut}(H_+\#L(i,2n-i))$$

where the outer automorphism group has order 16. Note that the automorphism ϕ is in the kernel of the homomorphism $\mathrm{Out}(G) \to O(q)$ induced by abelianization $G \to H_+\#L(i,2n-i)$. Using the above observations we see that $\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)$, even $\mathbf{A}(\mathrm{SL}(4n,\mathbf{R}))(G)$ for even i, maps onto $O(q)$. Thus the Quillen automorphism group $\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)$ has order 8 or 16. When

$i = n$ the automorphism ϕ is in $\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)$, even in $\mathbf{A}(\mathrm{SL}(4n,\mathbf{R}))(G)$, and when $i \neq n$, $\phi \notin \mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)$ as it does not preserve trace. Thus

$$|\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)| = \begin{cases} 16 & i = n \\ 8 & i \neq n \end{cases}$$

In any case the group $\mathbf{A}(\mathrm{SL}(4n,\mathbf{R}))(G)$ equals the group $\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)$ if and only if i is even. When i is odd, the automorphism ϕ_1 is induced from a matrix of negative determinant so that $N_{\mathrm{GL}(4n,\mathbf{R})}(G) \not\subset \mathrm{SL}(4n,\mathbf{R})$. According to *magma*, $\dim[H_+\#L(i,2n-i)]$ is 3 when i is even and 6 when i is odd. □

The six $\mathbf{F}_2\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(H_+\#L(i,2n-i))$-linear maps

eq:basisH+L (4.40)
$$\{df_+, df_0, f_0, df_+^D, df_0^D, f_0^D\}$$

given by

$$df_+[P > L] = \begin{cases} L & P = H_+ \\ 0 & \text{otherwise} \end{cases} \qquad df_0[P > L] = \begin{cases} P^{\mathbf{A}(P)} & P = H_+, q(L) = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_0[P > L] = \begin{cases} v_1 & [P,P] = 0, q(L) = 0 \\ 0 & \text{otherwise} \end{cases}$$

is a basis for the 6-dimensional $\mathbf{F}_2$-vector space $[H_+\#L(i,2n-i)]$ for i odd and $[H_+\#L(i,2n-i)] \times [(H_+\#L(i,2n-i))^D]$ for i even. Here, v_1 is one of the two non-zero vectors of $V^{\mathbf{A}(V)}$ that are not D-invariant when i is odd and the nonzero vector of $V^{\mathbf{A}(V)}$ when i is even where $V = H_+\#L(i,2n-i)$.

The quadratic function on $H_-\#L(i,n-i) \in \mathrm{Ob}(\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R})))$, $1 \leq i \leq [n/2]$, $q(v_1,v_2,v_3) = v_1^2 + v_1v_2 + v_2^2$, has automorphism group

$$O(q) = \begin{pmatrix} O^-(2,\mathbf{F}_2) & * \\ 0 & 1 \end{pmatrix}$$

of order $|O^-(2,\mathbf{F}_2)| \cdot 2^2 = 24$.

prop:H-L 4.41. PROPOSITION. *$H_-\#L(i,n-i) \neq (H_-\#L(i,n-i))^D$ for all $n \geq 2$. The Quillen automorphism group $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(H_-\#L(i,n-i)) = O(q)$ has order 24 and the dimension of the space of equivariant maps is $\dim[H_-\#L(i,n-i)] = 1$.*

PROOF. $H_-\#L(i,n-i) \subset \mathrm{PSL}(4n,\mathbf{R})$ is the quotient of

$$G = 2_-^{1+2} \times 2 = \langle \mathrm{diag}\left(\begin{pmatrix} 0 & -R \\ R & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & -R \\ R & 0 \end{pmatrix}\right), \mathrm{diag}\left(\begin{pmatrix} 0 & -T \\ T & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & -T \\ T & 0 \end{pmatrix}\right),$$

$$\mathrm{diag}\Big(\overbrace{\begin{pmatrix} -E & 0 \\ 0 & -E \end{pmatrix}, \dots, \begin{pmatrix} -E & 0 \\ 0 & -E \end{pmatrix}}^{i}, \overbrace{\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}, \dots \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}}^{n-i}\Big)\rangle = \langle g_1, g_2, g_3\rangle \subset \mathrm{SL}(4n,\mathbf{R})$$

The centralizer of G in $\mathrm{GL}(4n,\mathbf{R})$ is contained in the centralizer of its subgroup 2_-^{1+2} which is contained in $\mathrm{SL}(4n,\mathbf{R})$ (9.4). Observe that

- $\mathbf{A}(\mathrm{SL}(4,\mathbf{R}))(2_-^{1+2}) \cong O(q)$
- Conjugation with $\mathrm{diag}\Big(\overbrace{\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}, \dots, \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}}^{i}, \overbrace{\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}, \dots, \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}}^{n-i}\Big)$ induces the automorphism $(g_1,g_2,g_3) \xrightarrow{\phi_1} (g_1g_3, g_2, g_3)$
- Conjugation with $\mathrm{diag}\Big(\overbrace{\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}, \dots, \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}}^{i}, \overbrace{\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}, \dots, \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}}^{n-i}\Big)$ induces the automorphism $(g_1,g_2,g_3) \xrightarrow{\phi_2} (g_1, g_2g_3, g_3)$
- When $i = n/2$, conjugation with $\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$ induces $(g_1,g_2,g_3) \xrightarrow{\phi} (g_1,g_2,-g_3)$

Consider the automorphism groups

$$\mathbf{A}(\mathrm{SL}(4n,\mathbf{R}))(G)\subset\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)\subset\mathrm{Out}(G)\to O(q)\subset\mathrm{Aut}(H_-\#L(i,n-i))$$

where the outer automorphism group has order 48. Note that the automorphism ϕ is in the kernel of the homomorphism $\mathrm{Out}(G)\to O(q)$ induced by abelianization $G\to H_-\#L(i,n-i)$. Using the above observations we see that $\mathbf{A}(\mathrm{SL}(4n,\mathbf{R}))(G)$ maps onto $O(q)$. Thus the Quillen automorphism group $\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)$ has order 48 or 24. When n is even and $i=n/2$ the automorphism ϕ is in $\mathbf{A}(\mathrm{SL}(4n,\mathbf{R}))(G)$ and when $i<n/2$, ϕ is not in $\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)$ as it does not preserve trace. Thus

$$|\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)|=\begin{cases}48 & i=n/2\\ 24 & i<n/2\end{cases}$$

In any case, the group $\mathbf{A}(\mathrm{SL}(4n,\mathbf{R}))(G)$ equals $\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)$ so that $H_-\#L(i,n-i)\neq(H_-\#L(i,n-i))^D$ (9.2). According to *magma*, $\dim[H_-\#L(i,n-i)]=1$. □

The $\mathbf{F}_2\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(H_-\#L(i,n-i))$-linear map

eq:basisH-L

$$\{df_-\} \tag{4.42}$$

given by

$$df_-[P>L]=\begin{cases}L & P=H_-\\ 0 & \text{otherwise}\end{cases}$$

is a basis for the 1-dimensional $\mathbf{F}_2$-vector space $[H_-\#L(i,n-i)]$.

The quadratic function $q(v_1,v_2,v_3,v_4)=v_1^2+v_2v_3$ has automorphism group

$$O(q)=\begin{pmatrix}\mathrm{Sp}(2,\mathbf{F}_2) & *\\ 0 & 1\end{pmatrix},\qquad \mathrm{Sp}(2,\mathbf{F}_2)\cong\left\langle\begin{pmatrix}1&0&0\\1&1&1\\0&1&0\end{pmatrix},\begin{pmatrix}1&0&0\\1&1&1\\0&0&1\end{pmatrix}\right\rangle\subset\mathrm{GL}(3,\mathbf{F}_2)$$

of order 48.

lemma:intoV0L

4.43. Proposition. *The 4-dimensional object $V_0\#L(i,n-i)$, $1\leq i\leq[n/2]$, of the category $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))$ satisfies $V_0\#L(i,n-i)\neq(V_0\#L(i,n-i))^D$. It contains the objects objects V_0, $H_+\#L(2i,2n-2i)$, and $H_-\#L(i,n-i)$. The automorphism group $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(V_0\#L(i,n-i))=O(q)$ and the dimension of the space of equivariant maps is $\dim[V_0\#L(i,n-i)]=5$.*

Proof. $V_0\#L(i,n-i)\subset\mathrm{PSL}(4n,\mathbf{R})$ is (9.7) the quotient of

$$G=2_\pm^{1+2}\circ 4\times 2=$$

$$\Big\langle\mathrm{diag}\left(\begin{pmatrix}0&-E\\E&0\end{pmatrix},\dots,\begin{pmatrix}0&-E\\E&0\end{pmatrix}\right),\mathrm{diag}\left(\begin{pmatrix}R&0\\0&R\end{pmatrix},\dots,\begin{pmatrix}R&0\\0&R\end{pmatrix}\right),\mathrm{diag}\left(\begin{pmatrix}T&0\\0&T\end{pmatrix},\dots,\begin{pmatrix}T&0\\0&T\end{pmatrix}\right),$$

$$\mathrm{diag}\Big(\overbrace{\begin{pmatrix}-E&0\\0&-E\end{pmatrix},\dots,\begin{pmatrix}-E&0\\0&-E\end{pmatrix}}^{i},\overbrace{\begin{pmatrix}E&0\\0&E\end{pmatrix},\dots\begin{pmatrix}E&0\\0&E\end{pmatrix}}^{n-i}\Big)\Big\rangle=\langle g_1,g_2,g_3,g_4\rangle\subset\mathrm{SL}(4n,\mathbf{R})$$

The centralizer of G in $\mathrm{GL}(4n,\mathbf{R})$ is contained in the centralizer of its subgroup 2_-^{1+2} which is contained in $\mathrm{SL}(4n,\mathbf{R})$ (9.4). Observe that

- $\mathbf{A}(\mathrm{SL}(4,\mathbf{R}))(2_\pm^{1+2}\circ 4)=\mathrm{Out}(G)\cong\mathrm{Out}(C_4)\times\mathrm{Sp}(2,\mathbf{F}_2)$ (9.4)
- Conjugation with $\mathrm{diag}\Big(\overbrace{\begin{pmatrix}0&E\\E&0\end{pmatrix},\dots,\begin{pmatrix}0&E\\E&0\end{pmatrix}}^{i},\overbrace{\begin{pmatrix}E&0\\0&E\end{pmatrix},\dots,\begin{pmatrix}E&0\\0&E\end{pmatrix}}^{n-i}\Big)$ induces the automorphism $(g_1,g_2,g_3,g_4)\xrightarrow{\phi_1}(g_1g_4,g_2,g_3,g_4)$
- Conjugation with $\mathrm{diag}\Big(\overbrace{\begin{pmatrix}T&0\\0&T\end{pmatrix},\dots,\begin{pmatrix}T&0\\0&T\end{pmatrix}}^{i},\overbrace{\begin{pmatrix}E&0\\0&E\end{pmatrix},\dots,\begin{pmatrix}E&0\\0&E\end{pmatrix}}^{n-i}\Big)$ induces the automorphism $(g_1,g_2,g_3,g_4)\xrightarrow{\phi_2}(g_1,g_2g_4,g_3,g_4)$

- Conjugation with $\mathrm{diag}\Big(\overbrace{\begin{pmatrix} R & 0 \\ 0 & R\end{pmatrix},\dots,\begin{pmatrix} R & 0 \\ 0 & R\end{pmatrix}}^{i},\overbrace{\begin{pmatrix} E & 0 \\ 0 & E\end{pmatrix},\dots,\begin{pmatrix} E & 0 \\ 0 & E\end{pmatrix}}^{n-i}\Big)$ induces the automorphism $(g_1,g_2,g_3,g_4) \xrightarrow{\phi_3} (g_1,g_2,g_3g_4,g_4)$
- Conjugation with $\mathrm{diag}\Big(\begin{pmatrix} 0 & E \\ E & 0\end{pmatrix},\dots,\begin{pmatrix} 0 & E \\ E & 0\end{pmatrix}\Big)$ induces the automorphism $(g_1,g_2,g_3,g_4) \xrightarrow{\phi_4} (-g_1,g_2,g_3g_4)$
- When $i = n/2$, conjugation with $\begin{pmatrix} 0 & E \\ E & 0\end{pmatrix} \in \mathrm{SL}(4n,\mathbf{R})$ induces the automorphism $(g_1,g_2,g_3,g_4) \xrightarrow{\phi_5} (g_1,g_2,g_3,-g_4)$

Consider the automorphism groups

$$\mathbf{A}(\mathrm{SL}(4n,\mathbf{R}))(G) \subset \mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G) \subset \mathrm{Out}(G) \to O(q) \subset \mathrm{Aut}(V_0\#L(i,n-i))$$

where the outer automorphism group has order 196 and $O(q)$ order 48. Note that the automorphism ϕ_4 of order 2 is in the kernel of the homomorphism $\mathrm{Out}(G) \to O(q)$ induced by abelianization $G \to V_0\#L(i,n-i)$. Using the above observations we see that $\mathbf{A}(\mathrm{SL}(4n,\mathbf{R}))(G)$ maps onto $O(q)$ with a kernel of order at least 2. Thus the Quillen automorphism group $\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)$ has order 192 or 96. When n is even and $i = n/2$ the automorphism ϕ_5 is in $\mathbf{A}(\mathrm{SL}(4n,\mathbf{R}))(G)$ and when $i < n/2$, ϕ_5 is not in $\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)$ as it does not preserve trace. Thus

$$|\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)| = \begin{cases} 192 & i = n/2 \\ 96 & i < n/2 \end{cases}$$

In any case, the group $\mathbf{A}(\mathrm{SL}(4n,\mathbf{R}))(G)$ equals $\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)$ so that $V_0\#L(i,n-i) \neq (V_0\#L(i,n-i))^D$ (9.2). According to *magma*, $\dim[V_0\#L(i,n-i)] = 5$. □

The five $\mathbf{F}_2\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(V_0\#L(i,n-i))$-linear maps

eq:basisV0L (4.44) $$\{ddf_{+L(2i,2n-2i)}, ddf_{0L(2i,2n-2i)}, df_{0L(2i,2n-2i)}, ddf_{-L(i,n-i)}, df_{0V_0}\}$$

given by

$$ddf_{+L(2i,2n-2i)}[V > P > L] = \begin{cases} L & V = H_+\#L(2i,2n-2i), P = H_+ \\ 0 & \text{otherwise}\end{cases}$$

$$ddf_{0L(2i,2n-2i)}[V > P > L] = \begin{cases} P^{\mathbf{A}(P)} & V = H_+\#L(2i,2n-2i), P = H_+, q(L) = 0 \\ 0 & \text{otherwise}\end{cases}$$

$$df_{0L(2i,2n-2i)}[V > P > L] = \begin{cases} V^{\mathbf{A}(V)} & V = H_+\#L(2i,2n-2i), [P,P] = 0, q(L) = 0 \\ 0 & \text{otherwise}\end{cases}$$

$$ddf_{-L(i,n-i)}[V > P > L] = \begin{cases} L & V = H_-\#L(i,n-i), P = H_- \\ 0 & \text{otherwise}\end{cases}$$

$$df_{0V_0}[V > P > L] = \begin{cases} V^{\mathbf{A}(V)} & V = V_0, [P,P] = 0, q(L) = 0 \\ 0 & \text{otherwise}\end{cases}$$

constitute a basis for $[V_0\#L(i,n-i)]$.

lemma:intoH+P 4.45. Lemma. *The 4-dimensional object $H_+\#P(1,i-1,2n-i,0)$, $2 < i \le n$, of the category $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))$, $n > 2$, satisfies $H_+\#P(1,i-1,2n-i,0) = (H_+\#P(1,i-1,2n-i,0))^D$. It contains the 3-dimensional objects*

$$H_+\# \begin{cases} L(1,2n-1), L(i-1,2n-i+1), L(i-1,2n-i+1)^D, L(i,2n-i) & i \text{ odd} \\ L(1,2n-1), L(i-1,2n-i+1), L(i,2n-i), L(i,2n-i)^D & i \text{ even}\end{cases}$$

Its Quillen automorphism group is

$$\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(H_+\#P(1,i-1,2n-i,0)) =$$
$$\begin{cases} \left\langle \begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\ 0&1&1&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&1\\ 0&1&1&1\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix} \right\rangle & i>2 \text{ odd} \\ \left\langle \begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&1\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\ 0&1&0&1\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&1\\ 0&1&1&1\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix} \right\rangle & i>2 \text{ even} \end{cases}$$

of order 16*. The space of equivariant maps has dimension* $\dim[H_+\#P(1,i-1,2n-i,0)] = 16$*.*

PROOF. $H_+\#P(1,i-1,2n-i) \subset \mathrm{PSL}(4n,\mathbf{R})$ is (4.52) the quotient of

$$G = \langle \mathrm{diag}(R,\ldots,R), \mathrm{diag}(T,\ldots,T),$$
$$\mathrm{diag}(E,\overbrace{-E,\ldots,-E}^{i-1},\overbrace{E,\ldots,E}^{2n-i}), \mathrm{diag}(E,\overbrace{E,\ldots,E}^{i-1},\overbrace{-E,\ldots,-E}^{2n-i})\rangle = \langle g_1,g_2,g_3,g_4\rangle \subset \mathrm{SL}(4n,\mathbf{R})$$

The centralizer of G in $\mathrm{GL}(4n,\mathbf{R})$ is contained in the centralizer of its subgroup 2_+^{1+2} which is contained in $\mathrm{SL}(4n,\mathbf{R})$ (9.4). This means (9.3) that the elements of $\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G)$ and $\mathbf{A}(\mathrm{PGL}(4n,\mathbf{R}))(H_+\#P(1,i-1,2n-i,0))$ have a well-defined sign. The Quillen automorphism group is contained in the group $\begin{pmatrix} O^+(2,\mathbf{F}_2) & * \\ 0 & E \end{pmatrix}$ of order $2^5 = 32$. Observe that

- R and T are conjugate in $\mathrm{GL}(2,\mathbf{R})$ so that $(g_1,g_2,g_3,g_4) \xrightarrow{\phi_1} (g_2,g_1,g_3,g_4)$ is in the Quillen automorphism group and has sign $+1$
- Conjugation with $\mathrm{diag}(E,\overbrace{E,\ldots,E}^{i-1},\overbrace{T,\ldots,T}^{2n-i})$ induces $(g_1,g_2,g_3,g_4) \xrightarrow{\phi_2} (g_1g_4,g_2,g_3,g_4)$ of sign $(-1)^i$
- Conjugation with $\mathrm{diag}(E,\overbrace{T,\ldots,T}^{i-1},\overbrace{E,\ldots,E}^{2n-i})$ induces $(g_1,g_2,g_3,g_4) \xrightarrow{\phi_3} (g_1g_3,g_2,g_3,g_4)$ of sign $-(-1)^i$
- Conjugation with $\mathrm{diag}(E,\overbrace{E,\ldots,E}^{i-1},\overbrace{R,\ldots,R}^{2n-i})$ induces $(g_1,g_2,g_3,g_4) \xrightarrow{\phi_4} (g_1,g_2g_4,g_3,g_4)$ of sign $(-1)^i$
- Conjugation with $\mathrm{diag}(E,\overbrace{R,\ldots,R}^{i-1},\overbrace{E,\ldots,E}^{2n-i})$ induces $(g_1g_3,g_2,g_3,g_4) \xrightarrow{\phi_5} (g_1,g_2g_3,g_3,g_4)$ of sign $-(-1)^i$
- Conjugation with $\mathrm{diag}(E,\overbrace{RT,\ldots,RT}^{i-1},\overbrace{RT,\ldots,RT}^{2n-i})$ induces the automorphism given by $(g_1,g_2,g_3,g_4) \xrightarrow{\phi_6} (g_1g_3g_4,g_2g_3g_4,g_3,g_4)$ of sign $+1$.

It follows that $N_{\mathrm{GL}(4n,\mathbf{R})}(G) \not\subset \mathrm{SL}(4n,\mathbf{R})$ as this normalizer contains elements of negative determinant regardless of the parity of i. Also, $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(H_+\#P(1,i-1,2n-i,0))$ is generated by (the automorphisms induced by) ϕ_1, ϕ_2, ϕ_4, and ϕ_6 when i is even, and ϕ_1, ϕ_3, ϕ_5, and ϕ_6 when i is odd. □

The fourteen $\mathbf{F}_2\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(H_+\#P(1,i-1,2n-1))$-linear maps

eq:basispH+P

$$\begin{aligned}(4.46)\quad \{&ddf_{+L(i-1,2n-i+1)}, ddf^D_{+L(i-1,2n-i+1)}, ddf_{0L(i-1,2n-i+1)}, ddf^D_{0L(i-1,2n-i+10)},\\ &df_{0L(i-1,2n-i+1)}, df^D_{0L(i-1,2n-i+1)},\\ &ddf_{+L(i,2n-i)}, ddf^D_{+L(i,2n-i)}, ddf_{0L(i,2n-i)}, ddf^D_{0L(i,2n-i)}, df_{0L(i,2n-i)}, df^D_{0L(i,2n-i)},\\ &df_{0L(1,2n-1)}, df^D_{0L(1,2n-1)}\}\end{aligned}$$

form a partial basis for the 16-dimensional vector space $[H_+\#P(1,i-1,2n-1)]$, $2<i\leq n$. For $1<i\leq n$ and i odd,

$$ddf_{+L(i,2n-i)}[V>P>L]=\begin{cases} L & V=H_+\#L(i,2n-i), P=H_+ \\ 0 & \text{otherwise}\end{cases}$$

$$ddf^D_{+L(i,2n-i)}[V>P>L]=\begin{cases} L & V=H_+\#L(i,2n-i), P=H_+^D \\ 0 & \text{otherwise}\end{cases}$$

$$ddf_{0L(i,2n-i)}[V>P>L]=\begin{cases} P^{\mathbf{A}(P)} & V=H_+\#L(i,2n-i), P=H_+, q(L)=0 \\ 0 & \text{otherwise}\end{cases}$$

$$ddf^D_{0L(i,2n-i)}[V>P>L]=\begin{cases} P^{\mathbf{A}(P)} & V=H_+\#L(i,2n-i), P=H_+^D, q(L)=0 \\ 0 & \text{otherwise}\end{cases}$$

$$df_{01L(i,2n-i)}[V>P>L]=\begin{cases} V\cap O_1 & V=H_+\#L(i,2n-i), [P,P]=0, q(L)=0 \\ 0 & \text{otherwise}\end{cases}$$

$$df^D_{01L(i,2n-i)}[V>P>L]=\begin{cases} V\cap O_2 & V=H_+\#L(i,2n-i), [P,P]=0, q(L)=0 \\ 0 & \text{otherwise}\end{cases}$$

where (in the last two formulas), O_1 and O_2 are the two orbits of length 2 for the action of $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(H_+\#P(1,i-1,2n-i,0))$ on $H_+\#P(1,i-1,2n-i,0)$. Each of the hyperplanes isomorphic to $V=H_+\#L(i,2n-i)$ contains precisely one vector v_1 from O_1 and one vector v_2 from O_2 and $\{v_1,v_2\}$ is a basis for the fixed point group $V^{\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))(V)}$. For $1<i\leq n$ and i even,

$$ddf_{+L(i,2n-i)}[V>P>L]=\begin{cases} L & V=H_+\#L(i,2n-i), P=H_+ \\ 0 & \text{otherwise}\end{cases}$$

$$ddf^D_{+L(i,2n-i)}[V>P>L]=\begin{cases} L & V=(H_+\#L(i,2n-i))^D, P=(H_+)^D \\ 0 & \text{otherwise}\end{cases}$$

$$ddf_{0L(i,2n-i)}[V>P>L]=\begin{cases} P^{\mathbf{A}(P)} & V=H_+\#L(i,2n-i), P=H_+, q(L)=0 \\ 0 & \text{otherwise}\end{cases}$$

$$ddf^D_{0L(i,2n-i)}[V>P>L]=\begin{cases} P^{\mathbf{A}(P)} & V=(H_+\#L(i,2n-i))^D, P=(H_+)^D, q(L)=0 \\ 0 & \text{otherwise}\end{cases}$$

$$df_{0L(i,2n-i)}[V>P>L]=\begin{cases} V^{\mathbf{A}(V)} & V=H_+\#L(i,2n-i), [P,P]=0, q(L)=0 \\ 0 & \text{otherwise}\end{cases}$$

$$df^D_{0L(i,2n-i)}[V>P>L]=\begin{cases} V^{\mathbf{A}(V)} & V=(H_+\#L(i,2n-i))^D, [P,P]=0, q(L)=0 \\ 0 & \text{otherwise}\end{cases}$$

lemma:intoH+P1120

4.47. LEMMA. *The 4-dimensional object $H_+\#P(1,1,2,0)$ of the category $\mathbf{A}(\mathrm{PSL}(8,\mathbf{R}))$ satisfies $H_+\#P(1,1,2,0)=(H_+\#P(1,1,2,0))^D$. It contains the 3-dimensional objects*

$$H_+\#L(1,3), H_+\#L(2,2), (H_+\#L(2,2))^D$$

Its Quillen automorphism group is

$$\mathbf{A}(\mathrm{PSL}(8,\mathbf{R}))(H_+\#P(1,1,2,0))=$$
$$\left\langle \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&1&1\\0&1&1&1\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1\end{pmatrix} \right\rangle$$

of order 32, *and* $\dim[H_+\#P(1,i-1,2n-i,0)]=8$.

PROOF. The proof is similar to that of 4.45. $H_+\#P(1,1,2,0) \subset \mathrm{PSL}(8,\mathbf{R})$ is the quotient of

$$G = \langle \mathrm{diag}(R,R,R,R), \mathrm{diag}(T,T,T,T), \mathrm{diag}(E,-E,E,E), \mathrm{diag}(E,E,-E,-E)\rangle \subset \mathrm{SL}(8,\mathbf{R})$$

The extra element of $\mathbf{A}(\mathrm{PSL}(8,\mathbf{R}))(H_+\#P(1,1,2,0))$ is induced by conjugation with the matrix $\mathrm{diag}\left(\begin{pmatrix}0 & E\\ E & 0\end{pmatrix}, \begin{pmatrix}E & 0\\ 0 & E\end{pmatrix}\right) \in \mathrm{SL}(8,\mathbf{R})$. According to *magma*, $\dim[H_+\#P(1,1,2,0)] = 8$. □

The eight $\mathbf{F}_2\mathbf{A}(\mathrm{PSL}(8,\mathbf{R}))(H_+\#P(1,1,2,0))$-linear maps

eq:basisH+P112

$$\{ddf_{+L(2,2)}, ddf^D_{+L(2,2)}, ddf_{0L(2,2)}, ddf^D_{0L(2,2)}, df_{0L(2,2)}, df^D_{0L(2,2)}, df_{01L(1,3)}, df^D_{01L(1,3)}\} \tag{4.48}$$

is a basis for the vector space $[H_+\#P(1,1,2,0)]$.

We are now ready to describe the differentials d^1 and d^2 in Oliver's cochain complex (4.33) for computing the limits of the functor $\pi_1(BZC_{\mathrm{PSL}(4n,\mathbf{R})}(V)) = V$ on the category $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))$. The $6 \times (6n + 2[n/2] + 8)$ matrix for d^1 is of the following form (shown here for $n = 3$)

	$[H_+\#L(1,5)]$	$[H_+\#L(2,4)] \times [H_+\#L(2,4)]^D$	$H_+\#L(3,3)$
$[H_+]$	$\begin{pmatrix}A & 0\end{pmatrix}$	$\begin{pmatrix}A & 0\end{pmatrix}$	$\begin{pmatrix}A & 0\end{pmatrix}$
$[H_+]^D$	$\begin{pmatrix}0 & A\end{pmatrix}$	$\begin{pmatrix}0 & A\end{pmatrix}$	$\begin{pmatrix}0 & A\end{pmatrix}$
$[H_-]$			
$[H_-]^D$			

$[H_-\#L(1,1)] \times [H_-\#L(1,1)]^D$	$[V_0] \times [V_0]^D$	
	$\begin{pmatrix}H & 0\end{pmatrix}$	$[H_+]$
	$\begin{pmatrix}0 & H\end{pmatrix}$	$[H_+]^D$
$\begin{pmatrix}1 & 0\end{pmatrix}$	$\begin{pmatrix}B & 0\end{pmatrix}$	$[H_-]$
$\begin{pmatrix}0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & B\end{pmatrix}$	$[H_-]^D$

where

$$A = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}, \qquad H = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}, \qquad B = \begin{pmatrix}0 & 0 & 1 & 0\end{pmatrix}$$

is injective so $\lim^1 = 0$. Exactness is thus equivalent to

$$\dim(\mathrm{im}\, d^2) \geq 6n + 2[n/2] + 2$$

We shall show this by mapping the $n + [n/2] + 2[n/2] + 2$ objects of dimension 3,

$$H_+\#L(i, 2n-i), (H_+\#L(i,2n-i))^D (i \text{ even}), \quad 1 \leq i \leq n,$$
$$H_-\#L(i, n-i),\ (H_-\#L(i,n-i))^D, \quad 1 \leq i \leq [n/2], \qquad V_0,\ V_0^D,$$

of $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))$ to the $n - 2 + 2[n/2]$ objects of dimension 4,

$$H_+\#P(1, i-1, 2n-i, 0), \quad 2 < i \leq n, \qquad V_0\#L(n-i,i),\ (V_0\#L(n-i,i))^D, \quad 1 \leq i \leq [n/2],$$

for $n > 2$ and to

$$H_+\#P(1,1,2,0), \quad V_0\#L(1,1), \quad (V_0\#L(1,1))^D$$

when $n = 2$. The $(6n + 2[n/2] + 8) \times (16(n-2) + 10[n/2])$-matrix for d^2 (shown here for $n = 5$) is

	$[H_+\#P(1,2,7)]$	$[H_+\#P(1,3,6)]$	$[H_+\#P(1,4,5)]$
$[H_+\#L(1,9)]$	$\begin{pmatrix}A & A & B\end{pmatrix}$	$\begin{pmatrix}A & A & B\end{pmatrix}$	$\begin{pmatrix}A & A & B\end{pmatrix}$
$[H_+\#L(2,8)] \times [H_+\#L(2,8)]^D$	$\begin{pmatrix}E & 0 & 0\end{pmatrix}$		
$[H_+\#L(3,7)]$	$\begin{pmatrix}0 & E & 0\end{pmatrix}$	$\begin{pmatrix}E & 0 & 0\end{pmatrix}$	
$[H_+\#L(4,6)] \times [H_+\#L(4,6)]^D$		$\begin{pmatrix}0 & E & 0\end{pmatrix}$	$\begin{pmatrix}E & 0 & 0\end{pmatrix}$
$[H_+\#L(5,5)]$			$\begin{pmatrix}0 & E & 0\end{pmatrix}$
$[H_-\#L(1,4)] \times [H_-\#L(1,4)]^D$			
$[H_-\#L(2,3)] \times [H_-\#L(2,3)]^D$			
$[V_0] \times [V_0]^D$			

	$[V_0\#L(1,4)]$	$[V_0\#L(1,4)]^D$	$[V_0\#L(2,3)]$	$[V_0\#L(2,3)]^D$
$[H_+\#L(1,9)]$				
$[H_+\#L(2,8)] \times [H_+\#L(2,8)]^D$	$\begin{pmatrix} H \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ H \end{pmatrix}$		
$[H_+\#L(3,7)]$				
$[H_+\#L(4,6)] \times [H_+\#L(4,6)]^D$			$\begin{pmatrix} H \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ H \end{pmatrix}$
$[H_+\#L(5,5)]$				
$[H_-\#L(1,4)] \times [H_-\#L(1,4)]^D$	$\begin{pmatrix} L \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ L \end{pmatrix}$		
$[H_-\#L(2,3)] \times [H_-\#L(2,3)]^D$		$\begin{pmatrix} L \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ L \end{pmatrix}$	
$[V_0] \times [V_0]^D$	$\begin{pmatrix} K \\ 0 \end{pmatrix}$	$\begin{pmatrix} K \\ 0 \end{pmatrix}$	$\begin{pmatrix} K \\ 0 \end{pmatrix}$	$\begin{pmatrix} K \\ 0 \end{pmatrix}$

where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

while E is (6×6) unit matrix and 0 a zero matrix. These matrices are given with respect to the bases (4.38, 4.40, 4.42, 4.46, 4.44).

The case $n = 2$ of $\mathrm{PSL}(8, \mathbf{R})$ is special. Part of the matrix for d^2 is the (22×18)-matrix

	$[H_+\#P(1,1,2,0)]$	$[V_0\#L(1,1)]$	$[V_0\#L(1,1)]^D$
$[H_+\#L(1,3)]$	$\begin{pmatrix} A & B \end{pmatrix}$		
$[H_+\#L(2,2)] \times [H_+\#L(2,2)]^D$	$\begin{pmatrix} E & 0 \end{pmatrix}$	$\begin{pmatrix} H \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ H \end{pmatrix}$
$[H_-\#L(1,1)] \times [H_-\#L(1,1)]^D$		$\begin{pmatrix} L \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ L \end{pmatrix}$
$[V_0] \times [V_0]^D$		$\begin{pmatrix} K \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ K \end{pmatrix}$

where now

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

while E is 6×6 unit matrix and 0 a zero matrix. As (partial) bases we use the ordered sets (4.40, 4.42, 4.38, 4.48, 4.44, 4.40). This matrix has rank 16.

4.49. Corollary. *The partial differential*

$$\prod_{\substack{1\le i\le n\\ i\ odd}} [H_+\#L(i,2n-i)] \times \prod_{\substack{1\le i\le n\\ i\ even}} [H_+\#L(i,2n-i)] \times [H_+\#L(i,2n-i)]^D$$
$$\times \prod_{1\le i\le [n/2]} [H_-\#L(i,n-i)] \times [H_-\#L(i,n-i)]^D \times [V_0] \times [V_0]^D$$
$$\xrightarrow{d^2} \prod_{2<i\le n} [H_+\#P(1,i-1,2n-i,0)] \times \prod_{1\le i\le [n/2]} [V_0\#L(i,n-i)]$$

has rank $6n+2[n/2]+2$.

Proof. By now we know a matrix for this linear map so we simply check its rank. □

Proof of Lemma 4.32. For π_2 use that it is trivial on the objects with $[,]\neq 0$. □

sec:nontoral

7. The category $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))^{[\,,\,]\neq 0}_{\leq 4}$

We shall need information about all objects of $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))^{[\,,\,]\neq 0}$ of rank ≤ 3 and some objects of rank 4. If $V \subset \mathrm{PSL}(4n,\mathbf{R})$ is a nontoral elementary abelian 2-group with nontrivial inner product then its preimage $V^* \subset \mathrm{SL}(4n,\mathbf{R})$ is $P \times R(V)$ or $(C_4 \circ P) \times R(V)$ where P is an extraspecial 2-group, $C_4 \circ P$ a generalized extraspecial 2-group, and $\mho_1(V^*) = \langle -E \rangle$ (4.8). We manufacture all oriented real representations of these product groups as direct sums of tensor products of irreducible representations of the factors (9.6).

0innerprodrank2

4.50. Rank two objects with nontrivial inner product. The category $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))$ contains up to isomorphism four rank two objects with nontrivial inner product, $H_\pm$ and $H_\pm^D$. The elementary abelian 2-group $H_\pm \subset \mathrm{PSL}(4n,\mathbf{R})$ is the quotient of the extraspecial 2-group $2_\pm^{1+2} \subset \mathrm{SL}(4n,\mathbf{R})$ with $\mho_1(2_\pm^{1+2}) = \langle -E \rangle$ described in 9.4.(6) and 9.4.(7). Their centralizers [54, Proposition 4] in $\mathrm{SL}(4n,\mathbf{R})$ and $\mathrm{PSL}(4n,\mathbf{R})$ are

$$C_{\mathrm{SL}(4n,\mathbf{R})}(2_+^{1+2}) = \mathrm{GL}(2n,\mathbf{R}), \quad C_{\mathrm{PSL}(4n,\mathbf{R})}(H_+) = H_+ \times \mathrm{PGL}(2n,\mathbf{R}) = V_+ \times (\mathrm{PSL}(2n,\mathbf{R}) \rtimes C_2)$$
$$C_{\mathrm{SL}(4n,\mathbf{R})}(2_-^{1+2}) = \mathrm{GL}(n,\mathbf{H}), \quad C_{\mathrm{PSL}(4n,\mathbf{R})}(H_-) = H_- \times \mathrm{PGL}(n,\mathbf{H})$$

where H_+ and H_- are hyperbolic planes with quadratic functions $q_+(v_1,v_2) = v_1v_2$ and $q_-(v_1,v_2) = v_1^2 + v_1v_2 + v_2^2$ (9.5), respectively. In the first case, for instance, the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{PGL}(2n,\mathbf{R}) & \longrightarrow & C_{\mathrm{PSL}(4n,\mathbf{R})}(H_+) & \longrightarrow & H_+^\vee & \longrightarrow & 0 \\ & & & & \uparrow & \overset{\cong}{\nearrow}{\scriptstyle [\cdot,\cdot]} & & & \\ & & & & H_+ & & & & \end{array}$$

gives a central section of the short exact sequence from [47, 5.11].

0innerprodrank3

4.51. Rank three objects with nontrivial inner product. Let V be a rank three object of $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))$ with nontrivial inner product. Then V or V^D is isomorphic to $H_+\#L(i,2n-i)$ $(1 \leq i \leq n)$, $H_-\#L(i,n-i)$ $(1 \leq i \leq [n/2])$ or V_0. $H_+\#L(i,2n-i) \subset \mathrm{PSL}(4n,\mathbf{R})$ is defined to be the quotient of

$$\Big\langle \mathrm{diag}(R,\dots,R), \mathrm{diag}(T,\dots,T), \mathrm{diag}(\overbrace{-E,\dots,-E}^{i}, \overbrace{E,\dots,E}^{2n-i}) \Big\rangle \subset \mathrm{SL}(4n,\mathbf{R}),$$
$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

isomorphic to $2_+^{1+2} \times C_2$ and $H_-\#L(i,n-i) \subset \mathrm{PSL}(4n,\mathbf{R})$ to be the quotient of

$$\Big\langle \mathrm{diag}\left(\begin{pmatrix} 0 & -R \\ R & 0 \end{pmatrix},\dots,\begin{pmatrix} 0 & -R \\ R & 0 \end{pmatrix}\right), \mathrm{diag}\left(\begin{pmatrix} 0 & -T \\ T & 0 \end{pmatrix},\dots,\begin{pmatrix} 0 & -T \\ T & 0 \end{pmatrix}\right),$$
$$\mathrm{diag}\Big(\overbrace{\begin{pmatrix} -E & 0 \\ 0 & -E \end{pmatrix},\dots,\begin{pmatrix} -E & 0 \\ 0 & -E \end{pmatrix}}^{i}, \overbrace{\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix},\dots\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}}^{n-i}\Big)\Big\rangle \subset \mathrm{SL}(4n,\mathbf{R})$$

isomorphic to $2_-^{1+2} \times C_2$. The elementary abelian 2-group $V_0 \subset \mathrm{PSL}(4n,\mathbf{R})$ is the quotient of

$$\Big\langle \mathrm{diag}\left(\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix},\dots,\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}\right), \mathrm{diag}\left(\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix},\dots,\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}\right),$$
$$\mathrm{diag}\left(\begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix},\dots,\begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}\right)\Big\rangle$$

isomorphic to the generalized extraspecial 2-group $C_4 \circ 2_\pm^{1+2} \subset \mathrm{SL}(4n,\mathbf{R})$ as described in 9.4.(5).

innerprodrank4

4.52. Rank four objects with nontrivial inner product. The following partial census of rank four objects with nontrivial inner product suffices for our purposes. Define the elementary abelian 2-group $H_+\#P(1,i-1,2n-i) \subset \mathrm{PSL}(4n,\mathbf{R})$, $2 \le i \le n$, to be the quotient of

$$\big\langle \operatorname{diag}(R,\dots,R), \operatorname{diag}(T,\dots,T),\\ \operatorname{diag}(E,\overbrace{-E,\dots,-E}^{i-1},\overbrace{E,\dots,E}^{2n-i}), \operatorname{diag}(E,\overbrace{E,\dots,E}^{i-1},\overbrace{-E,\dots,-E}^{2n-i})\big\rangle \subset \mathrm{SL}(4n,\mathbf{R})$$

Define $V_0\#L(i,n-i) \subset \mathrm{PSL}(4n,\mathbf{R})$, $1 \le i \le [n/2]$, to be the quotient of

$$\Big\langle \operatorname{diag}\left(\begin{pmatrix}0 & -E\\ E & 0\end{pmatrix},\dots,\begin{pmatrix}0 & -E\\ E & 0\end{pmatrix}\right), \operatorname{diag}\left(\begin{pmatrix}R & 0\\ 0 & R\end{pmatrix},\dots,\begin{pmatrix}R & 0\\ 0 & R\end{pmatrix}\right),\\ \operatorname{diag}\left(\begin{pmatrix}T & 0\\ 0 & T\end{pmatrix},\dots,\begin{pmatrix}T & 0\\ 0 & T\end{pmatrix}\right),\\ \operatorname{diag}\Big(\overbrace{\begin{pmatrix}-E & 0\\ 0 & -E\end{pmatrix},\dots,\begin{pmatrix}-E & 0\\ 0 & -E\end{pmatrix}}^{i},\overbrace{\begin{pmatrix}E & 0\\ 0 & E\end{pmatrix},\dots\begin{pmatrix}E & 0\\ 0 & E\end{pmatrix}}^{n-i}\Big)\Big\rangle \subset \mathrm{SL}(4n,\mathbf{R})$$

isomorphic to $C_4 \circ 2^{1+2}_{\pm} \times C_2$.

sec:ZC

4.53. Centers of centralizers. For the computations in §6 we need to know the centers of the centralizers for some of the low dimensional objects of $\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))^{[\,,\,]\neq 0}$.

prop:ZC

4.54. PROPOSITION. *Let $V \in \mathrm{Ob}(\mathbf{A}(\mathrm{PSL}(4n,\mathbf{R}))^{[\,,\,]\neq 0})$ be one of the objects*

- *H_+, H_-,*
- *$H_+\#L(i,2n-i)$ $(1 \le i \le n)$, $H_-\#L(i,n-i)$ $(1 \le i \le [n/2])$, V_0, or*
- *$H_+\#P(1,i-1,2n-i,0)$ $(1 < i \le n)$, $V_0\#L(i,n-i)$ $(1 \le i \le [n/2])$*

introduced in 4.50–4.52. Then $ZC_{\mathrm{PSL}(4n,\mathbf{R})}(V) = V$.

PROOF. The proof is a case-by-case checking.
H_+ and H_- Since the centralizers of the rank two objects H_+ and H_- are $C_{\mathrm{PSL}(4n,\mathbf{R})}(H_+) = H_+ \times \mathrm{PGL}(2n,\mathbf{R})$ and $C_{\mathrm{PSL}(4n,\mathbf{R})}(H_-) = H_- \times \mathrm{PGL}(n,\mathbf{H})$, Proposition 4.54 is immediate in these cases.
$H_+\#L(i,2n-i)$ $(1 \le i \le n)$ and $H_+\#P(1,i-1,2n-i,0)$ $(1 < i \le n)$ We shall only prove the 2-dimensional case since the 3-dimensional case is similar. The centralizer of $H_+\#L(i,2n-i)$ is isomorphic to the product of H_+ with the centralizer of $L = L(i,2n-i)$ in $\mathrm{PGL}(2n,\mathbf{R})$. There is [47, 5.11] a short exact sequence

$$1 \to \frac{\mathrm{GL}(i,\mathbf{R}) \times \mathrm{GL}(2n-i,\mathbf{R})}{\langle -E\rangle} \to C_{\mathrm{PGL}(2n,\mathbf{R})}(L) \to \mathrm{Hom}(L,\langle -E\rangle)_\rho \to 1$$

where the group to the right consists of all homomorphisms $\phi\colon L \to \langle -E\rangle$ such that ρ and $\phi\cdot\rho$ are conjugate representations in $\mathrm{GL}(2n,\mathbf{R})$. By trace considerations, this group is trivial if $i<n$ and of order two if $i=n$. Hence

$$C_{\mathrm{PGL}(2n,\mathbf{R})}(L) = \begin{cases} \frac{\mathrm{GL}(i,\mathbf{R})\times\mathrm{GL}(2n-i,\mathbf{R})}{\langle -E\rangle} & i<n\\ \frac{\mathrm{GL}(n,\mathbf{R})^2}{\langle -E\rangle} \rtimes \langle C_1\rangle & i=n\end{cases}$$

where $C_1 = \begin{pmatrix}0 & E\\ E & 0\end{pmatrix}$ is the $(2n\times 2n)$-matrix that interchanges the two $\mathrm{GL}(n,\mathbf{R})$-factors. In case $i<n$, use 9.18. In case $i=n$, the center is (9.13) the pull-back of the group homomorphisms

$$\frac{\mathrm{GL}(n,\mathbf{R})\times\langle (E,-E)\rangle}{\langle -E\rangle} = \left(\frac{\mathrm{GL}(n,\mathbf{R})^2}{\langle -E\rangle}\right)^{\langle C_1\rangle} \to \mathrm{Aut}\left(\frac{\mathrm{GL}(n,\mathbf{R})^2}{\langle -E\rangle}\right) \leftarrow \langle C_1\rangle$$

which is $\frac{\mathrm{GL}(1,\mathbf{R})\times\langle(-E,E)\rangle}{\langle -E\rangle} = L$ again.

$\underline{V_0 \text{ and } V_0\#L(i,n-i)}$ The object $V_0 \subset \mathrm{PSL}(4n,\mathbf{R})$ is the quotient of $G = 4 \circ 2^{1+2}_{\pm} \subset \mathrm{SL}(4n,\mathbf{R})$ as described in 9.4.(5). As this representation $\rho = n(\chi + \overline{\chi})$ is the n-fold sum of an irreducible representation of complex type there are exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \frac{\mathrm{GL}(n,\mathbf{C})}{\langle -E\rangle} & \longrightarrow & C_{\mathrm{PSL}(4n,\mathbf{R})}(V_0) & \longrightarrow & \mathrm{Hom}(G,\langle -E\rangle)_\rho & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & Z(G)/G' & \longrightarrow & G/G' & \longrightarrow & G/Z(G) & \longrightarrow & 1 \end{array}$$

where the top row is [47, 5.11]. The elementary abelian group $\mathrm{Hom}(G,\langle -E\rangle)_\rho$, consisting of all homomorphisms $\phi\colon G \to \langle -E\rangle$ such that ρ and $\phi\cdot\rho$ are conjugate in $\mathrm{SL}(4n,\mathbf{R})$, equals all of $\mathrm{Hom}(G,\langle -E\rangle) = 2^3$ since conjugation with the first two of the generators from 4.51 and with

$$C_2 = \mathrm{diag}\left(\begin{pmatrix} E & 0 \\ 0 & -E\end{pmatrix},\dots,\begin{pmatrix} E & 0 \\ 0 & -E\end{pmatrix}\right)$$

induce three independent generators. Hence

$$C_{\mathrm{PSL}(4n,\mathbf{R})}(V_0) = \left(\frac{\mathrm{GL}(n,\mathbf{C})}{\langle -E\rangle} \times V_0/V_0^\perp\right) \rtimes \langle C_2\rangle$$

Note that conjugation with the matrix C_2 induces complex conjugation on $\mathrm{GL}(n,\mathbf{C})$. The center of this semi-direct product is (9.13) the pull-back of the group homomorphisms

$$\frac{\mathrm{GL}(n,\mathbf{R})\circ\langle i\rangle}{\langle -E\rangle} \times V_0/V_0^\perp = \left(\frac{\mathrm{GL}(n,\mathbf{C})}{\langle -E\rangle} \times V_0/V_0^\perp\right)^{\langle C_2\rangle} \to \mathrm{Aut}\left(\frac{\mathrm{GL}(n,\mathbf{C})}{\langle -E\rangle} \times V_0/V_0^\perp\right) \leftarrow \langle C_2\rangle$$

which is $\frac{\mathrm{GL}(1,\mathbf{R})\circ\langle i\rangle}{\langle -E\rangle} \times V_0/V_0^\perp = \frac{\langle i\rangle}{\langle -E\rangle} \times V_0/V_0^\perp = V_0$.

The case of $V_0\#L(i,n-i)$, $1 \leq i < [n/2]$, is quite similar. The centralizer is

$$C_{\mathrm{PSL}(4n,\mathbf{R})}(V_0\#L(i,n-i)) = \left(\frac{\mathrm{GL}(i,\mathbf{C})\times\mathrm{GL}(n-i,\mathbf{C})}{\langle -E\rangle} \times V_0/V_0^\perp\right) \rtimes \langle C_2\rangle$$

and its center is the pull-back of the homomorphisms

$$\frac{(\mathrm{GL}(i,\mathbf{R})\times\mathrm{GL}(n-i,\mathbf{R}))\circ\langle i\rangle}{\langle -E\rangle} \times V_0/V_0^\perp = \left(\frac{\mathrm{GL}(i,\mathbf{C})\times\mathrm{GL}(n-i,\mathbf{C})}{\langle -E\rangle} \times V_0/V_0^\perp\right)^{\langle C_2\rangle}$$
$$\to \mathrm{Aut}\left(\frac{\mathrm{GL}(i,\mathbf{C})\times\mathrm{GL}(n-i,\mathbf{C})}{\langle -E\rangle} \times V_0/V_0^\perp\right) \leftarrow \langle C_2\rangle$$

which is $ZC_{\mathrm{PSL}(4n,\mathbf{R})}(V_0\#L(i,n-i)) = \frac{(\mathrm{GL}(1,\mathbf{R})\times\mathrm{GL}(1,\mathbf{R}))\circ\langle i\rangle}{\langle -E\rangle} \times V_0/V_0^\perp = 2^2 \times V_0/V_0^\perp = V_0 \times L$.

If n is even and $i = n/2$, there is a short exact sequence

$$1 \to \frac{\mathrm{GL}(n,\mathbf{C})^2}{\langle -E\rangle} \to C_{\mathrm{PSL}(4n,\mathbf{R})}(V_0\times L) \to \mathrm{Hom}(G\times L,\langle -E\rangle)_\rho \to 1$$

where the elementary abelian group to the right is all of $\mathrm{Hom}(G\times L,\langle -E\rangle) = 2^4$. Hence the centralizer

$$C_{\mathrm{PSL}(4n,\mathbf{R})}(V_0\times L) = \left(\frac{\mathrm{GL}(n,\mathbf{C})^2}{\langle -E\rangle} \times V_0/V_0^\perp\right) \rtimes \langle C_1, C_2\rangle$$

where C_2 is as above and C_1 is the $(4n\times 4n)$-matrix $\begin{pmatrix} 0 & E \\ E & 0\end{pmatrix}$. The matrix C_2 commutes with $V_0/V_0^\perp$ and acts as complex conjugation on $\frac{\mathrm{GL}(n,\mathbf{C})^2}{\langle -E\rangle}$. The matrix C_1 commutes with $V_0/V_0^\perp$ and switches the two factors of $\mathrm{GL}(n,\mathbf{C})^2$. The center of the centralizer is the pull-back of the group homomorphisms

$$\frac{\mathrm{GL}(n,\mathbf{R})\circ\langle i\rangle\times\langle (E,-E)\rangle}{\langle -E\rangle} \times V_0/V_0^\perp = \left(\frac{\mathrm{GL}(n,\mathbf{C})^2}{\langle -E\rangle} \times V_0/V_0^\perp\right)^{\langle C_1,C_2\rangle}$$
$$\to \mathrm{Aut}\left(\frac{\mathrm{GL}(n,\mathbf{C})^2}{\langle -E\rangle} \times V_0/V_0^\perp\right) \leftarrow \langle C_1, C_2\rangle$$

which is $ZC_{\mathrm{PSL}(4n,\mathbf{R})}(V_0 \times L) = \frac{\mathrm{GL}(1,\mathbf{R})\circ\langle i\rangle\times\langle(E,-E)\rangle}{\langle -E\rangle} \times V_0/V_0^{\perp} = 2^2 \times V_0/V_0^{\perp} = V_0 \times L$.

$\underline{H_-\#L(i,n-i)}$ As above we have that

$$C_{\mathrm{PSL}(4n,\mathbf{R})}(H_- \times L) = \begin{cases} \frac{\mathrm{GL}(i,\mathbf{H})\times\mathrm{GL}(n-i,\mathbf{H})}{\langle -E\rangle} \times H_- & i < [n/2] \\ \frac{\mathrm{GL}(i,\mathbf{H})^2}{\langle -E\rangle} \rtimes \langle C_1\rangle \times H_- & n \text{ even and } i = n/2 \end{cases}$$

with center $ZC_{\mathrm{PSL}(4n,\mathbf{R})}(H_- \times L) = \frac{\mathrm{GL}(1,\mathbf{R})\times\mathrm{GL}(1,\mathbf{R})}{\langle -E\rangle} = 2 \times H_- = H_- \times L$ in case $i \neq n-i$. If n is even and $i = n/2$, then the center is the pull-back of the group homomorphisms

$$\frac{\mathrm{GL}(i,\mathbf{H}) \times \langle(-E,E)\rangle}{\langle -E\rangle} \times H_- = \frac{\mathrm{GL}(i,\mathbf{H})^2}{\langle -E\rangle} \times H_- \to \mathrm{Aut}\left(\frac{\mathrm{GL}(i,\mathbf{H})^2}{\langle -E\rangle} \times H_-\right) \leftarrow \langle C_1\rangle$$

which is $ZC_{\mathrm{PSL}(4n,\mathbf{R})}(H_- \times L) = \frac{\mathrm{GL}(1,\mathbf{R})\times\langle(-E,E)\rangle}{\langle -E\rangle} \times H_- = 2 \times H_- = H_- \times L$. □

CHAPTER 5

The B-family

sec:slodd

The B-family consists of the matrix groups

$$\mathrm{SL}(2n+1,\mathbf{R}), \quad n \geq 2,$$

of real $(2n+1)\times(2n+1)$ matrices of determinant $+1$. When $n=1$ we obtain the 2-compact group $\mathrm{SL}(3,\mathbf{R}) = \mathrm{PGL}(2,\mathbf{C})$ considered in Chapter 3. The embedding

$$\mathrm{GL}(2n,\mathbf{R}) \to \mathrm{SL}(2n+1,\mathbf{R})\colon A \to \begin{pmatrix} A & 0 \\ 0 & \det A \end{pmatrix}$$

permits us to consider $\mathrm{GL}(2n,\mathbf{R})$ as a maximal rank subgroup of $\mathrm{SL}(2n+1,\mathbf{R})$. The maximal torus normalizer for the subgroup $\mathrm{GL}(2n,\mathbf{R})$ is also the maximal torus normalizer for $\mathrm{SL}(2n+1,\mathbf{R})$, $N(\mathrm{SL}(2n+1,\mathbf{R})) = N(\mathrm{GL}(2n,\mathbf{R})$ (4.1), so that, in particular, the Weyl group $W(\mathrm{SL}(2n+1,\mathbf{R})) = W(\mathrm{GL}(2n,\mathbf{R})) = \Sigma_2 \wr \Sigma_n$ (4.2).

It is known that [**35**, 1.6] [**24**, Main Theorem]

$$H^0(W;\check{T}) = \mathbf{Z}/2, \quad H^1(W;\check{T}) = \begin{cases} \mathbf{Z}/2 & n=2 \\ \mathbf{Z}/2\times\mathbf{Z}/2 & n>2 \end{cases}$$

for these groups.

The full general linear group $\mathrm{GL}(2n+1,\mathbf{R}) = \mathrm{SL}(2n+1,\mathbf{R}) \times \langle -E\rangle$ is the direct product of $\mathrm{SL}(2n+1,\mathbf{R})$ with the opposite of the identity matrix so that $\mathrm{PGL}(2n+1,\mathbf{R}) = \mathrm{SL}(2n+1,\mathbf{R})$.

sec:structslodd

1. The structure of $\mathrm{SL}(2n+1,\mathbf{R})$

Consider the elementary abelian 2-groups

$$\begin{aligned} &\Delta_{2n+1} = \langle \mathrm{diag}(\pm1,\dots,\pm1)\rangle \subset \mathrm{GL}(2n+1,\mathbf{R}) \\ &S\Delta_{2n+1} = \mathrm{SL}(2n+1,\mathbf{R}) \cap \Delta_{2n+1} \subset \mathrm{SL}(2n+1,\mathbf{R}) \\ &t = t(\mathrm{SL}(2n+1,\mathbf{R})) = \Delta_{2n+1} \cap T(\mathrm{SL}(2n+1,\mathbf{R})) = \langle e_1,\dots,e_n\rangle \subset T(\mathrm{SL}(2n+1,\mathbf{R})) \end{aligned}$$

in $\mathrm{GL}(2n+1,\mathbf{R})$ and $\mathrm{SL}(2n+1,\mathbf{R})$.

lemma:sloddASL

5.1. Lemma. *The inclusion functors*

$$\mathbf{A}(\Sigma_{2n+1},\Delta_{2n+1}) \to \mathbf{A}(\mathrm{GL}(2n+1,\mathbf{R})), \qquad \mathbf{A}(\Sigma_{2n+1},S\Delta_{2n+1}) \to \mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R})),$$
$$\mathbf{A}(\Sigma_2\wr\Sigma_n,t) \to \mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}))^{\leq t}$$

are equivalences of categories.

Proof. Similar to 4.12. $\mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R})$ is a full subcategory of $\mathbf{A}(\mathrm{GL}(2n+1,\mathbf{R})$ since conjugation with the central element $-E$ of negative determinant is the identity. □

The Quillen categories $\mathbf{A}(\mathrm{GL}(2n,\mathbf{R})) = \mathbf{A}(\Sigma_{2n},\Delta_{2n})$ $\mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R})) = \mathbf{A}(\Sigma_{2n+1},\Delta_{2n+1})$ (4.12, 5.1) are not equivalent.

For any partition $i=(i_0,i_1)$, $i_0\geq 0$, $i_1>0$, of $2n+1$, let $L[i_0,i_1]\subset\Delta_{2n+1}$ be the subgroup generated by

$$\mathrm{diag}(\overbrace{+1,\dots,+1}^{i_0},\overbrace{-1,\dots,-1}^{i_1}) = (i_0\rho_0 + i_1\rho_1)(e_1)$$

For any partition (i_0, i_1, i_2, i_3) of $2n+1$ where at least two of i_1, i_2, i_3 are positive, let $P[i_0, i_1, i_2, i_3] \subset \Delta_{2n+1}$ be the subgroup generated by

$$\mathrm{diag}(\overbrace{+1,\dots,+1}^{i_0},\overbrace{-1,\dots,-1}^{i_1},\overbrace{+1,\dots,+1}^{i_2},\overbrace{-1,\dots,-1}^{i_3}) = (i_0\rho_0 + i_1\rho_1 + i_2\rho_2 + i_3\rho_3)(e_1)$$
$$\mathrm{diag}(\overbrace{+1,\dots,+1}^{i_0},\overbrace{+1,\dots,+1}^{i_1},\overbrace{-1,\dots,-1}^{i_2},\overbrace{-1,\dots,-1}^{i_3}) = (i_0\rho_0 + i_1\rho_1 + i_2\rho_2 + i_3\rho_3)(e_2)$$

Note that $L[i_0, i_1]$ is a subgroup of $S\Delta_{2n+1}$ if and only if i_1 is even, and that $P[i_0, i_1, i_2, i_3]$ is a subbgroup of $S\Delta_{2n+1}$ if and only of i_1, i_2, i_3 have the same parity, the opposite parity of i_0.

Let $P(k, r)$ denote the number of partitions of $k = i_0 + \cdots + i_{r-1}$ into sums of r positive integers $1 \leq i_0 \leq \cdots \leq i_{r-1}$. From the above discussion we conclude

p:sloddrankleq2 5.2. PROPOSITION. *The category* $\mathbf{A}(\mathrm{SL}(2n+1, \mathbf{R}))$ *contains precisely*

- n *isomorphism classes of rank one objects represented by the lines* $L[2i_0 + 1, 2i_1]$ *where* $0 \leq i_0 \leq n-1$ *and* $i_1 = n - i_0$.
- $\sum_{j=2}^{n} P(j,2) + \sum_{j=3}^{n} P(j,3)$ *isomorphism classes of toral rank two objects. They are represented by the subgroups* $P[2i_0+1, 2i_1, 2i_2, 0]$, *where* $0 \leq i_0 \leq n-2$ *and* (i_1, i_2) *is a partition of* $n - i_0$, *together with the subgroups* $P[2i_0+1, 2i_1, 2i_2, 2i_3]$, *where* $0 \leq i_0 \leq n-3$ *and* (i_1, i_2, i_3) *is a partition of* $n - i_0$.
- $\sum_{j=3}^{n+2} P(j,3)$ *isomorphism classes of nontoral rank two objects represented by the subgroups* $P[2i_0, 2i_1 - 1, 2i_2 - 1, 2i_3 - 1]$ *where* $0 \leq i_0 \leq n-1$ *and* (i_1, i_2, i_3) *is a partition of* $n - i_0 + 2$.

The centralizers of these objects are

eq:Lcent
$$C_{\mathrm{SL}(2n+1,\mathbf{R})}L[2i_0+1, 2i_1] = \mathrm{SL}(2n+1,\mathbf{R}) \cap \big(\mathrm{GL}(2i_0+1,\mathbf{R}) \times \mathrm{GL}(2i_1,\mathbf{R})\big) = \mathrm{SL}(2i_0+1,\mathbf{R}) \times \mathrm{GL}(2i_1,\mathbf{R}) \tag{5.3}$$

eq:Pcent
$$\begin{aligned} C_{\mathrm{SL}(2n+1,\mathbf{R})}P[i] &= \mathrm{SL}(2n+1,\mathbf{R}) \cap \prod_{j\in i} \mathrm{GL}(j,\mathbf{R}) \\ &= \begin{cases} \mathrm{SL}(2i_0+1,\mathbf{R}) \times \mathrm{GL}(2i_1,\mathbf{R}) \times \mathrm{GL}(2i_2,\mathbf{R}) \times \mathrm{GL}(2i_3,\mathbf{R}) & P[i] \text{ toral} \\ \mathrm{GL}(2i_0,\mathbf{R}) \times \mathrm{GL}(2i_1-1,\mathbf{R}) \times \mathrm{GL}(2i_2-1,\mathbf{R}) \times \mathrm{SL}(2i_3-1,\mathbf{R}) & P[i] \text{ nontoral} \end{cases} \end{aligned} \tag{5.4}$$

as, for instance,

$$\begin{aligned} &\mathrm{SL}(2n+1,\mathbf{R}) \cap \big(\mathrm{GL}(2i_0+1,\mathbf{R}) \times \mathrm{GL}(2i_1,\mathbf{R})\big) \\ &\quad = \mathrm{SL}(2n+1,\mathbf{R}) \cap \big(\mathrm{SL}(2i_0+1,\mathbf{R}) \times \langle -E\rangle \times \mathrm{SL}(2i_1,\mathbf{R}) \rtimes \langle D\rangle\big) \\ &\quad\quad = \mathrm{SL}(2i_0+1,\mathbf{R}) \times \mathrm{SL}(2i_1,\mathbf{R}) \rtimes \langle -D\rangle = \mathrm{SL}(2i_0+1,\mathbf{R}) \times \mathrm{GL}(2i_1,\mathbf{R}), \end{aligned}$$

and the centers of the centralizers are

eq:ZLcent
$$ZC_{\mathrm{SL}(2n+1,\mathbf{R})}L[2i_0+1, 2i_1] = L[2i_0+1, 2i_1], \tag{5.5}$$

eq:CPcent
$$ZC_{\mathrm{SL}(2n+1,\mathbf{R})}P[i] = \mathrm{SL}(2n+1,\mathbf{R}) \cap \prod_{i_j>0} Z\mathrm{GL}(i_j,\mathbf{R}) = \begin{cases} P[i] & \#\{j \mid i_j > 0\} = 3 \\ P[i] \times \mathbf{Z}/2 & \#\{j \mid i_j > 0\} = 4 \end{cases} \tag{5.6}$$

lemma:H0Vslodd 5.7. LEMMA. *For any nontrivial subgroup* $V \subset S\Delta_{2n+1}$ *there is a natural isomorphism*

$$ZC_{\mathrm{SL}(2n+1,\mathbf{R})}(V) = H^0(\Sigma_{2n+1}(V); S\Delta_{2n+1})$$

where $\Sigma_{2n+1}(V)$ *is the point-wise stabilizer subgroup (2.68).*

PROOF. Let $V \subset S\Delta_{2n+1}$ be any nontrivial subgroup of rank r. Then $V = V[i]$ is the image of $\sum_{\rho\in V^\vee} i_\rho \rho$ for some function $i\colon \mathrm{Hom}((\mathbf{Z}/2)^r, \mathbf{R}^\times) \to \mathbf{Z}$ where $\sum_{\rho\in V^\vee} i_\rho = 2n+1$ and

$$\begin{aligned} ZC_{\mathrm{SL}(2n+1,\mathbf{R})}V[i] &= Z\big(\mathrm{SL}(2n+1,\mathbf{R}) \cap \prod_{i_\rho>0} \mathrm{GL}(i_\rho,\mathbf{R})\big) \\ &= \mathrm{SL}(2n+1,\mathbf{R}) \cap \prod_{i_\rho>0} Z\mathrm{GL}(i_\rho,\mathbf{R}) = S\Delta_{2n+1} \cap \Delta_{2n+1}^{\prod \Sigma_{i_\rho}} = S\Delta_{2n+1}^{\Sigma_{2n+1}(V[i])} \end{aligned}$$

where the second equality can be proved by using that $C_{\mathrm{GL}(i,\mathbf{R})}\mathrm{SL}(i,\mathbf{R}) = Z\mathrm{GL}(i,\mathbf{R})$ and the final equality follows from the observation that the stabilizer subgroup $\Sigma_{2n+1}(V[i]) = \prod_{i_\rho>0}\Sigma_{i_\rho}$. □

cor:limislodd

5.8. COROLLARY. $\lim^i(\mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}),\pi_1(BZC_{\mathrm{SL}(2n+1,\mathbf{R})})) = 0$ *for all* $i>0$.

PROOF. Immediate from the general exactness theorem (2.69) for functors of the form as in 5.7. □

5.9. PROPOSITION. *Centralizers of objects of* $\mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}))^{\leq t}_{\leq 2}$ *are LHS.*

PROOF. Let X_1 and X_2 be connected Lie groups and π_1 and π_2 finite 2-groups acting on them. Suppose that the homomorphisms $\theta(X_1)^{\pi_1}$ and $\theta(X_1)^{\pi_1}$ (2.20) are surjective. Then also $\theta(X_1\times X_2)^{\pi_1\times\pi_2}$ is surjective and so the product $X_1\rtimes\pi_1\times X_2\rtimes\pi_2$ is LHS (2.28). This observation applies to the products (5.3, 5.4) since the θ-homomorphisms are surjective [**24**, 5.4] (2.29) for $\mathrm{SL}(2i+1,\mathbf{R})$, $i\geq 0$, and $\mathrm{SL}(2i,\mathbf{R})$, $i\geq 1$. □

2. The limit of the functor $H^1(W;\check{T})/H^1(\pi_0;\check{Z}(\;)_0)$ on $\mathbf{A}(\mathrm{PSL}(2n+1,\mathbf{R}))^{\leq t}_{\leq 2}$

sec:sloddlim0

In this subsection we check, using a modification of 2.53, that conditions (1) and (2) of 2.51 with $X=\mathrm{SL}(2n{+}1,\mathbf{R})$ are satisfied under the inductive assumptions that the connected 2-compact groups $\mathrm{SL}(2i+1,\mathbf{R})$, $0\leq i<n$, and $\mathrm{SL}(2i,\mathbf{R})$, $1\leq i\leq n$, are uniquely N-determined.

The objects $V\subset\mathrm{SL}(2n+1,\mathbf{R})$ of the category $\mathbf{A}(\mathrm{PSL}(2n+1,\mathbf{R}))^{\leq t}_{\leq 2}$ are the rank one objects $L[i_0,i_1]$ and the rank two objects $P[2i_0{+}1,2i_1,2i_2,0]$ and $P[2i_0{+}1,2i_1,2i_2,2i_3]$ as described in 5.2. The rank two object $P[2i_0{+}1,2i_1,2i_2,2i_3]$, $i_3\geq 0$, contains the three lines $L[2i_0{+}2i_1{+}1,2i_2{+}2i_3]$, $L[2i_0+2i_2+1,2i_1+2i_3]$, and $L[2i_0+2i_3+1,2i_1+2i_2]$. Their centralizers are described in (5.3) and (5.4). Note that there are functorial isomorphisms

eq:sloddsplit

$$\check{T}^{W_0(C_{\mathrm{SL}(2n+1,\mathbf{R})}(V))} = (\mathbf{Z}/2)^{\min\{i_0,1\}}\times\check{Z}(C_{\mathrm{SL}(2n+1,\mathbf{R})}(V)_0) \tag{5.11}$$

as modules over $\pi_0 C_{\mathrm{SL}(2n+1,\mathbf{R})}(V)$.

Condition (1) of 2.51 is satisfied as $C_X(V)$ has N-determined automorphisms and is N-determined for general reasons (2.39, 2.35, 2.40). This means that there are isomorphisms, α_V and f_V, such that the diagrams

$$\begin{array}{ccc} C_N(V) & \xrightarrow[\cong]{\alpha_V} & C_N(V) \\ \downarrow & & \downarrow \\ C_X(V) & \xrightarrow[f_V]{\cong} & C_{X'}(V) \end{array}$$

commute and $\alpha_V\in H^1(W;\check{T})(C_X(V))$. There may be more than choice for α_V but for each α_V there is just one possibility for f_V (2.13). The set of possible α_V for a given V is a $H^1(\pi_0;\check{Z}((\;)_0))(C_X(V))$-coset in $H^1(W;\check{T})(C_X(V))$ (2.37). The collection of the α_V for various V represents an element of the inverse limit

eq:sloddlim0

$$\lim{}^0\left(\mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}))^{\leq t}_{\leq 2},\frac{H^1(W;\check{T})}{H^1(\pi_0;\check{Z}((\;)_0))}\right) \tag{5.12}$$

of the quotient functor over the category $\mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}))^{\leq t}_{\leq 2}$. Condition (2) of 2.51 is satisfied if the restriction map from $H^1(W;\check{T})(\mathrm{SL}(2n+1,\mathbf{R}))$ to (5.12) is surjective. Because of the natural splitting (5.11) and because the centralizers $C_{\mathrm{SL}(2n+1,\mathbf{R})}(V)$ are LHS there is a short exact sequence

$$0\to\mathrm{Hom}(\pi_0,(\mathbf{Z}/2)^{\min\{i_0,1\}})\to\frac{H^1(W;\check{T})}{H^1(\pi_0;\check{Z}((\;)_0))}\to H^1(W_0;\check{T})^{\pi_0}\to 0$$

of functors on $\mathbf{A}(\mathrm{SL}(2n{+}1,\mathbf{R}))^{\leq t}_{\leq 2}$. If we apply the functor $\mathrm{Hom}(\pi_0,(\mathbf{Z}/2)^{\min\{i_0,1\}})$ to the morphisms

q:sloddmorphisms

$$L[2i_0+1,2i_1+2i_2]\to P[2i_0+1,2i_1,2i_2,0]\leftarrow L[2i_0+2i_1+1,2i_2] \tag{5.13}$$

we see that the induced morphisms are injective and that their images intersect trivially. Thus the inverse limit of this functor is trivial and from the above short exact sequence we obtain an injective map

$$\lim^0 \left(\mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}))^{\leq t}_{\leq 2}, \frac{H^1(W;\check{T})}{H^1(\pi_0;\check{Z}((\ )_0))}\right) \to \lim^0 \left(\mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}))^{\leq t}_{\leq 2}, H^1(W_0;\check{T})^{\pi_0}\right)$$

between the inverse limits. As the inverse limit to the right is a subgroup of the inverse limit of the functor $H^1(W_0;\check{T})$ we conclude that if the restriction map

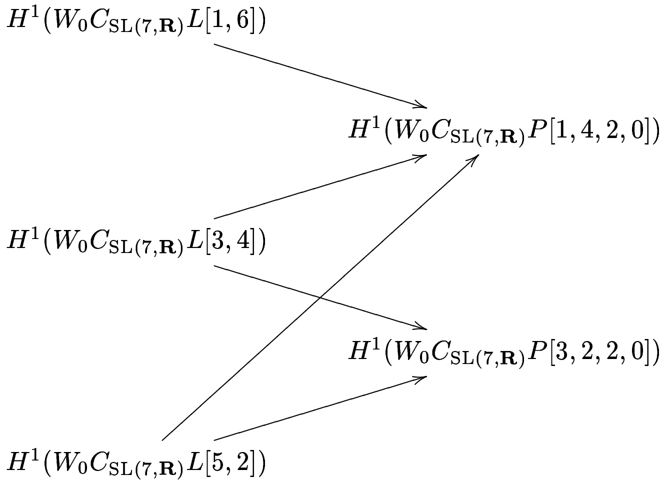
eq:sloddres (5.14) $$H^1(W_0;\check{T})(\mathrm{SL}(2n+1,\mathbf{R})) \to \lim^0 \left(\mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}))^{\leq t}_{\leq 2}, H^1(W_0;\check{T})\right)$$

is surjective, then condition (2) of 2.51 is satisfied.

ma:sloddlimH1W0 5.15. LEMMA. *The restriction homomorphism (5.14) is an isomorphism for all $n \geq 2$.*

PROOF. For $n = 2$, the image under the functor $H^1(W_0;\check{T})$ of the category $L[1,4] \to P[1,2,2,0] \leftarrow L[3,2]$ is $0 \to 0 \leftarrow \mathbf{Z}/2$ so that the limit of the functor $H^1(W_0;\check{T})$ is $\mathbf{Z}/2$. Since $\mathrm{SL}(3,\mathbf{R}) \times \mathrm{SL}(2;\mathbf{R}) \to \mathrm{SL}(5,\mathbf{R})$ turns out to induce an isomorphism on $H^1(W_0;\check{T})$ the claim follows in this case.

For $n = 3$, taking into account only the planes of type $P[2i_0-1, 2i_1, 2i_2, 0]$, we should compute the limit of the diagram



of $\mathbf{F}_2$-vector spaces. For each of the planes P take the intersections of the images in the cohomology groups $H^1(W_0C_{\mathrm{SL}(7,\mathbf{R})}P;\check{T})$ of $H^1(W_0C_{\mathrm{SL}(7,\mathbf{R})}L;\check{T})$ for each line $L \subset P$. Take the intersection of the pre-images in each $H^1(W_0C_{\mathrm{SL}(7,\mathbf{R})}L;\check{T})$ of these subspaces of $H^1(W_0C_{\mathrm{SL}(7,\mathbf{R})}P;\check{T})$. Using the computer program *magma* one may see that these subspaces have dimensions $1,2,2$ for $L = L[1,6], L[3,4], L[5,2]$, respectively, and that they equal the image of the restriction maps from $H^1(W_0;\check{T})(\mathrm{SL}(7,\mathbf{R}))$. This shows that the lemma is true in this case.

In general, the above mentioned subspaces of $H^1(W_0C_{\mathrm{SL}(7,\mathbf{R})}L;\check{T})$ have dimension 1 for $L = L[1,2n]$ and dimension 2 for the lines $L = L[2i+1, 2n-2i]$ with $1 \leq i \leq n-1$ and these subspaces equal the image of the restriction maps from $H^1(W_0;\check{T})(\mathrm{SL}(2n+1,\mathbf{R}))$. □

:sloddnontoral2

3. Rank two nontoral objects of $\mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}))$

The nontoral rank two objects of $\mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}))$ are represented by the subgroups $P[i] \subset S\Delta_{2n+1}$ generated by the elements

$$e_1 = \mathrm{diag}(\overbrace{+1,\dots,+1}^{2i_0},\overbrace{-1,\dots,-1}^{2i_1-1},\overbrace{+1,\dots,+1}^{2i_2-1},\overbrace{-1,\dots,-1}^{2i_3-1})$$
$$e_2 = \mathrm{diag}(\overbrace{+1,\dots,+1}^{2i_0},\overbrace{+1,\dots,+1}^{2i_1-1},\overbrace{-1,\dots,-1}^{2i_2-1},\overbrace{-1,\dots,-1}^{2i_3-1})$$

where $i = (2i_0, 2i_1 - 1, 2i_2 - 1, 2i_3 - 1)$, $0 \leq i_0 \leq n-1$ and (i_1, i_2, i_3) is a partition of $n + 2 - i_0$ (5.2). The generators of $P[i]$ may also be written as

eq:slodde1 (5.17)
$$e_1 = \mathrm{diag}(\overbrace{E,\dots,E}^{i_0-1}, E, \overbrace{-E,\dots,-E}^{i_1-1}, -R, \overbrace{E,\dots,E}^{i_2-1}, \overbrace{-E,\dots,-E}^{i_3-1}, -1), \qquad R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

eq:slodde2 (5.18)
$$e_2 = \mathrm{diag}(\overbrace{E,\dots,E}^{i_0-1}, E, \overbrace{E,\dots,E}^{i_1-1}, R, \overbrace{-E,\dots,-E}^{i_2-1}, \overbrace{-E,\dots,-E}^{i_3-1}, -1)$$

The centralizer of $P[i]$ is

$$\begin{aligned} C_{\mathrm{SL}(2n+1,\mathbf{R})}P[i] &= \mathrm{SL}(2n+1,\mathbf{R}) \cap \big(\mathrm{GL}(2i_0,\mathbf{R}) \times \mathrm{GL}(2i_1-1,\mathbf{R}) \times \mathrm{GL}(2i_2-1,\mathbf{R}) \times \mathrm{GL}(2i_3-1,\mathbf{R})\big) \\ &= \mathrm{GL}(2i_0,\mathbf{R}) \times \mathrm{GL}(2i_1-1,\mathbf{R}) \times \mathrm{GL}(2i_2-1,\mathbf{R}) \times \mathrm{SL}(2i_3-1,\mathbf{R}) \end{aligned}$$

Note that $P[i]$ is contained in the maximal torus normalizer $N(\mathrm{SL}(2n+1,\mathbf{R})) = \mathrm{GL}(2,\mathbf{R}) \wr \Sigma_n$. Since the centralizer of $P[i]$ in the maximal torus normalizer,

$$\begin{aligned} C_{\mathrm{GL}(2,\mathbf{R})\wr\Sigma_n}P[i] = \mathrm{GL}(2,\mathbf{R}) \wr \Sigma_{i_0} \times \mathrm{GL}(2,\mathbf{R}) \wr \Sigma_{i_1-1} \times \mathrm{GL}(1,\mathbf{R}) \\ \times \mathrm{GL}(1,\mathbf{R}) \times \mathrm{GL}(2,\mathbf{R}) \wr \Sigma_{i_2-1} \times \mathrm{GL}(2,\mathbf{R}) \wr \Sigma_{i_3-1}, \end{aligned}$$

is the maximal torus normalizer for the centralizer of $P[i]$, the lift $P[i] \subset N(\mathrm{SL}(2n+1,\mathbf{R}))$ is a preferred lift of $P[i] \subset \mathrm{SL}(2n+1,\mathbf{R})$ [45]. The two other preferred lifts are given by composing with the permutation matrices for the permutations $(1,2)(i_0+i_1, 2n+1)$ and $(1,2)(i_0+i_1+1, 2n+1)$ (assuming $i_0 > 0$) resulting in the lifts given by

$$\begin{aligned} e_1 &= \mathrm{diag}(\overbrace{E,\dots,E}^{i_0-1}, E, \overbrace{-E,\dots,-E}^{i_1-1}, -E, \overbrace{E,\dots,E}^{i_2-1}, \overbrace{-E,\dots,-E}^{i_3-1}, -1), \\ e_2 &= \mathrm{diag}(\overbrace{E,\dots,E}^{i_0-1}, E, \overbrace{E,\dots,E}^{i_1-1}, R, \overbrace{-E,\dots,-E}^{i_2-1}, \overbrace{-E,\dots,-E}^{i_3-1}, -1) \end{aligned}$$

and

$$\begin{aligned} e_1 &= \mathrm{diag}(\overbrace{E,\dots,E}^{i_0-1}, E, \overbrace{-E,\dots,-E}^{i_1-1}, R, \overbrace{E,\dots,E}^{i_2-1}, \overbrace{-E,\dots,-E}^{i_3-1}, -1), \\ e_2 &= \mathrm{diag}(\overbrace{E,\dots,E}^{i_0-1}, E, \overbrace{E,\dots,E}^{i_1-1}, -E, \overbrace{-E,\dots,-E}^{i_2-1}, \overbrace{-E,\dots,-E}^{i_3-1}, -1) \end{aligned}$$

respectively. These two lifts are also preferred lifts of $P[i] \subset \mathrm{SL}(2n+1,\mathbf{R})$. The three preferred lifts are not conjugate in $N(\mathrm{SL}(2n+1,\mathbf{R}))$ because the intersection with the maximal torus is generated by $e_1 + e_2$ in the first case and by e_1, respectively e_2, in the next two cases. Note that all three preferred lifts have the same maximal torus, $\mathrm{SL}(2,\mathbf{R})^{i_0} \times \mathrm{SL}(2,\mathbf{R})^{i_1-1} \times \mathrm{SL}(2,\mathbf{R})^{i_2-1} \times \mathrm{SL}(2,\mathbf{R})^{i_3-1}$.

Let $U = \langle e_1, e_2, e_3 \rangle$ be elementary abelian 2-group generated by e_1 and e_2 as in (5.17, 5.18) together with

$$e_3 = \mathrm{diag}(\overbrace{E,\dots,E}^{i_0-1}, R, \overbrace{E,\dots,E}^{i_1-1}, E, \overbrace{E,\dots,E}^{i_2-1}, \overbrace{E,\dots,E}^{i_3-1}, -1),$$

Note that the centralizer of U has a nontrivial identity component, that the inclusion $U \subset C_{\mathrm{SL}(2n+1,\mathbf{R})}P[i]$ induces an isomorphism on π_0.

Under the inductive assumption that $\mathrm{SL}(2i,\mathbf{R})$, $1 \leq i \leq n-1$, and $\mathrm{SL}(2i-1,\mathbf{R})$, $1 \leq i \leq n$, have $\pi_*(N)$-determined automorphisms (or using [31]) we conclude from 2.63 and 2.64 and (part of) [42, 5.2] that condition (3) of 2.51 is satisfied for $\mathrm{SL}(2n+1,\mathbf{R})$. (Namely, 2.63.(1) says that ν'_L does not depend on the choice of $L < V$. The difference $f^{-1}_{\nu,L_2} \circ f_{\nu,L_1}$ between any two of the maps $f_{\nu,L}$ from 2.51.(3) is an automorphismof $C_{\mathrm{SL}(2n+1,\mathbf{R})}P[i]$ that, by 2.63.(2), is the identity on the identity component and by the commutative diagram (2.64)

dia:pi0 (5.19)
$$\begin{array}{ccc} & U & \\ \swarrow & & \searrow \\ C_{\mathrm{SL}(2n+1,\mathbf{R})}P[i] & \xrightarrow[f^{-1}_{\nu,L_2} \circ f_{\nu,L_1}]{} & C_{\mathrm{SL}(2n+1,\mathbf{R})}P[i] \end{array}$$

also the identity on $\pi_0 C_{\mathrm{SL}(2n+1,\mathbf{R})} P[i]$. Any such automorphism of $C_{\mathrm{SL}(2n+1,\mathbf{R})} P[i]$ has [42, 5.2] the form $A \to \varphi(A)A$ where

$$\begin{aligned}\varphi\colon \mathrm{GL}(2i_0,\mathbf{R}) \times \mathrm{GL}(2i_1-1,\mathbf{R}) \times \mathrm{GL}(2i_2-1,\mathbf{R}) \times \mathrm{SL}(2i_3-1,\mathbf{R}) \to \\ \pi_0\big(\mathrm{GL}(2i_0,\mathbf{R}) \times \mathrm{GL}(2i_1-1,\mathbf{R}) \times \mathrm{GL}(2i_2-1,\mathbf{R}) \times \mathrm{SL}(2i_3-1,\mathbf{R})\big) \to Z\mathrm{GL}(2i_0,\mathbf{R})\end{aligned}$$

is some homomorphisms. Diagram (5.19) thus implies that the inclusion $U \to \mathrm{SL}(2n+1,\mathbf{R})$ and the monomorphism given by $e_i \to \varphi(e_i)e_i$, $1 \le i \le 3$, are conjugate. Since the trace of e_i, $1 \le i \le 3$, is odd (nonzero), φ must be trivial. Thus f_{ν,L_1} and f_{ν,L_2} are identical isomorphisms.)

CHAPTER 6

The C-family

sec:spn

Let $\mathbf{H} = \{a + bj | a, b \in \mathbf{C}\}$, where $j^2 = -1$ and $ja = \overline{a}j$ for $a \in \mathbf{C}$, be the quaternion algebra. The C-family consists of the matrix groups

$$\mathrm{PGL}(n, \mathbf{H}) = \mathrm{GL}(n, \mathbf{H}) / \langle -E \rangle, \quad n \geq 3,$$

of quaternion projective $n \times n$ matrices. (These 2-compact groups also exist for $n = 1$ or $n = 2$. However, $\mathrm{PGL}(1, \mathbf{H}) = \mathrm{SL}(3, \mathbf{R}) = \mathrm{PGL}(2, \mathbf{C})$ and $\mathrm{PGL}(2, \mathbf{H}) = \mathrm{SL}(5, \mathbf{R})$ (9.25) are already covered.)

The maximal torus normalizer for $\mathrm{GL}(1, \mathbf{H}) = \mathbf{H}^\times$, generated by the maximal torus $\mathrm{GL}(1, \mathbf{C}) = \mathbf{C}^\times$ and the element j, sits in the non-split extension

$$1 \to \mathrm{GL}(1, \mathbf{C}) \to N(\mathrm{GL}(1, \mathbf{H})) \to \langle j \rangle / \langle -1 \rangle \to 1$$

of Σ_2 by $\mathrm{GL}(1, \mathbf{C}) = \mathbf{C}^\times$. The maximal torus normalizer for $\mathrm{GL}(n, \mathbf{H})$ is the subgroup

$$N(\mathrm{GL}(n, \mathbf{H})) = N(\mathrm{GL}(1, \mathbf{H})) \wr \Sigma_n,$$

generated by $N(\mathrm{GL}(1, \mathbf{H}))^n \subset \mathrm{GL}(n, \mathbf{H})$ and the permutation matrices. The maximal torus normalizer for $\mathrm{PGL}(n, \mathbf{H})$, the quotient $N(\mathrm{GL}(n, \mathbf{H}))$ by the order two group $\langle -E \rangle$, sits in the extension

$$1 \to \frac{\mathrm{GL}(1, \mathbf{C})^n}{\langle -E \rangle} \to \frac{N(\mathrm{GL}(1, \mathbf{H}))^n}{\langle -E \rangle} \to \frac{N(\mathrm{GL}(1, \mathbf{H}))}{\mathrm{GL}(1, \mathbf{C})} \wr \Sigma_n \to 1$$

which does not split (for $n \geq 3$).

It is known that [35, 1.6] [24, Main Theorem]

$$H^0(W; \check{T})(\mathrm{PGL}(n, \mathbf{H})) = 0, \quad H^1(W; \check{T})(\mathrm{PGL}(n, \mathbf{H})) = \begin{cases} \mathbf{Z}/2 & n = 3, 4 \\ 0 & n > 4 \end{cases}$$

for the projective groups.

sec:NpglnH

1. The structure of PGL(n, H)

Let

$$\Delta_n = t(\mathrm{GL}(n, \mathbf{H})) = \langle \mathrm{diag}(\pm 1, \ldots, \pm 1) \rangle \subset \mathrm{GL}(n, \mathbf{H})$$

be the maximal elementary abelian 2-group in $\mathrm{GL}(n, \mathbf{H})$ and $C_4 = \langle I \rangle \subset \mathrm{GL}(n, \mathbf{H})$ the cyclic order four group generated by $I = \mathrm{diag}(i, \ldots, i)$. The maximal elementary abelian 2-group in $\mathrm{PGL}(n, \mathbf{H})$ is the quotient

$$t(\mathrm{PGL}(n, \mathbf{H})) = \frac{t(\mathrm{PGL}(n, \mathbf{H}))^*}{\langle -E \rangle}, \quad t(\mathrm{PGL}(n, \mathbf{H}))^* = C_4 \circ t(\mathrm{GL}(n, \mathbf{H}))$$

so that the toral part of the Quillen category is equivalent

$$\mathbf{A}(\mathrm{PGL}(n, \mathbf{H}))^{\leq t} = \mathbf{A}(C_2 \wr \Sigma_n, \frac{C_4 \circ \langle \mathrm{diag}(\pm 1, \ldots, \pm 1) \rangle}{\langle -E \rangle})$$

to the category whose objects are nontrivial subgroups of $t(\mathrm{PGL}(n, \mathbf{H}))$ and whose morphisms are induced from the action of the Weyl group.

For any partition $i = (i_0, i_1)$ of $n = i_0 + i_1$ into a sum of two positive integers $i_0 \geq i_1 \geq 1 > 0$ let $L[i] = L[i_0, i_1] \subset \mathrm{GL}(n, \mathbf{H})$ be the subgroup generated by

$$\mathrm{diag}(\overbrace{+1, \ldots, +1}^{i_0}, \overbrace{-1, \ldots, -1}^{i_1})$$

Then the centralizer

eq:cfamCL

$$C_{\mathrm{PGL}(n,\mathbf{H})}L[i_0,i_1] = \begin{cases} \frac{\mathrm{GL}(i_0,\mathbf{H})\times\mathrm{GL}(i_1,\mathbf{H})}{\langle -E\rangle} & i_0\neq i_1 \\ \frac{\mathrm{GL}(i_0,\mathbf{H})^2}{\langle -E\rangle} \rtimes \left\langle \begin{pmatrix} 0 & E \\ E & 0\end{pmatrix}\langle -E\rangle \right\rangle & i_0 = i_1 \end{cases} \tag{6.2}$$

so that the center $ZC_{\mathrm{PGL}(n,\mathbf{H})}L[i_0,i_1] = L[i_0,i_1]$ as in the proof of 4.54 and 9.18.

Let (also) $I\in\mathrm{PGL}(n,\mathbf{H})$ denote the order two element that is the image of the order four element $i\in\mathrm{GL}(n,\mathbf{H})$. Then

eq:cfamCI

$$C_{\mathrm{PGL}(n,\mathbf{H})}(I) = \frac{\mathrm{GL}(n,\mathbf{C})}{\langle -E\rangle} \rtimes \langle j\,\langle -E\rangle\rangle \tag{6.3}$$

so that the center $ZC_{\mathrm{PGL}(n,\mathbf{H})}(I) = \langle I\rangle$ as shown in the proof of 4.54.

For any partition $(i_0,i_1,i_2,0)$ of $n=i_0+i_1+i_2$ into a sum of three positive integers $i_0\geq i_1\geq i_2>0$ or any partition (i_0,i_1,i_2,i_3) of $n=i_0+i_1+i_2+i_3$ into a sum of four positive integers $i_0\geq i_1\geq i_2\geq i_3>0$ let $P[i_0,i_1,i_2,i_3]\subset \Delta_{2n+1}$ be the subgroup generated by the two elements

$$\mathrm{diag}(\overbrace{+1,\dots,+1}^{i_0},\overbrace{-1,\dots,-1}^{i_1},\overbrace{+1,\dots,+1}^{i_2},\overbrace{-1,\dots,-1}^{i_3})$$

$$\mathrm{diag}(\overbrace{+1,\dots,+1}^{i_0},\overbrace{+1,\dots,+1}^{i_1},\overbrace{-1,\dots,-1}^{i_2},\overbrace{-1,\dots,-1}^{i_3})$$

Then the centralizer

eq:cfamCP

$$C_{\mathrm{PGL}(n,\mathbf{H})}P[i] = \begin{cases} \frac{\mathrm{GL}(i_0,\mathbf{H})^4}{\langle -E\rangle}\rtimes (C_2\times C_2) & i=(i_0,i_0,i_0,i_0) \\ \frac{\mathrm{GL}(i_0,\mathbf{H})^2\times\mathrm{GL}(i_2,\mathbf{H})^2}{\langle -E\rangle}\rtimes C_2 & i=(i_0,i_0,i_2,i_2) \\ \frac{\mathrm{GL}(i_0,\mathbf{H})\times\mathrm{GL}(i_1,\mathbf{H})\times\mathrm{GL}(i_2,\mathbf{H})\times\mathrm{GL}(i_3,\mathbf{H})}{\langle -E\rangle} & \#i=4 \end{cases} \tag{6.4}$$

where the groups C_2 are generated by permutation matrices.

For any partition $i=(i_0,i_1)$ of $n=i_0+i_1$ into a sum of two positive integers $i_0\geq i_1>0$ let $I\#L[i_0,i_1]\subset\mathrm{PGL}(n,\mathbf{H})$ be the elementary abelian 2-group that is the quotient of

$$(I\#L[i_0,i_1])^* = \langle I, \mathrm{diag}(\overbrace{+1,\dots,+1}^{i_0},\overbrace{-1,\dots,-1}^{i_1})\rangle$$

Then the centralizer

eq:cfamCIL

$$C_{\mathrm{PGL}(n,\mathbf{H})}I\#L[i_0,i_1] = \begin{cases} \frac{\mathrm{GL}(i_0,\mathbf{C})\times\mathrm{GL}(i_1,\mathbf{C})}{\langle -E\rangle}\rtimes\langle j\,\langle -E\rangle\rangle & i_0\neq i_1 \\ \frac{\mathrm{GL}(i_0,\mathbf{C})^2}{\langle -E\rangle}\rtimes\left\langle j\,\langle -E\rangle, \begin{pmatrix} 0 & E\\ E & 0\end{pmatrix}\langle -E\rangle\right\rangle & i_0=i_1\end{cases} \tag{6.5}$$

6.6. PROPOSITION. *The category* $\mathbf{A}(\mathrm{PGL}(n,\mathbf{H}))$ *contains exactly*

- $[n/2]+1$ *rank one toral objects represented by the lines* $L[i,n-i]$, $1\leq i\leq [n/2]$ *(with* $q=0$*), and by the line* I *(with* $q\neq 0$*).*
- $P(n,3)+P(n,4)+[n/2]$ *rank two toral objects represented by the* $P(n,3)$ *planes* $P[i_0,i_1,i_2,0]$ *(with* $q=0$*), the* $P(n,4)$ *planes* $P[i_0,i_1,i_2,i_3]$ *(with* $q=0$*), and the* $[n/2]$ *planes* $I\#L[i,n-i]$, $1\leq i\leq[n/2]$ *(with* $q\neq 0$*).*

6.7. PROPOSITION. *Let* $V\subset\mathrm{PGL}(n,\mathbf{H})$ *be a nontrivial elementary abelian* 2*-group. Then*

$$V \textit{ is toral} \iff [V,V]\neq 0$$

PROOF. The proof is similar to 4.10 with the extra input that all elementary abelian 2-groups in $\mathrm{GL}(n,\mathbf{H})$ are toral by quaternion representation theory [1]. □

prop:cfamlimH1W

6.9. PROPOSITION. *Centralizers of objects of* $\mathbf{A}(\mathrm{GL}(n,\mathbf{H}))^{\leq t}_{\leq 2}$ *are LHS.*

PROOF. The centralizers $C=C_0\rtimes\pi$ in question are the nonconnected centralizers listed in (6.2), (6.3), (6.4), and (6.5). In fact, we only need to deal with

$$\frac{\mathrm{GL}(i,\mathbf{H})^2}{\langle -E\rangle}\rtimes C_2,\quad \frac{\mathrm{GL}(i_0,\mathbf{H})^2\times\mathrm{GL}(i_1,\mathbf{H})^2}{\langle -E\rangle}\rtimes C_2,\quad \frac{\mathrm{GL}(i,\mathbf{H})^4}{\langle -E\rangle}\rtimes (C_2\times C_2)$$

as the other cases are covered by 4.19. It suffices (2.28) to show that $\theta(C_0)^\pi$ (2.20) is surjective.

Computations with the program *magma* results in the table

$\frac{\mathrm{GL}(i,\mathbf{H})^2}{\langle -E\rangle} \rtimes C_2$	$\ker\theta$	$\mathrm{Hom}(W,\check{T}^W)$	$H^1(W;\check{T})$	θ	$H^1(W;\check{T})^\pi$
$1=i$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^2$	0	epi	0
$2=i$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^3$		$(\mathbf{Z}/2)^2$
$2<i$	0	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^4$	iso	$(\mathbf{Z}/2)^2$

From the table we see that θ^π is surjective unless $i=2$. In that exceptional case, more compute computations show that $H^1(\pi;\check{T}^W)=\mathbf{Z}/2$ and $H^1(W\rtimes C_2;\check{T})=(\mathbf{Z}/2)^3$ which means that also $\frac{\mathrm{GL}(2,\mathbf{H})^2}{\langle -E\rangle}\rtimes C_2$ is LHS.

Computations with the program *magma* results in the table

$\frac{\mathrm{GL}(i_0,\mathbf{H})^2\times\mathrm{GL}(i_1,\mathbf{H})^2}{\langle -E\rangle}$	$\ker\theta$	$\mathrm{Hom}(W,\check{T}^W)$	$H^1(W;\check{T})$	θ	$H^1(W;\check{T})^\pi$
$1=i_0, 2=i_1$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^{18}$	$(\mathbf{Z}/2)^{14}$	epi	$(\mathbf{Z}/2)^7$
$1=i_0, 2<i_1$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^{18}$	$(\mathbf{Z}/2)^{16}$	epi	$(\mathbf{Z}/2)^8$
$2=i_0<i_1$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^{24}$	$(\mathbf{Z}/2)^{22}$	epi	$(\mathbf{Z}/2)^{11}$
$3<i_0<i_1$	0	$(\mathbf{Z}/2)^{24}$	$(\mathbf{Z}/2)^{24}$	iso	$(\mathbf{Z}/2)^{12}$

Since θ is surjective and $H^{>0}(\pi;\ker\theta)=0$ because the action of π on $\ker\theta$ is induced from the trivial subgroup, θ^π is surjective.

Computations with the program *magma* results in the table

$\frac{\mathrm{GL}(i,\mathbf{H})^4}{\langle -E\rangle}\rtimes(C_2\times C_2)$	$\ker\theta$	$\mathrm{Hom}(W,\check{T}^W)$	$H^1(W;\check{T})$	θ	$H^1(W;\check{T})^\pi$
$1=i$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^{12}$	$(\mathbf{Z}/2)^8$	epi	$(\mathbf{Z}/2)^2$
$2=i$	$(\mathbf{Z}/2)^4$	$(\mathbf{Z}/2)^{24}$	$(\mathbf{Z}/2)^{20}$	epi	$(\mathbf{Z}/2)^5$
$2<i$	0	$(\mathbf{Z}/2)^{24}$	$(\mathbf{Z}/2)^{24}$	iso	$(\mathbf{Z}/2)^6$

Since θ is surjective and $H^{>0}(\pi;\ker\theta)=0$ because the action of π on $\ker\theta$ is induced from the trivial subgroup, θ^π is surjective. $\square$

sec:cfamlim0

2. The limit of the functor $H^1(W_0;\check{T})^{W/W_0}$ on $\mathbf{A}(\mathrm{PGL}(n,\mathbf{H})^{\le t}_{\le 2}$

Let $H^1(W_0;\check{T})\colon \mathbf{A}(\mathrm{PGL}(n,\mathbf{H}))^{\le t}_{\le t}\to\mathbf{Ab}$ be the functor that takes the toral elementary abelian 2-group $V\subset t(\mathrm{PGL}(n,\mathbf{H}))$ to the abelian group $H^1(W_0(C_{\mathrm{PGL}(n,\mathbf{H})}(V);\check{T}))$, and $H^1(W_0;\check{T})^{W/W_0}$ the functor that takes V to the the invariants for the action of the component group $\pi_0C_{\mathrm{PGL}(n,\mathbf{H})}(V)$ on this first cohomology group.

prop:cfamlimOH1

6.11. PROPOSITION. *The restriction map*

$$H^1(W(\mathrm{PGL}(n,\mathbf{H});\check{T})\to\lim{}^0(\mathbf{A}(\mathrm{PGL}(n,\mathbf{H}))^{\le t}_{\le 2},H^1(W_0;\check{T})^{W/W_0})$$

is an isomorphism for all $n>3$.

PROOF. PGL$(4,\mathbf{H})$: Computer computations show that the intersection of the images of the morphisms

$$H^1(W_0;\check{T})^{W/W_0}(L[1,3])\to H^1(W_0;\check{T})^{W/W_0}(I\#L[1,3])\xleftarrow{\cong}H^1(W_0;\check{T})^{W/W_0}(I)$$

is 1-dimensional and that its pre-image in $H^1(W_0;\check{T})^{W/W_0}(I)$ equals the image of the restriction map from $H^1(W,\check{T})(\mathrm{PGL}(4,\mathbf{H}))$. Similarly, the images of the monomorphisms

$$H^1(W_0;\check{T})^{W/W_0}(L[1,3])\hookrightarrow H^1(W_0;\check{T})^{W/W_0}(P[1,1,2,0])\hookleftarrow H^1(W_0;\check{T})^{W/W_0}(L[2,2])$$

meet in a 1-dimensional subspace whose inverse images in the cohomology groups to the right and to the left agree with the images of the restriction maps from $H^1(W,\check{T})(\mathrm{PGL}(4,\mathbf{H}))$.

PGL$(n,\mathbf{H})$, $n>4$: Computer computations show that the images of the morphisms

$$H^1(W_0;\check{T})^{W/W_0}(L[1,n-1])\to H^1(W_0;\check{T})^{W/W_0}(I\#L[1,n-1])\xleftarrow{\cong}H^1(W_0;\check{T})^{W/W_0}(I)$$

intersect trivially and that the arrow pointing left is an isomorphism. Similarly, the images of the injective morphisms

$$H^1(W_0;\check{T})^{W/W_0}(L[i,n-i]) \hookrightarrow H^1(W_0;\check{T})^{W/W_0}(P[i,1,n-i-1,0]) \\ \hookleftarrow H^1(W_0;\check{T})^{W/W_0}(L[i+1,n-i-1]), \qquad 1 \le i < [n/2],$$

intersect trivially. These observations imply that $\lim^0(\mathbf{A}(\mathrm{PGL}(n,\mathbf{H}))^{\le t}_{\le 2}, H^1(W_0;\check{T})^{W/W_0}) = 0$. □

sec:cfamnontoral

3. The category $\mathbf{A}(\mathrm{PGL}(n,\mathbf{H}))^{[\,,\,]\neq 0}_{\le 4}$

We shall need information about all nontoral objects of $\mathbf{A}(\mathrm{PGL}(n,\mathbf{H}))$ of rank ≤ 3 and some objects of rank 4. If $V \subset \mathrm{PGL}(n,\mathbf{H})$ is an elementary abelian 2-group with nontrivial inner product then its preimage $V^* \subset \mathrm{GL}(n,\mathbf{H})$ is $P \times R(V)$ or $(C_4 \circ P) \times R(V)$ where P is an extraspecial 2-group, $C_4 \circ P$ a generalized extraspecial 2-group, and $\mho_1(V^*) = \langle -E \rangle$ (4.8). We manufacture all oriented quaternion representations of these product groups as direct sums of tensor products of irreducible representations of the factors (9.6) as described in [1, 3.7, 3.65].

Note that the degrees of the faithful irreducible representations over $\mathbf{H}$ for the groups 2^{1+2}_+ and $C_4 \circ 2^{1+2}_\pm$ are even and that the quaternion group 2^{1+2}_- has a faithful irreducible representation over $\mathbf{H}$, namely the defining representation.

sec:cfamoddnont

6.12. The category $\mathbf{A}(\mathrm{PGL}(2n+1,\mathbf{H}))^{[\,,\,]\neq 0}_{\le 4}$. The category $\mathbf{A}(\mathrm{PGL}(2n+1,\mathbf{H}))$ contains up to isomorphism just one nontoral rank two object, H_-, whose inverse image in $\mathrm{GL}(2n+1,\mathbf{H})$ is

$$Q_8 = 2^{1+2}_- = \langle \mathrm{diag}(i,\dots,i), \mathrm{diag}(j,\dots,j) \rangle$$

As in 4.50, the centralizers [54, Proposition 4] of 2^{1+2}_- and H_- are

$$C_{\mathrm{GL}(2n+1,\mathbf{H})}(2^{1+2}_-) = \mathrm{GL}(2n+1,\mathbf{R}), \qquad C_{\mathrm{PGL}(2n+1,\mathbf{H})}(H_-) = H_- \times \mathrm{SL}(2n+1,\mathbf{R})$$

so that $ZC_{\mathrm{PGL}(2n+1,\mathbf{H})}(H_-) = H_-$.

There are n nontoral objects of rank three, $H_-\#L[i,2n+1-i]$, $1 \le i \le n$. The inverse image in $\mathrm{GL}(2n+1,\mathbf{H})$ of $H_-\#L[i,2n+1-i]$ is

$$\big\langle \mathrm{diag}(i,\dots,i), \mathrm{diag}(j,\dots,j), \mathrm{diag}(\overbrace{+1,\dots,+1}^{i}, \overbrace{-1,\dots,-1}^{2n+1-i}) \big\rangle$$

and the center of the centralizer, $C_{\mathrm{PGL}(2n+1,\mathbf{H})}(H_-\#L[i,2n+1-i]) = H_- \times C_{\mathrm{SL}(2n+1,\mathbf{R})}L[i,n-1]$, is $ZC_{\mathrm{PGL}(2n+1,\mathbf{H})}(H_-\#L[i,2n+1-i]) = H_-\#L[i,2n+1-i]$ according to (5.5).

The objects $H_-\#P[i_0,i_1,i_2,i_3]$, where $P[i_0,i_1,i_2,i_3]$ is as in 1, are rank four nontoral objects.

We need to know that the nontoral object H_- satisfies condition (3) of 2.51. Note that the conditions of 2.63 are satisfied because the identity component of $C_{\mathrm{PGL}(2n+1,\mathbf{H})}(H_-)$ is nontrivial and because the Quillen automorphism group $\mathbf{A}(\mathrm{PGL}(2n+1,\mathbf{H}))(H_-) = \mathrm{GL}(2,\mathbf{F}_2)$ acts transitively on the set preferred lifts $H_- \subset N(\mathrm{PGL}(2n+1,\mathbf{H}))$ of $H_- \subset \mathrm{PGL}(2n+1,\mathbf{H})$. Under the inductive assumption that $\mathrm{SL}(2n+1,\mathbf{R})$ has $\pi_*(N)$-determined automorphisms (or using [31]) we conclude from 2.51 and diagram (2.64) and (part of) [42, 5.2] that condition (3) of 2.51 is satisfied for the nontoral rank 2 object H_-. (Namely, 2.63.(1) says that ν'_L does not depend on the choice of $L < V$. The difference $f^{-1}_{\nu,L_2} \circ f_{\nu,L_1}$ between any two of the maps $f_{\nu,L}$ from 2.51.(3) is an automorphism of $C_{\mathrm{PGL}(2n+1,\mathbf{H})}(H_-)$ that, by 2.63.(2), is the identity on the identity component and by the commutative diagram (2.64)

dia:cfampi0H-

(6.13)
$$\begin{array}{ccc} & H_- & \\ \swarrow & & \searrow \\ C_{\mathrm{PGL}(2n+1,\mathbf{H})}(H_-) & \xrightarrow[f^{-1}_{\nu,L_2} \circ f_{\nu,L_1}]{} & C_{\mathrm{PGL}(2n+1,\mathbf{H})}(H_-) \end{array}$$

also the identity on $\pi_0 C_{\mathrm{PGL}(2n+1,\mathbf{H})}(H_-)$. Since the identity component $\mathrm{SL}(2n+1,\mathbf{R})$ of the centralizer $C_{\mathrm{PGL}(2n+1,\mathbf{H})}(H_-)$ has no center, this shows that $f^{-1}_{\nu,L_2} \circ f_{\nu,L_1}$ is the identity automorphism.)

c:cfamnontrank2

6.14. Rank two nontoral objects of $\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))$. The category $\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))$ contains up to isomorphism two nontoral rank two objects, H_+ and H_-, whose inverse images in $\mathrm{GL}(2n,\mathbf{H})$ are

$$2^{1+2}_+ = \langle \mathrm{diag}(R,\dots,R), \mathrm{diag}(T,\dots,T)\rangle, \qquad R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$2^{1+2}_- = \langle \mathrm{diag}(i,\dots,i), \mathrm{diag}(j,\dots,j)\rangle$$

where the representation of the dihedral group 2^{1+2}_+ is of real type and the representation of the quaternion group 2^{1+2}_- of quaternion type. This follows from 4.8 because 2^{1+2}_+ has one faithful irreducible $\mathbf{H}$-representation of degree 2 and 2^{1+2}_- has one faithful irreducible $\mathbf{H}$-representation of degree 1. The centralizers are [**54**, Proposition 4]

$$C_{\mathrm{GL}(2n,\mathbf{H})}(2^{1+2}_+) = \mathrm{GL}(n,\mathbf{H}), \qquad C_{\mathrm{PGL}(2n,\mathbf{H})}(H_+) = H_+ \times \mathrm{PGL}(n,\mathbf{H})$$
$$C_{\mathrm{GL}(2n,\mathbf{H})}(2^{1+2}_-) = \mathrm{GL}(2n,\mathbf{R}), \qquad C_{\mathrm{PGL}(2n,\mathbf{H})}(H_-) = H_- \times \mathrm{PGL}(2n,\mathbf{R})$$

as we see by an argument similar to that of 4.50. This implies (9.18) that $ZC_{\mathrm{PGL}(2n,\mathbf{H})}(H) = H$ for all nontoral rank two objects H of $\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))$.

We need to know that these nontoral objects satisfy condition (3) of 2.51. To see this we use 2.63.

$\underline{H_+}$: Condition (1) of 2.63 is clearly satisfied since the identity component of $C_{\mathrm{PGL}(2n,\mathbf{H})}(H_+)$ is nontrivial when $n \geq 3$. The group $H_+^* = 2^{1+2}_+$ is contained in $N(\mathrm{GL}(2n,\mathbf{H})) = N(\mathrm{GL}(1,\mathbf{H})) \wr \Sigma_{2n}$ and its centralizer there is

$$C_{N(\mathrm{GL}(2n,\mathbf{H}))}(2^{1+2}_+) = C_{N(\mathrm{GL}(1,\mathbf{H}))\wr\Sigma_{2n}}(2^{1+2}_+) = N(\mathrm{GL}(1,\mathbf{H})) \wr \Sigma_n = N(\mathrm{GL}(n,\mathbf{H}))$$

and therefore H_- is contained in $N(\mathrm{GL}(2n,\mathbf{H}))/\langle -E\rangle = N(\mathrm{PGL}(2n,\mathbf{H}))$ where its centralizer is

$$C_{N(\mathrm{PGL}(2n,\mathbf{H}))}(H_+) = H_+ \times N(\mathrm{PGL}(n,\mathbf{H})) = N(C_{\mathrm{GL}(2n,\mathbf{H})}(H_+))$$

as in 4.50. This means that $H_+ \subset N(\mathrm{PGL}(2n,\mathbf{H}))$ is a preferred lift [**45**] of $H_+ \subset \mathrm{GL}(2n,\mathbf{H})$. Precomposing the inclusion $H_+ \subset N(\mathrm{PGL}(2n,\mathbf{H}))$ with the nontrivial element of $\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))(H_+) = O^+(2,\mathbf{F}_2) \cong C_2$ (6.21) leads to another preferred lift. The third preferred lift is the quotient of

$$(2^{1+2}_+)^{\mathrm{diag}(B,\dots,B)} = \big\langle \mathrm{diag}(R^B,\dots,R^B), \mathrm{diag}((RT)^B,\dots,(RT)^B)\big\rangle,$$
$$B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \qquad R^B = T, \qquad (RT)^B = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Note that these three preferred lifts all have the same image in the Weyl group $\pi_0 N(\mathrm{GL}(2n,\mathbf{H})) = \pi_0(N(\mathrm{GL}(1,\mathbf{H}))) \wr \Sigma_{2n}$, namely the subgroup generated by the permutation $(1,2)(3,4)\cdots(2n-1,2n) \in \Sigma_{2n}$.

Under the inductive assumption that $\mathrm{PGL}(n,\mathbf{H})$ has $\pi_*(N)$-determined automorphisms (or using [**31**]) we conclude from 2.63 and diagram (2.64) and (part of) [**42**, 5.2] that condition (3) of 2.51 is satisfied for the nontoral rank 2 object H_+ of $\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))$. (Namely, 2.63.(1) says that ν'_L does not depend on the choice of $L < V$. The difference $f^{-1}_{\nu,L_2} \circ f_{\nu,L_1}$ between any two of the maps $f_{\nu,L}$ from 2.51.(3) is an automorphism of $C_{\mathrm{PGL}(2n,\mathbf{H})}(H_+)$ that, by 2.63.(2), is the identity on the identity component and by the commutative diagram (2.64)

dia:cfampiOH+

(6.15)
$$\begin{array}{ccc} & H_+ & \\ \swarrow & & \searrow \\ C_{\mathrm{PGL}(2n,\mathbf{H})}(H_+) & \xrightarrow[f^{-1}_{\nu,L_2}\circ f_{\nu,L_1}]{} & C_{\mathrm{PGL}(2n,\mathbf{H})}(H_+) \end{array}$$

also the identity on $\pi_0 C_{\mathrm{PGL}(2n,\mathbf{H})}(H_+)$. Since the identity component of $C_{\mathrm{PGL}(2n,\mathbf{H})}(H_+)$ has no center, this shows that $f^{-1}_{\nu,L_2} \circ f_{\nu,L_1}$ is the identity automorphism.)

$\underline{H_-}$: Condition (1) of 2.63 is clearly satisfied since the identity component of $C_{\mathrm{PGL}(2n,\mathbf{H})}(H_-)$ is nontrivial when $n \geq 3$. The group $H_-^* = 2^{1+2}_-$ is contained in $N(\mathrm{GL}(2n,\mathbf{H})) = N(\mathrm{GL}(1,\mathbf{H})) \wr \Sigma_{2n}$

and its centralizer there is

$$C_{N(\mathrm{GL}(1,\mathbf{H}))\wr\Sigma_{2n}}(2_-^{1+2}) \overset{9.10}{=} C_{N(\mathrm{GL}(1,\mathbf{H}))}(i,j)\wr\Sigma_n = \mathrm{GL}(1,\mathbf{R})\wr\Sigma_{2n} = N(\mathrm{GL}(2n,\mathbf{R}))$$

and therefore H_- is contained in $N(\mathrm{GL}(2n,\mathbf{H}))/\langle -E\rangle = N(\mathrm{PGL}(2n,\mathbf{H}))$ where its centralizer is

$$C_{N(\mathrm{PGL}(2n,\mathbf{H}))}(H_-) = H_- \times N(\mathrm{GL}(2n,\mathbf{R}))/\langle -E\rangle = H_- \times N(\mathrm{PGL}(2n,\mathbf{R})) = N(C_{\mathrm{PGL}(2n,\mathbf{H})}(H_-))$$

as in 4.50. This means that $H_- \subset N(\mathrm{PGL}(2n,\mathbf{H}))$ is a preferred lift [**45**] of $H_- \subset \mathrm{GL}(2n,\mathbf{H})$. Precomposing the inclusion $H_- \subset N(\mathrm{PGL}(2n,\mathbf{H}))$ with elements of $\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))(H_-) = O^-(2,\mathbf{F}_2) = \mathrm{GL}(2,\mathbf{F}_2)$ (6.21) leads to other two preferred lifts of H_-.

Under the inductive assumption that the identity component $\mathrm{PSL}(2n,\mathbf{R})$ of $\mathrm{PGL}(2n,\mathbf{R})$ has $\pi_*(N)$-determined automorphisms (or using [**31**]) we conclude from 2.63 and diagam (2.64) and (part of) [**42**, 5.2] that condition (3) of 2.51 is satisfied for the nontoral rank 2 object H_- of $\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))$. (The argument for this is the same as in case of H_+ with the little extra complication that $\pi_0 C_{\mathrm{PGL}(2n,\mathbf{H})}(H_-)$ has an extra generator so that we replace diagram (6.15) by

:cfampi0H-extra

(6.16)
$$\begin{array}{ccc} & \langle H_-, \mathrm{diag}(-1,1,\dots,1)\rangle & \\ \swarrow & & \searrow \\ C_{\mathrm{PGL}(2n,\mathbf{H})}(H_+) & \xrightarrow[f_{\nu,L_2}^{-1}\circ f_{\nu,L_1}]{} & C_{\mathrm{PGL}(2n,\mathbf{H})}(H_+) \end{array}$$

from (2.64) where the slanted arrows induce isomorphisms on the component groups.)

c:cfamnontrank3

6.17. Rank three nontoral objects of $\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))$. The nontoral rank three objects of the category $\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))$ are the quotients of $H_+\#L[i,n-i]$, $1\le i\le [n/2]$, $H_-\#L[i,2n-i]$, $1\le i\le n$, and V_0. These subgroups of $\mathrm{GL}(2n,\mathbf{H})$ are defined to be

$$\Big\langle\, \mathrm{diag}(\overbrace{R,\dots,R}^{n}), \mathrm{diag}(\overbrace{T,\dots,T}^{n}), \mathrm{diag}(\overbrace{E,\dots,E}^{i},\overbrace{-E,\dots,-E}^{n-i})\Big\rangle$$
$$\Big\langle\, \mathrm{diag}(\overbrace{i,\dots,i}^{2n}), \mathrm{diag}(\overbrace{j,\dots,j}^{2n}), \mathrm{diag}(\overbrace{1,\dots,1}^{i},\overbrace{-1,\dots,-1}^{2n-i})\Big\rangle$$
$$\Big\langle\, \mathrm{diag}\,(\overbrace{i,\dots,i}^{2n}), \mathrm{diag}(\overbrace{R,\dots,R}^{n}), \mathrm{diag}(\overbrace{T,\dots,T}^{n}),\Big\rangle$$

and their centralizers are

$$C_{\mathrm{PGL}(2n,\mathbf{H})}(H_+\#L[i,n-i]) = H_+ \times C_{\mathrm{PGL}(n,\mathbf{H})}L[i,n-i],$$
$$C_{\mathrm{PGL}(2n,\mathbf{H})}(H_-\#L[i,n-i]) = H_- \times C_{\mathrm{PGL}(2n,\mathbf{R})}L[i,2n-i],$$
$$C_{\mathrm{PGL}(2n,\mathbf{H})}(V_0) = H_+ \times C_{\mathrm{PGL}(n,\mathbf{H})}(I) \overset{6.3}{=} H_+ \times \frac{\mathrm{GL}(n,\mathbf{C})}{\langle -E\rangle} \rtimes \langle j\,\langle -E\rangle\rangle$$

so that (6.2, 4.54, 6.3) $ZC_{\mathrm{PGL}(2n,\mathbf{H})}(V) = V$ for all nontoral rank three objects V of $\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))$. The elements of $H_+\#L[i,n-i]$, $H_-\#L[i,2n-i]$, and V_0 have traces (computed in $\mathrm{GL}(4n,\mathbf{C})$) in the sets $\pm\{0,4n-8i,4n\}$, $\pm\{0,4n-4i,4n\}$, and $\pm\{0,4n\}$.

c:cfamnontrank4

6.18. Rank four nontoral objects of $\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))$. $H_-\#P[1,i-1,2n-i,0] \subset \mathrm{GL}(2n,\mathbf{H})$, $1<i\le n$, is

$$\langle \mathrm{diag}(\overbrace{i,\dots,i}^{2n}), \mathrm{diag}(\overbrace{j,\dots,j}^{2n}), \mathrm{diag}(1,\overbrace{-1,\dots,-1}^{i-1},\overbrace{1,\dots,1}^{2n-i}), \mathrm{diag}(1,\overbrace{1,\dots,1}^{i-1},\overbrace{-1,\dots,-1}^{2n-i})\rangle$$

The elements of P have traces $\{2n+2-2i,-2n+2i,2n+1\}$ and these three integers are all distinct so that the Quillen automorphism group (6.21) has order $3\cdot 2^5$. This nontoral rank four object contains the two nontoral rank three objects $H_-\#L[1,2n-1]$,$H_-\#L[2,2n-2]$ when $i=2$ and the three nontoral rank three objects $H_-\#L[1,2n-1]$,$H_-\#L[i-1,2n-i+1]$, $H_-\#L[i,2n-i]$ when $i>2$.

$V_0\#L[i,n-i] \subset \mathrm{GL}(2n,\mathbf{C}) \subset \mathrm{GL}(2n,\mathbf{H})$, $1\le i\le [n/2]$, is the subgroup

$$\Big\langle\, \mathrm{diag}(\overbrace{i,\dots,i}^{2n}), \mathrm{diag}(\overbrace{R,\dots,R}^{n}), \mathrm{diag}(\overbrace{T,\dots,T}^{n}), \mathrm{diag}\,(\overbrace{E,\dots,E}^{i},\overbrace{-E,\dots,-E}^{n-i})\Big\rangle$$

containing the three rank three objects $H_+\#L[i,n-i]$, $H_-\#L[2i,2n-2i]$, and V_0.

For these nontoral rank four objects $E \subset \mathrm{GL}(2n,\mathbf{H})$, the center of the centralizer is finite (4.54) and as, of course, $E \subset ZC_{\mathrm{PGL}(2n,\mathbf{H})}(E)$ we see that $\mathrm{Hom}_{\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))}(\mathrm{St}(E),E)$ is a subspace of $\mathrm{Hom}_{\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))}(\mathrm{St}(E),\pi_1 BZC_{\mathrm{PGL}(2n,\mathbf{H})}(E))$.

sec:cfamlim

4. Higher limits of the functor $\pi_i BZC_{\mathrm{PGL}(n,\mathbf{H})}$ on $\mathbf{A}(\mathrm{PGL}(n,\mathbf{H}))^{[\,,\,]\neq 0}$

In this section we compute the first higher limits of the functors $\pi_j BZC_{\mathrm{PGL}(n,\mathbf{H})}$, $j=1,2$.

lemma:cfamlim=0

6.19. LEMMA. *$\lim^1 \pi_1 BZC_{\mathrm{PGL}(n,\mathbf{H})} = 0 = \lim^2 \pi_1 BZC_{\mathrm{PGL}(n,\mathbf{H})}$ and $\lim^2 \pi_2 BZC_{\mathrm{PGL}(n,\mathbf{H})} = 0 = \lim^3 \pi_2 BZC_{\mathrm{PGL}(n,\mathbf{H})}$.*

The case $j=2$ is easy. Since $\pi_2 BZC_{\mathrm{PGL}(n,\mathbf{H})}$ has value 0 on all objects of $\mathbf{A}(\mathrm{PGL}(n,\mathbf{H}))^{[\,,\,]\neq 0}$ of rank ≤ 4 it is immediate from Oliver's cochain complex [53] that $\lim^2$ and $\lim^3$ of this functor are trivial. We shall therefore now concentrate on the case $j=1$.

For any elementary abelian 2-group E in $\mathrm{PGL}(n,\mathbf{H})$ we shall write

defn:cfam[E]

$$[E] = \mathrm{Hom}_{\mathbf{A}(\mathrm{PGL}(n,\mathbf{H})(E))}(\mathrm{St}(E),E) \tag{6.20}$$

for the $\mathbf{F}_2$-vector space of $\mathbf{F}_2\mathbf{A}(\mathrm{PGL}(n,\mathbf{H})(E)$-equivariant maps from the Steinberg representation $\mathrm{St}(E)$ over $\mathbf{F}_2$ of $\mathrm{GL}(E)$ to E. Olivers cochain complex has the form (4.33).

prop:cfamH-L

6.21. PROPOSITION. *Regardless of the parity of n, the Quillen automorphism groups*

$$\mathbf{A}(\mathrm{PGL}(n,\mathbf{H}))(H_-) = O^-(2,\mathbf{F}_2)$$

$$\mathbf{A}(\mathrm{PGL}(n,\mathbf{H}))(H_-\#V) = \begin{pmatrix} O^-(2,\mathbf{F}_2) & * \\ 0 & \mathbf{A}(\mathrm{GL}(n,\mathbf{R}))(V) \end{pmatrix}$$

and $\dim_{\mathbf{F}_2}[H_-] = 1 = \dim_{\mathbf{F}_2}[H_-\#L[i,2n+1-i]]$ as described in 4.34 and 4.41.

PROOF. $\mathbf{A}(\mathrm{GL}(n,\mathbf{H}))(2_-^{1+2}) = \mathrm{Out}(2_-^{1+2})$ since all automorphisms of 2_-^{1+2} preserve the trace. This group maps (isomorphically) to the subgroup $O^-(2,\mathbf{F}_2) \subset \mathrm{GL}(H_-)$ of automorphisms that preserve the quadratic function q on H_-. The Quillen automorphism group of $H_-\#V$ consists of the automorphisms that lift to trace preserving automorphisms of $2_-^{1+2}\#V$. The dimension of the vector spaces of equivariant maps was computed by *magma*. □

In the odd case of $\mathrm{GL}(2n+1,\mathbf{H})$ the cochain complex (4.33) takes the form

eq:cfam0lcccodd

$$0 \to [H_-] \xrightarrow{d^1} \prod_{1\leq i\leq n} [H_-\#L[i,2n+1-i]] \xrightarrow{d^2} \prod_{|E|=2^4} [E] \xrightarrow{d^3} \cdots \tag{6.22}$$

and we need to show that d^1 is injective and that $\dim(\operatorname{im} d^2) \geq n-1$.

If $E = H_-\#P[i]$, where $P[i]$ is as in (6.4), then

$$\mathbf{A}(\mathrm{PGL}(2n+1,\mathbf{H}))(H_-\#P[i]) = \begin{pmatrix} O^-(2,\mathbf{F}_2) & * \\ 0 & \mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}))(P[i]) \end{pmatrix}$$

where $\mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}))(P[i])$ is the group of trace preserving automorphisms of $P[i]$. It turns out that

$$\dim_{\mathbf{F}_2}[H_-\#P[i_0,i_1,i_2,i_3]] = \begin{cases} 2 & \mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}))(P[i]) = \{E\} \\ 1 & \mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}))(P[i]) = C_2 \\ 0 & \mathbf{A}(\mathrm{SL}(2n+1,\mathbf{R}))(P[i]) = \mathrm{GL}(2,\mathbf{F}_2) \end{cases}$$

When $n=1$ or $n=2$, the cochain complex (6.22) has the form

$$0 \to [H_-] \xrightarrow{d^1} [H_-\#L[1,2]] \xrightarrow{d^2} [H_-\#P[1,1,1,0]] \to \cdots$$

$$0 \to [H_-] \xrightarrow{d^1} [H_-\#L[1,4]] \times [H_-\#L[2,3]] \xrightarrow{d^2} [H_-\#P[1,1,3,0]] \times [H_-\#P[1,2,2,0]] \to \cdots$$

where all vector spaces are one-dimensional. In the case of $n=1$, d^1 is an isomorphism, and in the case $n=2$, d^1 has matrix $\begin{pmatrix}1 & 1\end{pmatrix}$ and d^2 has matrix $\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$. In case $n\geq 3$, it is enough to show that d^1 is injective and d^2 has rank $n-1$ in the cochain complex

$$0 \to [H_-] \xrightarrow{d^1} \prod_{1\le i\le n} [H_-\#L[i,2n+1-i]] \xrightarrow{d^2} \prod_{2<i\le n} [H_-\#P[1,i-1,2n-i+1,0]]$$

that agrees with (6.22) in degrees 1, a product of one-dimensional vector spaces, and 2, a product of two-dimensional vector spaces. The elementary abelian 2-group $H_-\#P[1,i-1,2n-i+1,0] \subset \mathrm{GL}(2n+1,\mathbf{H})$ contains the nontoral subspaces $H_-\#L[1,2n]$, $H_-\#L[i-1,2n-i+2]$, and $H_-\#L[i,2n-i+1]$. The map f_-, defined exactly as in 4.36, is the nonzero element of $[H_-]$ and the maps df_-, defined exactly as in 4.42, are nonzero in $H_-\#L[i,2n+1-i]$. Thus d^1 is injective. A *magma* computation reveals that $\{ddf_{-L[i-1,2n-i+2]}, ddf_{-L[i,2n-i+1]}\}$, where these $\mathbf{F}_2\mathbf{A}(\mathrm{GL}(2n+1,\mathbf{H}))(H_-\#P[1,i-1,2n-i+1,0])$-maps are defined as in 4.44, is a basis for the two-dimensional space $H_-\#P[1,i-1,2n-i+1,0]$ and that $ddf_{-L[1,2n]} = ddf_{-L[i-1,2n-i+2]} + ddf_{-L[i,2n-i+1]}$. This shows that d^2 has rank $n-1$.

In the even case of $\mathrm{GL}(2n,\mathbf{H})$ the cochain complex (4.33) takes the form

$$0 \to [H_-]\times[H_+] \xrightarrow{d^1} \prod_{1\le i\le n} [H_-\#L[i,2n-i]] \times \prod_{1\le i\le [n/2]} [H_+\#L[i,n-i]] \times [V_0] \xrightarrow{d^2} \prod_{|E|=2^4} [E]$$

prop:cfamH+L

6.23. PROPOSITION. *The automorphism groups of the low-degree nontoral objects of the Quillen category* $\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))$ *are:*

$$\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))(H_+) = O^+(2,\mathbf{F}_2) \quad \mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))(H_+\#V) = \begin{pmatrix} O^+(2,\mathbf{F}_2) & * \\ 0 & \mathbf{A}(\mathrm{GL}(n,\mathbf{H}))(V)\end{pmatrix}$$

$$\mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))(V_0) \cong \mathrm{Sp}(2,\mathbf{F}_2) \qquad \mathbf{A}(\mathrm{PGL}(2n,\mathbf{H}))(V_0\#L[i,n-i]) \cong \begin{pmatrix} \mathrm{Sp}(2,\mathbf{F}_2) & * \\ 0 & 1\end{pmatrix}$$

and $\dim_{\mathbf{F}_2}[H_+]=2$, $\dim_{\mathbf{F}_2}[H_+\#L[i,n-i]]=3$, $\dim_{\mathbf{F}_2}[V_0]=4$, *and* $\dim_{\mathbf{F}_2}[V_0\#L[i,n-i]=5$ *as described in 4.34, 4.39, and 4.37, and 4.44.*

PROOF. The Quillen automorphism groups of the dihedral group 2_+^{1+2} and the generalized extraspecial group $4\circ 2_\pm^{1+2}$ are the full outer automorphism groups because the traces are nonzero only on the derived groups which are characteristic. The images in $\mathrm{GL}(H_+)$, respectively, $\mathrm{GL}(V_0)$, isomorphic to $O^+(2,\mathbf{F}_2)\cong C_2$ and to $\mathrm{Sp}(2,\mathbf{F}_2)=\mathrm{GL}(2,\mathbf{F}_2)$, are the Quillen automorphism groups for H_+ and V_0. For the middle formula, recall that the trace of $H_\pm\#V$ is the product of the traces. □

As in the real case (Chp 4) we get that d^1 embeds $[H_-]\times[H_+]$ into $[V_0]$. The only problem is to show that the rank of d^2 is $\ge n+3[n/2]+4-3 = n+3[n/2]+1$. We have to show that

$$\dim(\operatorname{im} d^2) \ge n+3[n/2]+1$$

We show this by mapping the $n+[n/2]+1$ nontoral rank three objects (6.17),

- $[H_-\#L[i,2n-i]]$, $1\le i\le n$, with basis $\{df_-\}$ as in (4.42),
- $[H_+\#L[i,n-i]]$, $1\le i\le [n/2]$, with basis $\{df_+, df_0, f_0\}$ as in (4.40), and
- $[V_0]$ with basis $\{df_+, df_0, df_-, f_0\}$ as in (4.38)

into the $(n-2)+[n/2]$ nontoral rank four objects (6.18)

- $H_-\#P[1,i-1,2n+1-i]$, $2<i\le n$, with basis $\{ddf_{-L[i-1,2n+1-i]}, ddf_{-L[i,2n-i]}\}$ where these maps are defined as the similar maps in (4.44),
- $V_0\#L[i,n-i]$, $1\le i\le [n/2]$, with basis

$$\{ddf_{+L[i,n-i]}, ddf_{0L[i,n-i]}, df_{0L[i,n-i]}, ddf_{-L[2i,2n-2i]}, df_{0V_0}\}$$

as in (4.44)

Computations with *magma* shows that the resulting $(n+3[n/2]+4)\times(2n+5[n/2])$-matrix has rank $n+3[n/2]+1$. The matrix has the form (shown here for $n=5$)

	$[H_-\#P[1,2,7]]$	$[H_-\#P[1,3,6]]$	$[H_-\#P[1,4,5]]$
$H_-\#L[1,9]$	$\begin{pmatrix}1 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 1\end{pmatrix}$
$H_-\#L[2,8]$	$\begin{pmatrix}1 & 0\end{pmatrix}$		
$H_-\#L[3,7]$	$\begin{pmatrix}0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0\end{pmatrix}$	
$H_-\#L[4,6]$		$\begin{pmatrix}0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0\end{pmatrix}$
$H_-\#L[5,5]$			$\begin{pmatrix}0 & 1\end{pmatrix}$
$H_+\#L[1,4]$			
$H_+\#L[2,3]$			
V_0			

$V_0\#L[1,4]$	$V_0\#L[2,3]$	
		$H_-\#L[1,9]$
$\begin{pmatrix}0 & 0 & 0 & 1 & 0\end{pmatrix}$		$H_-\#L[2,8]$
		$H_-\#L[3,7]$
	$\begin{pmatrix}0 & 0 & 0 & 1 & 0\end{pmatrix}$	$H_-\#L[4,5]$
		$H_-\#L[5,5]$
A		$H_+\#L[1,4]$
	A	$H_+\#L[2,3]$
B	B	V_0

where

$$A = \begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\end{pmatrix}, \qquad B = \begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}$$

CHAPTER 7

The 2-compact groups G_2, DI(4) and F_4

chp:di4andf4

We use the material of the previous chapters to (re)prove that the 2-compact groups G_2, DI(4) and F_4 are uniquely N-determined.

1. The 2-compact group G_2

sec:G2

BG_2 is a rank two 2-compact group containing a rank three elementary abelian 2-group $E_3 \subset G_2$ such that $\mathbf{A}(G_2)(E_3) = GL(3, \mathbf{F}_2)$ [23, 6.1] [20, 5.3] and

$$H^*(BG_2 : \mathbf{F}_2) \cong H^*(BE_3; \mathbf{F}_2)^{GL(3,\mathbf{F}_2)} \cong \mathbf{F}_2[c_4, c_6, c_7]$$

realizes the mod 2 rank 3 Dickson algebra [36]. The Quillen category $\mathbf{A}(G_2))$ contains exactly one isomorphism class of objects E_1, E_2, E_3 of ranks $1, 2, 3$ as Lannes theory [32] implies that the inclusion functor $\mathbf{A}(E_3, GL(3, \mathbf{F}_2)) \to \mathbf{A}(G_2)$ is an equivalence of categories. The centralizers of $E_1 \subset E_2 \subset E_3$ are

$$SO(4) \supset T \rtimes \langle -E \rangle \supset E_3,$$

In all three cases, $ZC_{G_2}(E_i) = E_i$ so that $\pi_2 BZC_{G_2} = 0$ and $\pi_1 BZC_{G_2} = H^0(GL(3, \mathbf{F}_2)(-); E_3)$. Thus $\pi_1 BZC_{G_2}$ is an exact functor (2.69) with $\lim^0 \pi_1 BZC_{G_2} = H^0(GL(3, \mathbf{F}_2); E_3) = 0$.

The Weyl $W(G_2) \subset GL(2, \mathbf{Z}) \subset GL(2, \mathbf{Z}_2)$, of order 12, is generated by the two matrices [4, VI.4.13]

$$\begin{pmatrix} -1 & 0 \\ 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

and the maximal torus normalizer $N(G_2)$ is the semi-direct product of the maximal torus and the Weyl group [10].

It is known that $H^0(W; \check{T})(G_2) = 0$, $H^1(W; \check{T})(G_2) = 0$, and $H^2(W; \check{T})(G_2) = 0$ [24, 26].

Proof of Theorem 1.4. The rank one centralizer, $SL(4, \mathbf{R}) = SL(2, \mathbf{C}) \circ SL(2, \mathbf{C})$, is uniquely N-determined (1.2). Condition 2.51.(2) is satisfied because $H^1(W(X); \check{T}(X)) = 0$ for $X = G_2$, $SL(4, \mathbf{R})$ [24], 2.51.(1) and 2.51.(3) because the only rank two object in G_2 is toral and its centralizer is a 2-compact toral group. We noted above that the higher limits vanish. Now, 2.48 and 2.51 show that G_2 is uniquely N-determined.

We have $\mathrm{Aut}(G_2) = W(G_2) \backslash N_{GL(2,\mathbf{Z}_2)}(W(G_2))$ (2.17) as the extension class $e(G_2) = 0$ [10]. The exact sequence (2.8) can be used to calculate the automorphism group. Using the description of the root system from [4, VI.4.13] with short root $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and long root $\alpha_2 = 2\varepsilon - \varepsilon_2 - \varepsilon_3$ generating the integral lattice in $\mathbf{Z}_2^3$ one finds that

$$N_{GL(2,\mathbf{Z}_2)}(W(G_2)) = \langle \mathbf{Z}_2^\times, A, W(G_2) \rangle, \quad A = \sqrt{-3} \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}$$

and therefore $\mathrm{Aut}(G_2) = \mathbf{Z}_2^\times / \mathbf{Z}^\times \times C_2$ where the cyclic group of order two is generated by the exotic automorphism A interchanging the two roots. □

2. The 2-compact group DI(4)

sec:di4

$BDI(4)$ is a rank three 2-compact group containing a rank four elementary abelian 2-group $E_4 \subset DI(4)$ such that $\mathbf{A}(DI(4))(E_4) = GL(4, \mathbf{F}_2)$ and [16]

$$H^*(BDI(4); \mathbf{F}_2) \cong H^*(BE_4; \mathbf{F}_2)^{GL(4,\mathbf{F}_2)} \cong \mathbf{F}_2[c_8, c_{12}, c_{14}, c_{15}]$$

realizes the mod 2 rank 4 Dickson algebra. Lannes theory [32] implies that the Quillen category $\mathbf{A}(\mathrm{DI}(4))$ is equivalent to $\mathbf{A}(\mathrm{GL}(4,\mathbf{F}_2),E_4)$ with exactly one elementary abelian 2-group (isomorphism class), $E_1,\dots,E_4$, of each rank $1,\dots,4$. The centralizers of the toral subgroups E_1,E_2,E_3 and the nontoral subgrouop E_4 are, respectively,

$$\mathrm{Spin}(7)\supset \mathrm{SU}(2)^3/\langle(-E,-E,-E)\rangle\supset T\rtimes\langle -E\rangle\supset E_4$$

and $ZC_{\mathrm{DI}(4)}(E_i)=E_i$ in all four cases so that the functor $\pi_jBZC_{\mathrm{DI}(4)}\colon \mathbf{A}(\mathrm{GL}(4,\mathbf{F}_2),E_4)\to\mathbf{Ab}$ is the 0-functor for $j=2$ and equivalent to the functor $H^0(\mathrm{GL}(4,\mathbf{F}_2)(-);E_4)$ for $j=1$. This is an exact functor (2.69) and $\lim^0\pi_1BZC_{\mathrm{DI}(4)}=H^0(\mathrm{GL}(4,\mathbf{F}_2);E_4)=0$.

As may be seen from [57], the Weyl group $W(\mathrm{DI}(4))\subset\mathrm{GL}(3,\mathbf{Z}_2)$ of order $2|\mathrm{GL}(3,\mathbf{F}_2)|=336$ is generated by the matrices

$$\begin{pmatrix}1&0&0\\2&-1&-1\\0&0&1\end{pmatrix},\quad\begin{pmatrix}1&0&0\\0&0&1\\2&-1&-1\end{pmatrix},\quad\begin{pmatrix}-1&1&1\\0&1&0\\0&0&1\end{pmatrix},\quad\begin{pmatrix}-v&0&v^2+v\\-1&1&v\\-2v&0&v\end{pmatrix}$$

where $v\in\mathbf{Z}_2$ is the unique 2-adic integer with $2v^2-v+1=0$. The first three matrices generate $W(\mathrm{Spin}(7))$ [7, 3.9, 3.11]. Since $W(\mathrm{DI}(4))$ is isomorphic to $\mathrm{GL}(3,\mathbf{F}_2)\times\langle -E\rangle$,

$$H^n(W;\check{T})(\mathrm{DI}(4))=\bigoplus_{2i\le n}H^{n-2i}(\mathrm{GL}(3,\mathbf{F}_2);H^{2i}(\langle -E\rangle;\check{T})=\bigoplus_{2i\le n}H^{n-2i}(\mathrm{GL}(3,\mathbf{F}_2);(\mathbf{Z}/2)^3)$$

and, in particular,

$$H^0(W;\check{T})(\mathrm{DI}(4))=0,\quad H^1(W;\check{T})(\mathrm{DI}(4))=\mathbf{Z}/2,\quad H^2(W;\check{T})(\mathrm{DI}(4))=\mathbf{Z}/2$$

We may characterize the maximal torus normalizer short exact sequence for DI(4) as the unique nonsplit extension of $\check{T}$ by $W(\mathrm{DI}(4))$; it is nonsplit because the restriction to $W(\mathrm{Spin}(7))\subset W(\mathrm{DI}(4))$ is nonsplit [10].

We can not use 2.51 as it stands because condition (2) fails: The restriction map

$$\mathbf{Z}/2=H^1(W;\check{T})(\mathrm{DI}(4))\to H^1(W;\check{T})(\mathrm{Spin}(7))\overset{[24]}{=}(\mathbf{Z}/2)^2$$

is not surjective. Note, however, that the proof of 2.51 goes through with only insignificant changes if we replace hypotheses (1) and (2) by

(1 & 2) The centralizer of any toral $(V,\nu)\in\mathrm{Ob}(\mathbf{A}(X)^{\le t}_{\le 2})$ is uniquely N-determined.

and leave the other conditions unchanged.

Proof of Theorem 1.5. Condition (1 & 2) is satisfied for DI(4) since the connected 2-compact groups Spin(7) and $\mathrm{SU}(2)^2/\Delta$ are uniquely N-determined (1.2, 1.3) and the general results of 2.§2. Since also the relevant higher limits vanish [16, 2.4], DI(4) is uniquely N-determined by 2.48 and 2.51. Since $\mathrm{Out}_{\mathrm{tr}}(W(\mathrm{DI}(4)))$ is trivial and $Z(W(\mathrm{DI}(4)))=\langle -E\rangle$ has order two, Aut(DI(4)) can be read off from 2.17 and the short exact sequence (2.9). □

sec:f4

3. The 2-compact group F_4

BF_4 is a rank four 2-compact group containing a rank five elementary abelian 2-group $E_5\subset F_4$ such that [59, 2.1]

$$H^*(BF_4;\mathbf{F}_2)\cong H^*(BE_5;\mathbf{F}_2)^{\mathbf{A}(F_4)(E_5)}\cong\mathbf{F}_2[y_4,y_6,y_7,y_{16},y_{24}]$$

where the Quillen automorphism group is the parabolic subgroup

$$\mathbf{A}(F_4)(E_5)=\begin{pmatrix}\mathrm{GL}(2,\mathbf{F}_2)&*\\0&\mathrm{GL}(3,\mathbf{F}_2)\end{pmatrix}\subset\mathrm{GL}(5,\mathbf{F}_2)$$

of order $2^6|\mathrm{GL}(2,\mathbf{F}_2)||\mathrm{GL}(3,\mathbf{F}_2)|$. The inclusion functor $\mathbf{A}(\mathbf{A}(F_4)(E_5),E_5)\to\mathbf{A}(F_4)$ is a category equivalence by Lannes theory [32]. Inspection of the list of centralizers of elementary abelian 2-groups in F_4 [59, 3.2] shows that $ZC_{F_4}(V)=V$ for each nontrivial $V\subset E_5$ so that the functor $\pi_2BZC_{F_4}=0$ and $\pi_1BZC_{F_4}=H^0(\mathbf{A}(F_4)(E_5)(-);E_5)$. Thus $\pi_1BZC_{F_4}$ is an exact functor (2.69) and $\lim^0\pi_1BZC_{F_4}=H^0(\mathbf{A}(F_4)(E_5);E_5)=0$.

It is known that $H^0(W;\check{T})(F_4)=0$, $H^1(W;\check{T})(F_4)=0$, and $H^2(W;\check{T})(F_4)=\mathbf{Z}/2$ [24, 26].

PROOF OF THEOREM 1.6. Condition (1 & 2) is satisfied for F_4 because centralizers of rank one objects [59, 3.2]

$$\frac{SU(2)\times Sp(3)}{\mathbf{Z}/2}, \quad Spin(9)$$

and centralizers of rank two objects [59, 3.2],

$$\frac{U(1)\times U(3)}{\mathbf{Z}/2} \rtimes \mathbf{Z}/2, \quad \frac{Spin(4)\times Spin(5)}{\mathbf{Z}/2}, \quad Spin(8)$$

have uniquely N-determined centralizers. This follows from 2.§2 as the simple factors are uniquely N-determined (1.2, 1.3). All elementary abelian 2-groups in F_4 of order at most four are toral. Since also the relevant higher limits vanish, F_4 is uniquely N-determined by 2.48 and 2.51.

The automorphism group of the 2-compact group F_4 is (2.17) the middle term of the exact sequence (2.9). (All automorphisms of F_4 preserve the extension class $e(F_4)$ which is the nontrivial element of $H^2(W;\check{T}) = \mathbf{Z}/2$ [10, 34].) The group $\mathrm{Out}_{\mathrm{tr}}(W(F_4))$ of trace preserving outer automorphisms is cyclic of order two but the nontrivial outer automorphism of $W(F_4)$ can not be realized as conjugating with an element of $N_{\mathrm{GL}(L)}(W)$. The center of $W(F_4)$ is $C_2 = \langle -E\rangle$. We conclude that $\mathrm{Aut}(F_4) = \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times$ consists entirely of unstable Adams operations. □

CHAPTER 8

Proofs of the main theorems

cha:proofs

This chapter contains the proofs the main results stated in the Introduction.

1. Proof of Theorem 1.2

We show that $\mathrm{PGL}(n+1,\mathbf{C})$ is uniquely N-determined by induction over n. The induction step is provided by Lemma 8.1 and the start of the induction by Proposition 8.2.

vercond

8.1. LEMMA. *Suppose that* $\mathrm{PGL}(r+1,\mathbf{C})$ *is uniquely* N*-determined for all* $0 \leq r < n$. *Then* $\mathrm{PGL}(n+1,\mathbf{C})$, $n \geq 1$, *satisfies conditions 2.48.(1) (for* $\pi_*(N)$*-determined automorphisms), 2.51.(1), 2.51.(2), and 2.51.(3).*

PROOF. Condition (1) of 2.48 (for $\pi_*(N)$-determined automorphisms) is concerned with centralizers $C_{\mathrm{PGL}(n+1,\mathbf{C})}(L,\lambda)$ of rank one objects (3.5). The condition is satisfied for all connected rank one centralizers by the induction hypothesis and 2.42, 2.39. The condition is satisfied for the nonconnected rank one centralizer (when $n+1$ is even) by 2.35 since $H^1(C_2;\mathbf{Z}/2^\infty)=0$ for the nontrivial action of the cyclic group C_2 of order two on $\mathbf{Z}/2^\infty$.

We use 2.54 to verify conditions (1) and (2) of 2.51. Let (V,ν) be a toral elementary abelian 2-subgroup of $\mathrm{PGL}(n+1,\mathbf{C})$ of rank ≤ 2 and $C(\nu)=C_{\mathrm{PGL}(n+1,\mathbf{C})}(\nu)$ its centralizer. We have seen that $C(\nu)$ is LHS (§2) and that $\check{Z}(C(\nu)_0)=\check{Z}(N_0(C(\nu)))$ as $C(\nu)_0$ does not contain a direct factor isomorphic to $\mathrm{GL}(2,\mathbf{C})/\mathrm{GL}(1,\mathbf{C})=\mathrm{SO}(3)$ (2.32, (3.5)). The identity component $C(\nu)_0$ has $\pi_*(N)$-determined automorphisms according to 2.38 and 2.39, and $C(\nu)$ has N-determined automorphisms by 2.35. The identity component $C(\nu)_0$ is N-determined according to 2.42 and 2.43, and $C(\nu)$ is N-determined by 2.40. Thus $C(\nu)$ is LHS and totally N-determined.

The functor $H^1(W/W_0;\check{T}_0^W)$ is zero on $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\leq t}_{\leq 2}$ except on the object $(V,\nu)=(i_0,i_0,i_0,i_0)$, when $n+1=4i_0$, where it has value $\mathbf{Z}/2$. However, this object has Quillen automorphism group $\mathrm{GL}(V)$ and since the only $\mathrm{GL}(V)$-equivariant homomorphism $\mathrm{St}(V)=V\to\mathbf{Z}/2$ is the trivial homomorphism, $\lim^1(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\leq t}_{\leq 2};H^1(W/W_0;\check{T}_0^W))=0$ follows from Oliver's cochain complex [**53**].

We now turn to condition (3) of 2.51. When $n+1$ is odd there are no nontoral rank two objects (3.3) and so there is nothing to prove. When $n+1=2m$ is even, let (H,ν) be the unique nontoral rank two object of $\mathbf{A}(\mathrm{PGL}(2m,\mathbf{C}))$ (3.17.(1)). Let X' be a connected 2-compact group with maximal torus normalizer $j'\colon N(\mathrm{PGL}(2m,\mathbf{C})\to X'$. We must show that ν'_L and $f_{\nu,L}\colon C_{\mathrm{PGL}(2m,\mathbf{C})}(H,\nu)\to C_{X'}(H,\nu'_L)$ as defined in 2.51.(3) are independent of the choice of rank one subgroup $L\subset V$. When $m=1$, the claim follows from 2.59, 2.60, 2.62 (where $\overline{\nu}(V)$ and $\overline{\nu}'(V)$ are isomorphisms in this case) since $\mathrm{PGL}(2,\mathbf{C})$ does contain a unique rank one elementary abelian 2-group with nonconnected centralizer (3.9) and a unique nontoral rank two elementary abelian 2-group (3.17.(1)). When $m>1$, we use 2.63 which immediately yields that ν'_L is independent of the choice of $L<V$. There exists a torus $T_\nu\to C_N(V,\nu_L^N)$ as in 2.63.(2) because the three preferred lifts ν_L^N, $L<V$, differ by an automorphism of H (the Quillen automorphism group of (H,ν) is the full automorphism group $\mathrm{Aut}(H)$ of H (3.17.(1))). Since the identity component of $C_{\mathrm{PGL}(n+1,\mathbf{C})}(H,\nu)$ is uniquely N-determined by induction hypothesis, the restriction $(f_{\nu,L})_0$ of $f_{\nu,L}$ to the identity components is independent of the choice of L (2.15.(2)). Also $\pi_0(f_{\nu,L})$ is independent of the choice of $L<V$ by 2.62 (where $\pi_0(\overline{\nu}(V))$ and $\pi_0(\overline{\nu}'(V))$ are isomorphisms). But since $\mathrm{PGL}(m,\mathbf{C})$ is centerfree $f_{\nu,L}$ is in fact determined (use one half of [**42**, 5.2]) by $(f_{\nu,L})_0$ and $\pi_0(f_{\nu,L})$. We conclude that $f_{\nu,L}$ is independent of the choice of $L<V$.

□

pgl2C

8.2. PROPOSITION. *The* 2*-compact group* $\mathrm{PGL}(2,\mathbf{C})$ *is uniquely* N*-determined.*

PROOF. The centralizer cofunctor $C_{\mathrm{PGL}(2,\mathbf{C})}$ takes the Quillen category of $\mathrm{PGL}(2,\mathbf{C})$, consisting (3.9, 3.17) of one toral line, L, and one nontoral plane, H,

Qforpgl2C (8.3)
$$L \longrightarrow H \circlearrowleft \mathrm{GL}(H)$$

to the diagram

(8.4)
$$\mathrm{GL}(1,\mathbf{C})^2/\mathrm{GL}(1,\mathbf{C}) \rtimes C_2 \longleftarrow H \circlearrowleft \mathrm{GL}(H)^{\mathrm{op}}$$

of uniquely N-determined 2-compact groups. The 2-compact toral group to the left is uniquely N-determined because (2.41) $H^1(C_2;\mathbf{Z}/2^\infty) = 0$ for the nontrivial action of C_2 on $\mathbf{Z}/2^\infty$. The center cofunctor takes this diagram back to the starting point (8.3) for which the higher limits vanish (2.69). $\mathrm{PGL}(2,\mathbf{C})$ is thus uniquely N-determined by 2.48 and 2.51. □

PROOF OF THEOREM 1.2. The proof is by induction over $n \geq 1$. The start of the induction is provided by 8.2. The induction step is provided by 8.1 and 3.18 using 2.48 and 2.51.

According to 2.17, the automorphism group

$$\mathrm{Aut}(\mathrm{PGL}(n+1,\mathbf{C})) = W\backslash N_{\mathrm{GL}(L)}(W) = W\backslash\langle \mathbf{Z}_2^\times, W\rangle = Z(W)\backslash \mathbf{Z}_2^\times$$

is isomorphic to $\mathbf{Z}^\times\backslash\mathbf{Z}_2^\times$ for $n=1$ and to $\mathbf{Z}_2^\times$ for $n>1$. Here we use [39] or the exact sequence (2.9) where we note that $\mathrm{Out}_{\mathrm{tr}}(W)$, is trivial for all $n \geq 1$; $\mathrm{Out}(\Sigma_{n+1})$ is trivial for all $n \neq 5$ [28, II.5.5] and the nontrivial outer automorphism of Σ_6 does not preserve trace. □

PROOF OF COROLLARY 1.7. Let $X = \mathrm{GL}(n,\mathbf{C})$, $n \geq 1$, and write $\check{T}$, W and L for $\check{T}(X)$, $W(X)$, and $L(X)$. Since the adjoint form $PX = PGL(n,\mathbf{C})$ of X is uniquely N-determined (1.2), so is X (2.38, 2.42). The extension class $e(X) \in H^2(W;\check{T})$ (§2.4) is the zero class since the maximal torus normalizer $N(X) = \mathrm{GL}(1,\mathbf{C}) \wr \Sigma_n$ splits. Therefore, $\mathrm{Aut}(X)$ is isomorphic to $W\backslash N_{\mathrm{GL}(L)}(W)$ (2.17). Using the exact sequence (2.9), we conclude, as in the proof of Theorem 1.2, that $\mathrm{Aut}(X) \cong Z(W)\backslash \mathrm{Aut}_{\mathbf{Z}_2 W}(L) = Z(W)\backslash \mathrm{Aut}_{\mathbf{Z}_2\Sigma_n}(\mathbf{Z}_2^n)$. □

2. Proof of Theorem 1.3

The proof of Theorem 1.3 uses induction over n simultaneously applied to the three infinite families $\mathrm{PSL}(2n,\mathbf{R})$, $\mathrm{SL}(2n+1,\mathbf{R})$, and $\mathrm{PGL}(n,\mathbf{H})$.

PROOF OF THEOREM 1.3. The statement of the theorem means (2.11) that the 2-compact groups

- $\mathrm{PSL}(2n,\mathbf{R})$, $\mathrm{SL}(2n+1,\mathbf{R})$, and $\mathrm{PGL}(n,\mathbf{H})$ have $\pi_*(N)$-determined automorphisms,
- $\mathrm{PSL}(2n,\mathbf{R})$, $\mathrm{SL}(2n+1,\mathbf{R})$, and $\mathrm{PGL}(n,\mathbf{H})$ are N-determined.

We may inductively assume that the connected 2-compact groups $\mathrm{PSL}(2i,\mathbf{R})$, $1 \leq i \leq n-1$, $\mathrm{SL}(2i+1,\mathbf{R})$, $1 \leq i < n-1$, and $\mathrm{PGL}(i,\mathbf{H})$, $1 \leq i < n$, are uniquely N-determined. From Theorem 1.2 we know that $\mathrm{PGL}(i,\mathbf{C})$ is uniquely N-determined for all $i \geq 1$. The plan is now to use 2.48 and 2.51 inductively.

Consider first the connected, centerless 2-compact group $\mathrm{PSL}(2n,\mathbf{R})$.

$\mathrm{PSL}(2n,\mathbf{R})$ has N-determined automorphisms: According to 2.48 it suffices to show that

(1) $C_{\mathrm{PSL}(2n,\mathbf{R})}(L)$ has N-determined automorphism for any rank one elementary abelian 2-group $L \subset \mathrm{PSL}(2n,\mathbf{R})$.

(2) $\lim^1(\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R})), \pi_1 BZC_{\mathrm{PSL}(2n,\mathbf{R})}) = 0 = \lim^2(\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R})), \pi_2 BZC_{\mathrm{PSL}(2n,\mathbf{R})})$.

Item (2) is proved in 4.32. The centralizers that occur in item (1) are listed in (4.14) and (4.15). That the centralizers of (4.14) have N-determined automorphisms follows, under the induction hypothesis that the 2-compact groups $\mathrm{PSL}(2i,\mathbf{R})$, $1 \leq i \leq n-1$, have N-determined automorphisms, from general hereditary properties of N-determined 2-compact groups (§2). Note here that $\check{Z}(C_0) = \check{T}(C_0)^{W(C_0)}$ for $C = C_{\mathrm{PSL}(2n,\mathbf{R})}(L)$ by [35, 1.6]. Similarly, the centralizers of (4.15) have N-determined automorphisms because the 2-compact groups $\mathrm{PGL}(n,\mathbf{C})$, $1 \leq n < \infty$, have N-determined automorphisms (1.2).

$\mathrm{PSL}(2n,\mathbf{R})$ is N-determined: We verify the four conditions of 2.51. Let $V \subset \mathrm{PSL}(2n,\mathbf{R})$ be a toral elementary abelian 2-group of rank at most 2. The centralizer $C = C_{\mathrm{PSL}(2n,\mathbf{R})}(V)$ is one of the 2-compact groups listed in (4.14), (4.16), (4.15), or (4.17), so it is LHS (4.19). The identity component C_0 of C satisfies the equation $\check{Z}(C_0) = \check{T}(C_0)^{W(C_0)}$ [35, 1.6] and the adjoint form

$$PC_0 = \begin{cases} \mathrm{PSL}(2i_0, \mathbf{R}) \times \mathrm{PSL}(2i_1, \mathbf{R}) & i_0 + i_1 = n \\ \mathrm{PSL}(2i_0, \mathbf{R}) \times \mathrm{PSL}(2i_1, \mathbf{R}) \times \mathrm{PSL}(2i_2, \mathbf{R}) \times \mathrm{PSL}(2i_3, \mathbf{R}) & i_0 + i_1 + i_2 + i_3 = n \\ \mathrm{PGL}(n, \mathbf{C}) & \\ \mathrm{PGL}(i_0, \mathbf{C}) \times \mathrm{PGL}(i_1, \mathbf{C}) & i_0 + i_1 = n \end{cases}$$

in these four cases. The induction hypothesis and the general results of §2 imply that C_0 is uniquely N-determined and that C is totally N-determined. Since also the homomorphism $H^1(W; \check{T}) \twoheadrightarrow \lim^1(\mathbf{A}(\mathrm{PSL}(2n, \mathbf{R}))^{\leq t}_{\leq 2}; H^1(W_0; \check{T})^{W/W_0})$ is surjective (4.§3), we get from 2.54 that the first two conditions of 2.51 are satisfied. The third condition has been verified in 4.§4 and the fourth, and final, condition in 4.32.

$\mathrm{PSL}(2n, \mathbf{R})$ has $\pi_*(N)$-determined automorphisms: This means that the only automorphism of $\mathrm{PSL}(2n, \mathbf{R})$ that restricts to the identity on the maximal torus is the identity, ie that

$$H^1(W; \check{T})(\mathrm{PSL}(2n, \mathbf{R})) \cap \mathrm{AM}(\mathrm{Aut}(\mathrm{PSL}(2n, \mathbf{R}))) = \{0\}$$

where AM is the Adams–Mahmud homomorphism (§2.4). For $n > 4$, $H^1(W; \check{T})(\mathrm{PSL}(2n, \mathbf{R})) = 0$, and there is nothing to prove. Consider the case $n = 4$. Let f be an automorphism of $\mathrm{PSL}(8, \mathbf{R})$ such that $\mathrm{AM}(f) \in H^1(W; \check{T})$. Let $L \subset \mathrm{PSL}(8, \mathbf{R})$ be any rank one elementary abelian 2-group. Since f is the identity on the maximal torus, $f(L)$ is conjugate to L so that f restricts to an automorphism of $C_{\mathrm{PSL}(8,\mathbf{R})}(L)$ and to an automorphism of the identity component of $C_{\mathrm{PSL}(8,\mathbf{R})}(L)$. Since $C_{\mathrm{PSL}(8,\mathbf{R})}(L)_0$ has $\pi_*(N)$-determined automorphisms by 2.38 and 2.39, $f \in H^1(W; \check{T})(\mathrm{PSL}(2n, \mathbf{R}))$ restricts to 0 in $H^1(W; \check{T})(C_{\mathrm{PSL}(8,\mathbf{R})}(L)_0)$. However, the restriction map is injective (see the proof of 4.20) so that $f = 0$. This shows that $\mathrm{PSL}(8, \mathbf{R})$ has $\pi_*(N)$-determined automorphisms.

Consider next the 2-compact group $\mathrm{SL}(2m + 1, \mathbf{R})$ where $m = n - 1$.

$\mathrm{SL}(2m + 1, \mathbf{R})$ has N-determined automorphisms: We verify the conditions of 2.48. Let $L \subset \mathrm{SL}(2m + 1, \mathbf{R})$ be an elementary abelian 2-group of rank 1. The centralizer $C = C_{\mathrm{SL}(2m+1,\mathbf{R})}(L)$ is given in (5.3). According to §2, C has N-determined automorphisms. (Use the natural splitting of (5.11) in connection with 2.35.) See 5.8 for the vanishing of the higher limits.

$\mathrm{SL}(2m + 1, \mathbf{R})$ is N-determined: Conditions (1) and (2) of 2.51 are verified in 5.§2, condition (3) in 5.§3, and condition (4) in 5.8.

$\mathrm{SL}(2m + 1, \mathbf{R})$ has $\pi_*(N)$-determined automorphisms: To prove this, it suffices to find a rank one elementary abelian 2-group $L \subset \mathrm{SL}(2m + 1, \mathbf{R})$ such that $C_{\mathrm{SL}(2m+1,\mathbf{R})}(L)_0$ has $\pi_*(N)$-determined automorphisms and such that $C_{\mathrm{SL}(2m+1,\mathbf{R})}(L)_0 \to \mathrm{SL}(2m + 1, \mathbf{R})$ induces a monomorphism on $H^1(W; \check{T})$. Such a line is provided by $L = L[2m - 1, 2]$ with centralizer identity component $C_{\mathrm{SL}(2m+1,\mathbf{R})}(L)_0 = \mathrm{SL}(2m - 1, \mathbf{R}) \times \mathrm{SL}(2, \mathbf{R})$; see the proof of 5.15.

Consider finally the 2-compact group $\mathrm{PGL}(n, \mathbf{H})$ for $n \geq 3$.

$\mathrm{PGL}(n, \mathbf{H})$ has N-determined automorphisms: We verify the conditions of 2.48. Let $L \subset \mathrm{PGL}(n, \mathbf{H})$ be an elementary abelian 2-group of rank 1. The centralizer $C = C_{\mathrm{PGL}(n,\mathbf{H})}(L)$ is given in (6.2) and (6.3). According to the general results of §2, C has N-determined automorphisms and according to 6.19 the higher limits vanish.

$\mathrm{PGL}(n, \mathbf{H})$ is N-determined: Note that $\mathrm{PGL}(3, \mathbf{H})$ satisfies condition (1 & 2) of 7.§2 so that we may apply the same variant of 2.51 used for DI(4) (7.§2). When $n > 3$, conditions (1) and (2) of 2.51 follow if we can verify that the conditions of 2.54 are satisfied. That the centralizer $C_{\mathrm{PGL}(n,\mathbf{H})}(V)$ (6.2, 6.3, 6.4, 6.5), where V is an elementary abelian 2-group of rank at most two, satisfies the conditions of 2.54 is a consequence of the general results of §2 and 6.9, 6.11. See 6.12 and 6.14 for condition (3) and 6.19 for condition (4) of 2.51.

$\mathrm{PGL}(n, \mathbf{H})$ has $\pi_*(N)$-determined automorphisms: We only need to consider the cases $n = 3$ and $n = 4$ as $H^1(W; \check{T})(\mathrm{PGL}(n, \mathbf{H})) = 0$ for $n > 4$ [24]. In those two cases, it suffices, as above, to find a rank one elementary abelian 2-group $L \subset \mathrm{PGL}(n, \mathbf{H})$ such that $C_{\mathrm{PGL}(n,\mathbf{H})}(L)_0$ has $\pi_*(N)$-determined automorphisms and such that $C_{\mathrm{PGL}(n,\mathbf{H}))}(L)_0 \to \mathrm{PGL}(n, \mathbf{H})$ induces a monomorphism on $H^1(W; \check{T})$. Such a line is provided by $L = I$ for which $C_{\mathrm{PGL}(n,\mathbf{H})}(I)_0 = \mathrm{GL}(n, \mathbf{C})/\langle -E \rangle$ (6.3).

Since $\mathrm{PSL}(2n,\mathbf{R})$, $n \geq 4$, is uniquely N-determined and has a split maximal torus normalizer, its automorphism group is isomorphic to $W\backslash N_{\mathrm{GL}(L)}(W)$ by 2.17. When $n=4$, the group, $\mathrm{Out}_{\mathrm{tr}}(W)$, to the right in the exact sequence (2.9) is the permutation group Σ_3. There are Lie group outer automorphisms inducing Σ_3. When $n>4$,

$$\mathrm{Aut}(\mathrm{PSL}(2n,\mathbf{R})) \cong W\backslash N_{\mathrm{GL}(L)}(W) = W\backslash\langle \mathbf{Z}_2^*, W(\mathrm{PGL}(2n,\mathbf{R}))\rangle = W\backslash\left\langle \mathbf{Z}_2^\times, W, c_1\right\rangle$$
$$= (W\cap\left\langle \mathbf{Z}_2^\times, c_1\right\rangle)\backslash\left\langle \mathbf{Z}_2^\times, c_1\right\rangle = \begin{cases}\langle -c_1\rangle\backslash\left\langle \mathbf{Z}_2^\times, c_1\right\rangle = \mathbf{Z}_2^\times & n \text{ odd}\\ \langle -1\rangle\backslash\left\langle \mathbf{Z}_2^\times, c_1\right\rangle = \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times\times\langle c_1\rangle & n \text{ even}\end{cases}$$

Similarly,

$$\mathrm{Aut}(\mathrm{SL}(2n+1,\mathbf{R})) \cong W\backslash N_{\mathrm{GL}(L)}(W) = W\backslash\left\langle \mathbf{Z}_2^\times, W\right\rangle = (W\cap\mathbf{Z}_2^\times)\backslash\mathbf{Z}_2^\times = \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times$$

for $n\geq 2$ by 2.17.

The automorphism group $\mathrm{Aut}(\mathrm{PGL}(n,\mathbf{H}))$, $n\geq 3$, is (2.17) contained in $W\backslash N_{\mathrm{GL}(L)}(W)\cong \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times$. Since $H^2(W;\check{T})$ is an elementary abelian 2-group [**34**], it is isomorphic to the second cohomology group $H^2(W;t(\mathrm{PGL}(n,\mathbf{H})))$ with coefficient module $t(\mathrm{PGL}(n,\mathbf{H}))$, the maximal elementary abelian 2-group in the maximal torus. The unstable Adams operations with index in $\mathbf{Z}_2^\times$ act trivially here since they act as coefficient group automorphisms. Thus all elements of $W\backslash N_{\mathrm{GL}(L)}(W)$ preserve the extension class $e\in H^2(W;\check{T})$ and we conclude that $\mathrm{Aut}(\mathrm{PGL}(n,\mathbf{H}))\cong \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times$. □

Proof of Corollary 1.8. Note first that $\mathrm{GL}(n,\mathbf{R})$ is LHS for all $n\geq 1$. If n is odd, $\mathrm{GL}(n,\mathbf{R}) = \mathrm{SL}(n,\mathbf{R})\times\langle -E\rangle$ is LHS because its Weyl group is the direct product of the Weyl group of the identity component with the component group. If n is even, see 2.29.(5). According to 2.35 and 2.40, $\mathrm{GL}(n,\mathbf{R})$ is totally N-determined.

If n is odd, the identity component has trivial center, so that $\mathrm{Aut}(\mathrm{GL}(n,\mathbf{R})) = \mathrm{Aut}(\mathrm{SL}(n,\mathbf{R})) = \mathbf{Z}^\times\backslash\mathbf{Z}_2^\times$ by the short exact sequence [**42**, 5.2].

Suppose next that $n=2m$ is even. When $m=1$, $\mathrm{Aut}(\mathrm{GL}(2,\mathbf{R})) = \mathrm{Aut}(\mathbf{Z}/2,\mathbf{Z}/2^\infty,0) = \mathrm{Aut}(\mathbf{Z}/2^\infty) = \mathbf{Z}_2^\times$ according to (2.6). When $m>1$, $H^1(\pi;\check{Z}(\mathrm{SL}(2m,\mathbf{R}))) = H^1(\pi;\langle -E\rangle)$ is the order two subgroup $\langle\delta\rangle$ of $\mathrm{Aut}(\mathrm{GL}(2m,R))$ generated by the group isomorphism $\delta(A) = (\det A)A$, $A\in\mathrm{GL}(2m,\mathbf{R})$, and $H^1(W;\check{T}) = \mathrm{Hom}(W_{\mathrm{ab}},\langle -E\rangle) = \mathbf{Z}/2\times\mathbf{Z}/2$ (for $m>2$) [**24, 34**] is the middle group of an exact sequence

$$0\to H^1(\pi;\langle -E\rangle)\to H^1(W;\check{T})\to H^1(W_0;\check{T})\to 0$$

because $\mathrm{GL}(2m,\mathbf{R})$ is LHS. (For $m=2$, $H^1(W_0;\check{T})=0$ and $H^1(\pi;\check{Z}(\mathrm{SL}(2m,\mathbf{R}))) = \mathbf{Z}/2$, though.) In the exact sequence (2.6) for the automorphism group of $N = N(\mathrm{GL}(2m,\mathbf{R})) = N(\mathrm{SL}(2m+1,\mathbf{R}))$, the group on the right hand side is $\mathrm{Aut}(W,\check{T},0) = \left\langle W,\mathbf{Z}_2^\times\right\rangle$ as for $\mathrm{SL}(2m+1,\mathbf{R})$. Thus $\mathrm{Aut}(N)$ is generated by $H^1(W;\check{T})$, W, and $\mathbf{Z}_2^\times$ so that $\mathrm{Aut}(N,N_0) = \mathrm{Aut}(N)$ as W_0 is normal in W. Note that these three subgroups of $\mathrm{Aut}(N,N_0)$ commute because of the special form of the elements of $H^1(W;\check{T}) = \mathrm{Hom}(W_{\mathrm{ab}},\langle -E\rangle)$. Hence

$$\frac{\mathrm{Aut}(N,N_0)}{W_0} = \frac{\left\langle H^1(W;\check{T}),W,\mathbf{Z}_2^\times\right\rangle}{W_0} = \frac{\left\langle H^1(W;\check{T}),W_0,c_1,\mathbf{Z}_2^\times\right\rangle}{W_0} = \frac{\left\langle H^1(W;\check{T}),c_1,\mathbf{Z}_2^\times\right\rangle}{W_0\cap\left\langle H^1(W;\check{T}),c_1,\mathbf{Z}_2^\times\right\rangle}$$
$$= \begin{cases}\frac{\left\langle H^1(W;\check{T}),c_1,\mathbf{Z}_2^\times\right\rangle}{\langle -c_1\rangle} = H^1(W;\check{T})\times\mathbf{Z}_2^\times & m \text{ odd}\\ \frac{\left\langle H^1(W;\check{T}),c_1,\mathbf{Z}_2^\times\right\rangle}{\langle -1\rangle} = H^1(W;\check{T})\times\langle c_1\rangle\times\mathbf{Z}^\times\backslash\mathbf{Z}_2^\times & m \text{ even}\end{cases}$$

According to 2.18, the automorphism group $\mathrm{Aut}(\mathrm{GL}(2m,\mathbf{R}))$ is a subgroup of the above group and

$$\mathrm{Aut}(\mathrm{GL}(2m,\mathbf{R})) = \begin{cases}\langle\delta\rangle\times\mathbf{Z}_2^\times & m \text{ odd}\\ \langle\delta\rangle\times\langle c_1\rangle\times\mathbf{Z}^\times\backslash\mathbf{Z}_2^\times & m \text{ even}\end{cases}$$

for $m>1$. □

CHAPTER 9

Miscellaneous

sec:misc

This chapter contains standard facts used at various places in this paper.

sec:realreps

1. Real representation theory

Real representations are semi-simple and determined by their characters [29, 2.11, 3.12.(c)]. Any simple real representation arises from a simple complex representation in the following way: Let χ be the character of a simple complex representation of a finite group G. Then [29, 13.1, 13.11, 13.12]

$\chi \neq \overline{\chi}, \varepsilon_2(\chi) = 0$**:** $\psi = \chi + \overline{\chi}$ is the character of a simple $\mathbf{R}$-module of complex type.
$\chi = \overline{\chi}, \varepsilon_2(\chi) = +1$**:** χ is the character of a simple $\mathbf{R}$-module of real type.
$\chi = \overline{\chi}, \varepsilon_2(\chi) = -1$**:** $\psi = 2\chi$ is the character of simple $\mathbf{R}$-module of quaternion type.

where $\varepsilon_2(\chi) = \frac{1}{|G|}\sum_{g\in G}\chi(g^2)$.

9.1. Example. (1) The character table of the cyclic group C_4 of order 4

C_4	ε_2	1	-1	i	$-i$
χ_1	$+$	1	1	1	1
χ_2	$+$	1	1	-1	-1
χ_3	0	1	-1	i	$-i$
χ_4	0	1	-1	$-i$	i

shows that there are two linear real representations and one 2-dimensional simple real faithful representation of complex type with character $\psi = \chi_3 + \chi_4 = (2, -2, 0, 0)$.
(2) The character table of the dihedral group $D_8 = 2_+^{1+2}$

D_8	ε_2	1	-1	R_1	R_2	i
χ_1	$+$	1	1	1	1	1
χ_2	$+$	1	1	-1	1	-1
χ_3	$+$	1	1	1	-1	-1
χ_4	$+$	1	1	-1	-1	1
χ_5	$+$	2	-2	0	0	0

shows that there are four linear real representations and one 2-dimensional simple real faithful representation of real type with character $\chi_5 = (2, -2, 0, 0, 0)$.
(3) The character table of the quaternion group $Q_8 = 2_-^{1+2}$ (identical to the one for D_8 except for one value of ε_2)

Q_8	ε_2	1	-1	k	j	i
χ_1	$+$	1	1	1	1	1
χ_2	$+$	1	1	-1	1	-1
χ_3	$+$	1	1	1	-1	-1
χ_4	$+$	1	1	-1	-1	1
χ_5	$-$	2	-2	0	0	0

shows that there are four linear real representations and one 4-dimensional simple real faithful representation of quaternion type with character $\psi = 2\chi_5 = (4, -4, 0, 0, 0)$.

We are interested in real oriented representations, i.e. homomorphisms of finite groups into the special linear group $\mathrm{SL}(2n, \mathbf{R})$ (as opposed to homomorphisms into the general linear group $\mathrm{GL}(2n, \mathbf{R})$). The outer automorphism of $\mathrm{SL}(2n, \mathbf{R})$ is conjugation by any orientation reversing matrix such as $D = \mathrm{diag}(-1, 1, \ldots, 1)$.

lemma:intoSL

9.2. LEMMA. *Let* $V \subset \mathrm{PSL}(2n,\mathbf{R})$ *be an object of* $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))$ *and* $G = V^* \subset \mathrm{SL}(2n,\mathbf{R})$ *its inverse image in* $\mathrm{SL}(2n,\mathbf{R})$ *as in 4.8. Then*

$$V \text{ and } V^D \text{ are nonisomorphic objects of } \mathbf{A}(\mathrm{PSL}(2n,\mathbf{R})) \iff N_{\mathrm{GL}(2n,\mathbf{R})}(G) \subset \mathrm{SL}(2n,\mathbf{R})$$

PROOF. We note that

$$\begin{aligned} V,\, V^D \text{ are isomorphic objects of } \mathbf{A}(\mathrm{PSL}(2n,\mathbf{R})) &\iff G,\, G^D \text{ are conjugate subgroups of } \mathrm{SL}(2n,\mathbf{R}) \\ &\iff G \in G^{D\mathrm{SL}(2n,\mathbf{R})} \\ &\iff N_{\mathrm{GL}(2n,\mathbf{R})}(G) \cap D\mathrm{SL}(2n,\mathbf{R}) \neq \emptyset \\ &\iff N_{\mathrm{GL}(2n,\mathbf{R})}(G) \not\subset \mathrm{SL}(2n,\mathbf{R}) \end{aligned}$$

for any nontrivial elementary abelian 2-group $V \subset \mathrm{PSL}(2n,\mathbf{R})$. □

For instance, all representations of elementary abelian p-groups are conjugate in $\mathrm{SL}(2n,\mathbf{R})$ if and only if they are conjugate in $\mathrm{GL}(2n,\mathbf{R})$ - as in 4.12.

Let $\mathbf{A}(\mathrm{GL}(2n,\mathbf{R}))(G)$ be the subgroup of $\mathrm{Out}(G)$ consisting of all outer automorphisms of G induced by conjugation with some element of $\mathrm{GL}(2n,\mathbf{R})$ [47, 5.8] (ie $\mathbf{A}(\mathrm{GL}(2n,\mathbf{R}))(G)$ is the group $\mathrm{Out}_{\mathrm{tr}}(G)$ of all trace preserving outer automorphisms of G) and $\mathbf{A}(\mathrm{SL}(2n,\mathbf{R}))(G)$ the subgroup of $\mathrm{Out}(G)$ consisting of all outer automorphisms of G induced by conjugation with some element of $\mathrm{SL}(2n,\mathbf{R})$. Since

eq:NOut

$$N_{\mathrm{GL}(2n,\mathbf{R})}(G)/GC_{\mathrm{GL}(2n,\mathbf{R})}(G) \xrightarrow{\cong} \mathbf{A}(\mathrm{GL}(2n,\mathbf{R}))(G) \tag{9.3}$$

we conclude from 9.2 that

$$\begin{aligned} G,\, G^D \text{ are nonconjugate subgroups of } \mathrm{SL}(2n,\mathbf{R}) &\iff N_{\mathrm{GL}(2n,\mathbf{R})}(G) \subset \mathrm{SL}(2n,\mathbf{R}) \\ &\iff \begin{cases} C_{\mathrm{GL}(2n,\mathbf{R})}(G) \subset \mathrm{SL}(2n,\mathbf{R}) \\ \mathbf{A}(\mathrm{SL}(2n,\mathbf{R}))(G) = \mathbf{A}(\mathrm{GL}(2n,\mathbf{R}))(G) \end{cases} \end{aligned}$$

Let V and E be objects of $\mathbf{A}(\mathrm{PSL}(2n,\mathbf{R}))$ such that $\dim V + 1 = \dim E$. If there are morphisms $V \to E$ and $V^D \to E$, then (some representative of) $E = \langle V, V^D\rangle$ is generated by the images of (some representatives of) V and V^D so that $E = E^D$. Conversely, if $E = E^D$ and there is morphism $V \to E$ then there is also a morphism $V^D \to E^D = E$.

We have $2(\phi^D) = 2\phi \in \mathrm{Rep}(G, \mathrm{SL}(4n,\mathbf{R}))$ for any oriented real degree $2n$ representation $\phi \in \mathrm{Rep}(G,\mathrm{SL}(2n,\mathbf{R}))$ as the conjugating matrix $2D$ is orientation preserving.

exmp:IDQ

9.4. EXAMPLE. (1) Let $G \subset \mathrm{SL}(2d,\mathbf{R})$ be a finite group making $\mathbf{R}^{2d}$ a simple $\mathbf{R}G$-module of complex type. Consider the image of $G \subset \mathrm{SL}(2nd,\mathbf{R})$ of G under the n-fold diagonal $\mathrm{SL}(2d,\mathbf{R}) \xrightarrow{\Delta_n} \mathrm{SL}(2dn,\mathbf{R})$. The centralizer $C_{\mathrm{GL}(2nd,\mathbf{R})}(G) = \mathrm{GL}(n,\mathbf{C})$ is connected, hence contained in $\mathrm{SL}(2dn,\mathbf{R})$. Since $C_{\mathrm{GL}(2d,\mathbf{R})}(G) = \mathrm{GL}(1,\mathbf{C})$, the elements of G commute with $i \in \mathrm{GL}(1,\mathbf{C})$ and we may factor the inclusion of G into $\mathrm{SL}(2dn,\mathbf{R})$ as

$$G \to C_{\mathrm{GL}(2d,\mathbf{R})}(i) = \mathrm{GL}(d,\mathbf{C}) \xrightarrow{\Delta_n} \mathrm{GL}(dn,\mathbf{C}) \to \mathrm{SL}(2dn,\mathbf{R})$$

Let χ be the character for G in $\mathrm{GL}(d,\mathbf{C})$ so that the character for G in $\mathrm{GL}(2d,\mathbf{R})$ is $\chi+\overline{\chi}$. There are inclusions

$$\begin{array}{ccccc} \mathbf{A}(\mathrm{GL}(2dn,\mathbf{R}))(G) & = & \mathbf{A}(\mathrm{GL}(2d,\mathbf{R}))(G) & = & \mathrm{Out}_{\chi+\overline{\chi}}(G) \\ \uparrow & & \uparrow & & \uparrow \\ \mathbf{A}(\mathrm{SL}(2dn,\mathbf{R}))(G) & \hookleftarrow & \mathbf{A}(\mathrm{GL}(d,\mathbf{C}))(G) & = & \mathrm{Out}_{\chi}(G) \end{array}$$

where $\mathrm{Out}_{\phi}(G)$ is the group of all outer automorphisms that respect the function ϕ.

exmp:IDQ2

(2) Let $G \subset \mathrm{GL}(d,\mathbf{R})$ be a finite group making $\mathbf{R}^d$ a simple $\mathbf{R}G$-module of real type. Consider the image of $G \subset \mathrm{SL}(2nd,\mathbf{R})$ of G under the $2n$-fold diagonal $\mathrm{GL}(d,\mathbf{R}) \xrightarrow{\Delta_{2n}} \mathrm{SL}(2dn,\mathbf{R})$. The centralizer $C_{\mathrm{GL}(2nd,\mathbf{R})}(G) = \mathrm{GL}(2n,\mathbf{R})$ is contained in $\mathrm{SL}(2dn,\mathbf{R})$ when d is even. We my factor the inclusion of G into $\mathrm{SL}(2dn,\mathbf{R})$ as

$$G \to \mathrm{GL}(d,\mathbf{R}) \to \mathrm{GL}(d,\mathbf{C}) \xrightarrow{\Delta_n} \mathrm{GL}(nd,\mathbf{C}) \to \mathrm{SL}(2nd,\mathbf{R})$$

and as the trace functions for G in $\mathrm{GL}(d,\mathbf{C})$ and $\mathrm{GL}(2nd,\mathbf{R})$ are proportional $\mathbf{A}(\mathrm{GL}(2nd,\mathbf{R}))(G) = \mathbf{A}(\mathrm{GL}(d,\mathbf{C}))(G) \subset \mathbf{A}(\mathrm{SL}(2dn,\mathbf{R}))(G)$. Hence $G \neq G^D$ in $\mathrm{SL}(2nd,\mathbf{R})$.

exmp:IDQ3 (3) Let $G \subset \mathrm{SL}(4d,\mathbf{R})$ be a finite group making $\mathbf{R}^{4d}$ a simple $\mathbf{R}G$-module of quaternion type. Consider the image of $G \subset \mathrm{SL}(4nd,\mathbf{R})$ of G under the n-fold diagonal $\mathrm{SL}(4d,\mathbf{R}) \xrightarrow{\Delta_n} \mathrm{SL}(4dn,\mathbf{R})$. The centralizer $C_{\mathrm{GL}(4dn,\mathbf{R})}(G) = \mathrm{GL}(n,\mathbf{H})$ is connected so it is contained in $\mathrm{SL}(4dn,\mathbf{R})$. Since $C_{\mathrm{GL}(4d,\mathbf{R})}(G) = \mathrm{GL}(1,\mathbf{H}) \subset \mathrm{GL}(2,\mathbf{C})$ the elements of G commute with $i \in \mathrm{GL}(2,\mathbf{C})$ and we may factor the inclusion of G into $\mathrm{SL}(4dn,\mathbf{R})$ as

$$G \to C_{\mathrm{GL}(4d,\mathbf{R})}(i) = \mathrm{GL}(2d,\mathbf{C}) \xrightarrow{\Delta_n} \mathrm{GL}(2nd,\mathbf{C}) \to \mathrm{SL}(4nd,\mathbf{R})$$

and as the trace functions for G in $\mathrm{GL}(2d,\mathbf{C})$ and $\mathrm{GL}(4nd,\mathbf{R})$ are proportional $\mathbf{A}(\mathrm{GL}(4nd,\mathbf{R}))(G) = \mathbf{A}(\mathrm{GL}(2d,\mathbf{C}))(G) \subset \mathbf{A}(\mathrm{SL}(4dn,\mathbf{R}))(G)$. Hence $G \neq G^D$ in $\mathrm{SL}(4nd,\mathbf{R})$.

exmp:IDQ4 (4) $\mathbf{R}^2$ is a simple $\mathbf{R}C_4$-module of complex type with respect to the group

$$C_4 = \langle I \rangle \subset \mathrm{SL}(2,\mathbf{R}), \qquad I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Consider the image of C_4 in $\mathrm{SL}(2n,\mathbf{R})$ under the n-fold diagonal. The Quillen automorphism group $\mathbf{A}(\mathrm{GL}(2n,\mathbf{R}))(C_4) = \mathbf{A}(\mathrm{GL}(2,\mathbf{R}))(C_4) = \mathrm{Out}(C_4)$ since the trace lives on $\mho_1(C_4) = \langle -E \rangle$ only. However,

$$\mathbf{A}(\mathrm{SL}(2n,\mathbf{R}))(C_4) = \begin{cases} \mathrm{Out}(C_4) & n \text{ even} \\ \{1\} & n \text{ odd} \end{cases}$$

so that $C_4 \neq C_4^D \iff n$ even.

exmp:IDQ5 (5) $\mathbf{R}^4$ is a simple $\mathbf{R}G_{16}$-module of complex type with respect to the group

$$G_{16} = 4{\circ}2_{\pm}^{1+2} = \left\langle \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}, \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}, \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} \right\rangle \subset \mathrm{SL}(4,\mathbf{R}), \quad R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Consider the image of G_{16} in $\mathrm{SL}(4n,\mathbf{R})$ under the n-fold diagonal. The Quillen automorphism group $\mathbf{A}(\mathrm{GL}(4n,\mathbf{R}))(G_{16}) = \mathbf{A}(\mathrm{GL}(4,\mathbf{R}))(G_{16}) = \mathrm{Out}(G_{16}) \cong \mathrm{Out}(C_4) \times \mathrm{Sp}(2,\mathbf{F}_2)$ since the trace lives on the derived group $[G_{16},G_{16}] = \langle -E \rangle$ only. In fact, $\mathbf{A}(\mathrm{GL}(2,\mathbf{C}))(G_{16})$ is the factor $\mathrm{Sp}(2,\mathbf{F}_2)$ and since the generator of the factor $\mathrm{Out}(C_4)$ is induced from conjugation with the matrix $\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$ of $\mathrm{SL}(4,\mathbf{R})$, we see that also $\mathbf{A}(\mathrm{SL}(4,\mathbf{R}))(G_{16}) \subset \mathbf{A}(\mathrm{SL}(4n,\mathbf{R}))(G_{16})$ is the full outer automorphism group of G_{16}. Hence $G_{16} \neq G_{16}^D$ in $\mathrm{SL}(4n,\mathbf{R})$.

exmp:IDQ6 (6) $\mathbf{R}^2$ is a simple $\mathbf{R}G$-module of real type with respect to the group

$$G = 2_+^{1+2} = \langle R, T \rangle \subset \mathrm{GL}(2,\mathbf{R})$$

Consider the image of G in $\mathrm{SL}(4n,\mathbf{R})$ under the $2n$-fold diagonal map. Then $G \neq G^D$ in $\mathrm{SL}(4n,\mathbf{R})$.

exmp:IDQ7 (7) $\mathbf{R}^4$ is a simple $\mathbf{R}G$-module of quaternion type with respect to the group

$$G = 2_-^{1+2} = \left\langle \begin{pmatrix} 0 & -R \\ R & 0 \end{pmatrix}, \begin{pmatrix} 0 & -T \\ T & 0 \end{pmatrix} \right\rangle \subset \mathrm{SL}(4,\mathbf{R})$$

Consider the image of G in $\mathrm{SL}(4n,\mathbf{R})$ under the n-fold diagonal map. Then $G \neq G^D$ in $\mathrm{SL}(4n,\mathbf{R})$.

ec:xtraspecreps

9.5. Representations of (generalized) extraspecial 2-groups. The extraspecial 2-groups $G = 2_{\pm}^{1+2d}$ have [29, 7.5] 2^d linear characters (that vanish on $\mho_1(G) = G' = Z(G) = C_2$) and one simple complex character

$$\chi(g) = \begin{cases} 0 & g \notin Z(G) \\ 2^d\lambda(g) & g \in Z(G) \end{cases}$$

induced from the nontrivial linear character (group isomorphism) $\lambda\colon Z(G) \to \{\pm 1\}$.

If $G = 2_+^{1+2d}$ is of positive type, $\varepsilon_2(\chi) = +1$ and $\chi\alpha = \chi$ for all $\alpha \in \mathrm{Out}(G)$, isomorphic to $O^+(2d,2)$ [28, III.13.9.b]. This complex character is also the character of the unique simple real representation $G \to \mathrm{GL}(2^d,\mathbf{R})$ which is of real type; when d is even this representation actually takes values in $\mathrm{SL}(2^d,\mathbf{R})$ but when d is odd this representation is not oriented. The unique faithful real representation $G \to \mathrm{GL}(2 \cdot 2^d,\mathbf{R})$ with central $\mho_1$ has character 2χ and it splits into two distinct oriented real faithful representations $\psi, \psi^D\colon G \to \mathrm{SL}(2\cdot 2^d, \mathbf{R})$ invariant under the action of $\mathrm{Out}(G)$ (9.4.(2)).

If $G = 2_-^{1+2d}$ is of negative type, $\varepsilon_2(\chi) = -1$ and $\chi\alpha = \chi$ for all $\alpha \in \mathrm{Out}(G)$, isomorphic to $O^-(2d, 2)$ [**28**, III.13.9.b)]. The unique simple real representation $G \to \mathrm{GL}(2\cdot 2^d, \mathbf{R})$ with character 2χ is of quaternion type. It splits into two distinct oriented representations $\psi, \psi^D\colon G \to \mathrm{SL}(2\cdot 2^d, \mathbf{R})$ invariant under the action of $\mathrm{Out}(G)$ (9.4.(3)).

The generalized extraspecial 2-group $G = 4 \circ 2_\pm^{1+2d}$ has [**29**, 7.5] 2^{1+d} linear characters (that vanish on $\mho_1(G) = G' = C_2 \subsetneq Z(G) = C_4$) and two simple complex characters

$$\chi(g) = \begin{cases} 0 & g \notin Z(G) \\ 2^d\lambda(g) & g \in Z(G) \end{cases}$$

induced from the two faithful linear characters $\lambda\colon Z(G) \to \langle i \rangle = C_4$. These two degree 2^d simple characters, χ and $\overline{\chi}$, are interchanged by the action of $\mathrm{Out}(G) = \mathrm{Out}(C_4) \times \mathrm{Sp}(2d, 2)$ [**22**] (interchanged by the first factor $\mathrm{Out}(C_4)$ and preserved by the second factor $\mathrm{Sp}(2d, 2)$). The unique simple real representation $G \to \mathrm{GL}(2\cdot 2^d, \mathbf{R})$ has character $\chi + \overline{\chi}$ and is of complex type as $\varepsilon_2(\chi) = 0$. It splits up into two distinct oriented representations $\psi, \psi^D\colon G \to \mathrm{SL}(2\cdot 2^d, \mathbf{R})$ invariant under the action of $\mathrm{Out}(G)$ (9.4.(1)).

These irreducible faithful representations have easy explicit constructions that we now explain.

Let E be a nontrivial elementary abelian 2-group of rank $d \geq 1$ and $\mathbf{R}[E]$ its real group algebra. For $\zeta \in E^\vee = \mathrm{Hom}(E, \mathbf{R}^\times)$ and $u \in v$, let $R_\zeta, T_u \in \mathrm{GL}(\mathbf{R}[E])$ be the linear automorphisms given by $R_\zeta(v) = \zeta(v)v$ and $T_u(v) = u + v$ for all $v \in E$. The computation

$$R_\zeta T_u(v) = R_\zeta(u\cdot v) = \zeta(u)\zeta(v)(u\cdot v) = \zeta(u)T_u(\zeta(v)v) = \zeta(u)T_uR_\zeta(v)$$

shows that $R_\zeta T_u = \zeta(u)T_uR_\zeta$ or, equivalently, $[R_\zeta, T_u] = \zeta(u)$.

The group $2_+^{1+2d} = \langle R_\zeta, T_u \rangle \subset \mathrm{GL}(\mathbf{R}[E]) \subset \mathrm{GL}(\mathbf{C}[E]) \overset{\tau}{\subset} \mathrm{SL}(2^{d+1}, \mathbf{R})$ is extraspecial and the quadratic form on its abelianization 2^{2d} is given by

$$q(x_1, \ldots, x_d, y_1, \ldots, y_d) = x_1y_1 + \cdots + x_dy_d$$

because

$$(R_1^{x_1}\cdots R_d^{x_d}T_1^{y_1}\cdots T_d^{y_d})^2 = \prod_{i=1}^d (R_i^{x_i}T_i^{y_i})^2 = \prod_{i=1}^d (-E)^{x_iy_i}$$

where $T_1, \ldots, T_d$ correspond to a basis of E, $R_1, \ldots, R_d$ correspond to the dual basis, and $x_i, y_i \in \{0, 1\} = \mathbf{F}_2$. This is the unique faithful complex representation of degree 2^d. It is also the character of a simple real representation $G \to \mathrm{GL}(2^d, \mathbf{R})$, even $G \to \mathrm{SL}(2^d, \mathbf{R})$ when d is even, of real type.

The group $2_-^{1+2d} = \langle R_1, \ldots, R_{d-1}, iR_d, T_1, \ldots, T_{d-1}, iT_d \rangle \subset \mathrm{GL}(\mathbf{C}[E]) \overset{\tau}{\subset} \mathrm{SL}(2^{d+1}, \mathbf{R})$ is extraspecial and the quadratic form on its abelianization 2^{2d} is given by

$$q(x_1, \ldots, x_d, y_1, \ldots, y_d) = x_1y_1 + \cdots + x_{d-1}y_{d-1} + x_d^2 + x_dy_d + y_d^2$$

because

$$(R_1^{x_1}\cdots R_{d-1}^{x_{d-1}}(iR_d)^{x_d}T_1^{y_1}\cdots T_{d-1}^{y_{d-1}}(iT_d)^{y_d})^2 = (-E)^{x_d^2+y_d^2}(R_1^{x_1}\cdots R_d^{x_d}T_1^{y_1}\cdots T_d^{y_d})^2$$

where $x_i, y_i \in \{0, 1\} = \mathbf{F}_2$. This is the unique faithful complex representation of degree 2^d.

The group $4 \circ 2_\pm^{1+2d} = 4 \circ 2_+^{1+2d} = \langle i, R_\zeta, T_u \rangle = \langle 2_+^{1+2d}, 2_-^{1+2d} \rangle = 4 \circ 2_-^{1+2d} \subset \mathrm{GL}(\mathbf{C}[E]) \overset{\tau}{\subset} \mathrm{SL}(2^{d+1}, \mathbf{R})$ is generalized extraspecial with center $C_4 = \langle i \rangle$, derived group $[4 \circ 2_\pm^{1+2d}, 4 \circ 2_\pm^{1+2d}] = \mho_1(4 \circ 2_\pm^{1+2d}) = C_2 \subset C_4 = Z(4 \circ 2_\pm^{1+2d})$, and elementary abelian abelianization 2×2^{2d}. The quadratic form on its abelianization is given by

$$q(z, x_1, \ldots, x_d, y_1, \ldots, y_d) = z^2 + \sum_{i=1}^d x_iy_i$$

because

$$(i^zR_1^{x_1}\cdots R_d^{x_d}T_1^{y_1}\cdots T_d^{y_d})^2 = (-E)^{z^2}\prod_{i=1}^d (R_i^{x_i}T_i^{y_i})^2 = (-E)^{z^2}(R_1^{x_1}\cdots R_d^{x_d}T_1^{y_1}\cdots T_d^{y_d})^2$$

where $z, x_i, y_i \in \{0,1\}$. This representation and its conjugate are the two faithful complex representation of degree 2^d.

In the first two cases the associated symplectic inner product is

$$[(x_1,\dots,x_d,y_1,\dots,y_d),(x'_1,\dots,x'_d,y'_1,\dots,y'_d)] = \sum_{i=1}^{d}(x_iy'_i + x'_iy_i)$$

while it is

$$[(z,x_1,\dots,x_d,y_1,\dots,y_d),(z',x'_1,\dots,x'_d,y'_1,\dots,y'_d)] = \sum_{i=1}^{d}(x_iy'_i + x'_iy_i)$$

in the last case.

sec:tensor

9.6. Tensor products of real representations. Suppose that $\mathbf{R}^m$ is an $\mathbf{R}G$-module with trace χ and $\mathbf{R}^n$ an $\mathbf{R}H$ module with trace ρ. Consider $\mathbf{R}^{mn} = \mathbf{R}^m \otimes \mathbf{R}^n$ as an $\mathbf{R}(G\times H)$-module in the usual way where $(g,h)(u\otimes v) = gu \otimes hv$. The trace of this representation is $\chi\#\rho(g,h) = \chi(g)\rho(h)$ and the determinant is $\det(g,h) = (\det g)^n(\det h)^m$. This means that if

- m and n are both even, or if
- m is even and $\mathbf{R}^m$ an oriented G-representation

then $\mathbf{R}^{mn}$ is a real oriented $G\times H$-representation.

sec:glnCtosl2nR

9.7. Embedding $\mathrm{GL}(n,\mathbf{C})$ in $\mathrm{SL}(2n,\mathbf{R})$. Here are two embeddings $\tau\colon \mathrm{GL}(n,\mathbf{C}) \to \mathrm{GL}^+(2n,\mathbf{R})$ with the property that $\mathrm{tr}(\tau(A)) = \mathrm{tr}(A) + \overline{\mathrm{tr}(A)}$ for all $\mathbf{A} \in \mathrm{GL}(n,\mathbf{C})$.

If we write $\mathbf{C}^n = (\mathbf{R}+i\mathbf{R})^n$, then

$$\mathrm{GL}(n,\mathbf{C}) \ni A+iB \xrightarrow{\tau} \left(\begin{pmatrix} a_{ij} & -b_{ij} \\ b_{ij} & a_{ij}\end{pmatrix}\right)_{1\le i,j\le n} \in \mathrm{GL}^+(2n,\mathbf{R})$$

In particular, $i \in \mathrm{GL}(n,\mathbf{C})$ is sent to $\mathrm{diag}(I,\cdots,I) \in \mathrm{SL}(2n,\mathbf{R})$ and $C_{\mathrm{SL}(2n,\mathbf{R})}(i)$ consists of matrices with 2×2 blocks as above. For (2×2)-matrices this embedding has the form

$$\mathrm{GL}(2,\mathbf{C}) \ni \begin{pmatrix} a_1+ia_2 & b_1+ib_2 \\ c_1+ic_2 & d_1+id_2\end{pmatrix} \xrightarrow{\tau} \begin{pmatrix} \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1\end{pmatrix} & \begin{pmatrix} b_1 & -b_2 \\ b_2 & b_1\end{pmatrix} \\ \begin{pmatrix} c_1 & -c_2 \\ c_2 & c_1\end{pmatrix} & \begin{pmatrix} d_1 & -d_2 \\ d_2 & d_1\end{pmatrix}\end{pmatrix} \in \mathrm{SL}(4,\mathbf{R})$$

and with this convention the six subgroups of $\mathrm{SL}(4,\mathbf{R})$ isomorphic to D_8, Q_8, $G_{16} = 4\circ 2_{\pm}^{1+2}$ are

$$D_8 = \left\langle \begin{pmatrix} E & 0 \\ 0 & -E\end{pmatrix}, \begin{pmatrix} 0 & E \\ E & 0\end{pmatrix}\right\rangle, \qquad D_8^D = \left\langle \begin{pmatrix} E & 0 \\ 0 & -E\end{pmatrix}, \begin{pmatrix} 0 & -R \\ -R & 0\end{pmatrix}\right\rangle$$

$$Q_8 = \left\langle \begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix}, \begin{pmatrix} 0 & I \\ I & 0\end{pmatrix}\right\rangle, \qquad Q_8^D = \left\langle \begin{pmatrix} -I & 0 \\ 0 & -I\end{pmatrix}, \begin{pmatrix} 0 & T \\ -T & 0\end{pmatrix}\right\rangle$$

$$G_{16} = \left\langle \begin{pmatrix} E & 0 \\ 0 & -E\end{pmatrix}, \begin{pmatrix} 0 & E \\ E & 0\end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix}\right\rangle, \qquad G_{16}^D = \left\langle \begin{pmatrix} E & 0 \\ 0 & -E\end{pmatrix}, \begin{pmatrix} 0 & -R \\ -R & 0\end{pmatrix}, \begin{pmatrix} -I & 0 \\ 0 & I\end{pmatrix}\right\rangle$$

If we write $\mathbf{C}^n = \mathbf{R}^n + i\mathbf{R}^n$ then

$$\mathrm{GL}(n,\mathbf{C}) \ni A+iB \xrightarrow{\tau} \begin{pmatrix} A & -B \\ B & A\end{pmatrix} \in \mathrm{GL}^+(2n,\mathbf{R})$$

In particular, $i \in \mathrm{GL}(n,\mathbf{C})$ is sent to $\begin{pmatrix} 0 & -E \\ E & 0\end{pmatrix} \in \mathrm{SL}(2n,\mathbf{R})$ and $C_{\mathrm{SL}(2n,\mathbf{R})}(i)$ consists of block matrices of the form as above. For (2×2)-matrices this embedding has the form

$$\mathrm{GL}(2,\mathbf{C}) \ni \begin{pmatrix} a_1+ia_2 & b_1+ib_2 \\ c_1+ic_2 & d_1+id_2\end{pmatrix} \xrightarrow{\tau} \begin{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1\end{pmatrix} & -\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2\end{pmatrix} \\ \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2\end{pmatrix} & \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1\end{pmatrix}\end{pmatrix} \in \mathrm{SL}(4,\mathbf{R})$$

and with this convention the six subgroups of $\mathrm{SL}(4,\mathbf{R})$ isomorphic to D_8, Q_8, $G_{16} = 4\circ 2^{1+2}_{\pm}$ are

$$D_8 = \left\langle \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}, \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \right\rangle, \qquad D_8^D = \left\langle \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}, \begin{pmatrix} -T & 0 \\ 0 & T \end{pmatrix} \right\rangle$$

$$Q_8 = \left\langle \begin{pmatrix} 0 & -R \\ R & 0 \end{pmatrix}, \begin{pmatrix} 0 & -T \\ T & 0 \end{pmatrix} \right\rangle, \qquad Q_8^D = \left\langle \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}, \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} \right\rangle$$

$$G_{16} = \left\langle \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}, \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}, \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} \right\rangle, \qquad G_{16}^D = \left\langle \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}, \begin{pmatrix} -T & 0 \\ 0 & T \end{pmatrix}, \begin{pmatrix} 0 & R \\ -R & 0 \end{pmatrix} \right\rangle$$

sec:lietheory

2. Lie group theory

Some facts from Lie theory are collected here.

sec:centerings

9.8. Centerings. There are centerings [55]

$$L(\mathrm{pin}(2n)) \xrightarrow{P} L(\mathrm{GL}(2n,\mathbf{R})) \xrightarrow{Q} L(\mathrm{PGL}(2n,\mathbf{R}))$$

where

$$P(x_1,x_2,\ldots,x_n) = (2x_1-x_2-\cdots-x_n,x_2,\ldots,x_n), \quad Q(x_1,x_2,\ldots,x_n) = (2x_1,x_1-x_2,\ldots,x_1-x_n)$$

The expression for P is worked out in [7, pp. 174–175]. The expression for Q follows from the commutative diagram

$$\begin{array}{ccc} T(\mathrm{GL}(2n,\mathbf{R})) = \mathrm{U}(1)^n & \longrightarrow & \mathrm{U}(1)^n/\langle(-1,\ldots,-1)\rangle = T(\mathrm{PGL}(2n,\mathbf{R}) \\ & \searrow^{\varphi} & \downarrow\cong \\ & & \mathrm{U}(1)^n \end{array}$$

where $\varphi(z_1,z_2,\ldots,z_n) = (z_1^2, z_1z_2^{-1},\ldots,z_1z_n^{-1})$ is surjective with kernel $C_2 = \langle(-1,\ldots,-1)\rangle$. Since the action of the Weyl group $C_2\wr\Sigma_n$ is known in $L(\mathrm{GL}(2n,\mathbf{R}))$, the two other actions can be worked out as well. The action in $L(\mathrm{pin}(2n))$ is $P^{-1}(C_2\wr\Sigma_n)P$ and the action in $L(\mathrm{PGL}(2n,\mathbf{R}))$ is $Q(C_2\wr\Sigma_n)Q^{-1}$. Here,

$$P^{-1}(u_1,u_2,\ldots,u_n) = (\frac{1}{2}(u_1+\cdots+u_n),u_2,\ldots,u_n),$$
$$Q^{-1}(u_1,u_2,\ldots,u_n) = (\frac{1}{2}u_1,\frac{1}{2}u_1-u_2,\ldots,\frac{1}{2}u_1-u_n)$$

are the inverses.

sec:semicent

9.9. Centralizers in semi-direct products. Let $G\rtimes W$ be the semi-direct product for a group action of W on G. The following lemma is elementary.

lemma:centrgw

9.10. LEMMA. *For any $g\in G$ and $w\in W$,*

$$C_{G\rtimes W}(g,w) = \{(h,v)\mid \exists w\in C_W(w)\colon g(wh)=h(vg)\},$$
$$C_{G\rtimes W}(g) = \{(h,v)\in G\rtimes W\mid vg=g^h\}, \qquad C_{G\rtimes W}(w) = G^w\rtimes C_W(w)$$

where G^w is the fixed point group for the action of w on G. If G is abelian then

$$C_{G\rtimes W}(g) = G\rtimes W(g)$$

where $W(g)=\{w\in W\mid wg=g\}$ is is the isotropy subgroup at g.

Let $\mu\colon V\to\check{T}\rtimes W$ be a group homomorphism of an elementary abelian 2-group V into the semi-direct product of a discrete 2-compact torus $\check{T}$ and a group W. Write $\mu = (\check{T}(\mu),W(\mu))$ for the two coordinates of μ. Then $W(\mu)\colon V\to W$ is a group homomorphism and $\check{T}(\mu)\colon V\to\check{T}$ a crossed homomorphism into the V-module $V\xrightarrow{W(\mu)} W\to\mathrm{Aut}(\check{T})$. Let $H^1(V;\check{T})$ be the first cohomology group for this V-module and $[\check{T}(\mu)]\in H^1(V;\check{T})$ the cohomology class represented by the crossed homomorphism $\check{T}(\mu)$.

9.11. LEMMA. *There is a short exact sequence*

$$0 \to H^0(V;\check{T}) \to C_{\check{T}\rtimes W}(\mu) \to C_W(W(\mu))_{[\check{T}(\mu)]} \to 1$$

where $C_W(W(\mu))_{[\check{T}(\mu)]}$ *is the isotopy subgroup at* $[\check{T}(\mu)]$ *for the action of* $C_W(W(\mu))$ *on* $H^1(V;\check{T})$*.*

PROOF. We first determine the kernel of the homomorphism $C_{\check{T}\rtimes W}(\mu) \to C_W(W(\mu))$. Let $t \in \check{T}$. Then

$$\begin{aligned}(t,1) \text{ commutes with } V &\iff \forall v \in V\colon (t,1)(\check{T}(\mu)(v), W(\mu)(v)) = (\check{T}(\mu)(v), W(\mu)(v))(t,1)\\ &\iff \forall v \in V\colon t + \check{T}(\mu)(v) = \check{T}(\mu)(v) + W(\mu)(v)(t)\\ &\iff \forall v \in V\colon W(\mu)(v)t = t\\ &\iff t \in H^0(V;\check{T})\end{aligned}$$

More generally, for any element $(t,w) \in \check{T} \rtimes W$ we have

$$\begin{aligned}(t,w) \text{ commutes with } V &\iff \forall v \in V\colon (t,w)(\check{T}(\mu)(v), W(\mu)(v)) = (\check{T}(\mu)(v), W(\mu)(v))(t,w)\\ &\iff \forall v \in V\colon s + w\check{T}(\mu)(v) = \check{T}(\mu)(v) + W(\mu)(v)(t), wW(\mu)(v) = W(\mu)(v)w\\ &\iff w \in C_W(W(\mu)) \text{ and } \forall v \in V\colon (1-w)\check{T}(\mu)(v) = (1-W(\mu)(v))t\\ &\iff w \in C_W(W(\mu)) \text{ and } \forall v \in V\colon w\check{T}(\mu)(v) = \check{T}(\mu)(v) - (1-W(\mu)(v))t\end{aligned}$$

It follows that for $w \in C_W(W(\mu))$ we have

$$\begin{aligned}w \in \operatorname{im}\big(C_{\check{T}\rtimes W}(\mu) \to C_W(W(\mu))\big) &\iff \exists t \in \check{T}\colon (t,w) \in C_{\check{T}\rtimes W}(\mu)\\ &\iff w[\check{T}(\mu)] = [\check{T}(\mu)]\end{aligned}$$

i.e. that w fixes the crossed homomorphism $\check{T}(\mu)$ up to a principal crossed homomorphism. □

sec:centsemi

9.12. Centers of semi-direct products. Let $G \rtimes \Sigma$ be the semi-direct product for the action $\Sigma \to \mathrm{Aut}(G)$ of the group Σ on the group G. Let $G^\Sigma = \{g \in G | \Sigma g = g\}$ and $\Sigma_G = \{\sigma \in \Sigma | \sigma(g) = g \text{ for all } g \in G\}$.

semicenter

9.13. LEMMA. *The center* $Z(G \rtimes \Sigma) = G^\Sigma \times_{\mathrm{Aut}(G)} Z(\Sigma)$ *of* $G \rtimes \Sigma$ *is the pull-back*

$$\begin{array}{ccc} Z(G\rtimes\Sigma) & \longrightarrow & Z(\Sigma)\\ \downarrow & & \downarrow\\ G^\Sigma & \longrightarrow & \mathrm{Aut}(G)\end{array}$$

of the action map restricted to the center of Σ *along the map* $G^\Sigma \to \mathrm{Aut}(G)$ *given by inner automorphisms.*

PROOF. Suppose that $(g,\sigma) \in G \times \Sigma$ is in the center of $G \rtimes \Sigma$. Since

$$(g,\sigma)\cdot(1,\tau) = (g,\sigma\tau) = (1,\tau)\cdot(g,\sigma) = (\tau(g),\tau\sigma)$$

for all $\tau \in \Sigma$, g is fixed by Σ and σ is central in Σ. Moreover, from

$$(g,\sigma)\cdot(h,1) = (g\cdot\sigma(h),\sigma) = (h,1)\cdot(g,\sigma) = (hg,\sigma)$$

we see that $\sigma(h) = h^g$ for all $h \in G$. □

cor:cent

9.14. COROLLARY. *If the center of* Σ *acts faithfully on* G *through automorphisms that are not inner, then* $Z(G\rtimes\Sigma) = Z(G)^\Sigma$*. If* G *is abelian, then* $Z(G\rtimes\Sigma) = G^\Sigma \times Z(\Sigma)_G$ *is a direct product.*

PROOF. In the first case, the vertical map $Z(\Sigma) \to \mathrm{Aut}(G)$ is injective and its image intersects trivially with the image of the horizontal map $G^\Sigma \to \mathrm{Aut}(G)$. So the pull-back is $G^\Sigma \cap Z(G) = Z(G)^\Sigma$. In the second case, the bottom horizontal homomorphism $G^\Sigma \to \mathrm{Aut}(G)$ is trivial. □

cycliccenter

9.15. COROLLARY. *Let* G *be a group and* $Z \neq G$ *a central subgroup. Let the cyclic group* C_p *of prime order* p *act on* G^p/Z *by cyclic permutation. Then*

$$Z(G)/Z \times \{z \in Z | z^p = 1\} \cong Z(G^p/Z \rtimes C_p)$$

via the isomorphism that takes the element $z \in Z$ *of order* p *to* $(1,z,\dots,z^{p-1})Z \in G^p/Z$ *and is the diagonal on* $Z(G)/Z$*.*

PROOF. Observe that

$$G/Z \times \{z \in Z | z^p = 1\} \xrightarrow{\cong} \left(G^p/Z\right)^{C_p}$$

via the isomorphism that takes (gZ, z) to $g(1, z, \dots, z^{p-1})Z$. To see this, consider an element $(g_1, \dots, g_p)Z$ which is fixed by C_p. Then $(g_1, g_2, \dots, g_p)Z = (g_p, g_1, \dots, g_{p-1})Z$ so there exists an element $z \in Z$ so that $g_2 = g_1 z, g_3 = g_2 z = g_1 z^2, \dots, g_p = g_1 z^{p-1}, g_1 = g_1 z^p$. Therefore, $z^p = 1$ and $(g_1, g_2, \dots, g_p) = g_1(1, z, \dots, z^{p-1})$.

Thus $Z(G^p/Z \rtimes C_p)$ is the pull back of the group homomorphisms

$$G/Z \times \{z \in Z | z^p = 1\} \xrightarrow{\varphi} \mathrm{Aut}(G^p/Z) \leftarrow C_p$$

where $\varphi(gZ, z)((g_1, \dots, g_p)Z) = (g_1^g, \dots, g_p^g)Z$. Let $((gZ, z), \sigma)$ be an element of the pull back. Assume that σ is non-trivial. Since p is a prime number, σ has no fixed points. The equation

$$\forall g_1, \dots, g_p \in G \colon (g_1^g, \dots, g_p^g)Z = (g_{\sigma(1)}, \dots, g_{\sigma(p)})Z$$

shows that $g_1^g Z = g_{\sigma(1)} Z$. This is impossible unless σ is the identity since otherwise we can find a $g_1 \in Z$ and a $g_{\sigma(1)} \notin Z$. Thus the permutation σ must be the identity. The requirement for $((gZ, z), 1)$ to be in the pull back is that

$$\forall (g_1, \dots, g_p) \in G^p \exists u \in Z \colon (g_1^g, g_2^g, \dots, g_p^g) = (g_1 u, g_2 u, \dots, g_p u)$$

which implies that $[g_1, g] = u = [g_2, g]$ for all $g_1, g_2 \in G$. If we take $g_1 = 1$ to be the identity, we see that g must be central. □

sec:centLie

9.16. Centers of Lie groups and p-compact groups. Let Y be a compact connected Lie group and ZY its center. Let BY denote the p-completed classifying space of Y, ie the p-compact group associated to Y. Lie group multiplication $ZY \times Y \to Y$ induces a homotopy equivalence $BZY \to \mathrm{map}(BY, BY)_{B1}$ [18, 1.4] of the p-completed classifying space BZY to the the mapping space component containing the identity map. We need a version that holds for nonconnected Lie groups as well.

Let G be a possible nonconnected Lie group and ZG its center. Let BZG and BG denote the $\mathbf{F}_p$-localized classifying spaces of ZG and G, respectively. The space $\mathrm{map}(BG, BG)_{B1}$ is the center of the p-compact group BG [18, 1.3].

lemma:ZG

9.17. LEMMA. *The map*

$$BZG \to \mathrm{map}(BG, BG)_{B1}$$

induced by Lie group multiplication $ZG \times G \to G$, is a weak homotopy equivalence.

PROOF. Let Y be the identity component of G and $\pi = G/Y$ the group of components. Note that the group π acts on the center ZY of Y and that there is an exact sequence of abelian groups

$$1 \to H^0(\pi; ZY) \to ZG \to Z\pi \to H^1(\pi; ZY)$$

relating the centers ZY, ZG, and $Z\pi$, of Y, G, and π. The abelian Lie group ZG, a product of a torus and a finite abelian group, is described by the data

$$\pi_1(ZG) \otimes \mathbf{Q} \cong H^0(\pi; \pi_1(ZY) \otimes \mathbf{Q})$$

$$1 \to H^0(\pi; \pi_0(ZY)) \times H^1(\pi; \pi_1(ZY)) \to \pi_0(ZG) \to Z\pi \to H^1(\pi; \pi_0(ZY)) \times H^2(\pi; \pi_1(ZY))$$

where the second line is an exact sequence.

Similarly, there is a fibration of mapping spaces

$$\mathrm{map}(BY, BY)^{h\pi} \to \mathrm{map}(BG, BG) \to \mathrm{map}(BG, B\pi)$$

where the fibre over $BG \xrightarrow{B\pi_0} B\pi$ is the space $\mathrm{map}(BY, BY)^{h\pi}$ of self-maps of BG over $B\pi$. If we restrict to a single component of the total space, we obtain a fibration

$$\mathrm{map}(BG, BG)_{B1} \to \mathrm{map}(BG, B\pi)_{B\pi_0} \simeq \mathrm{map}(B\pi, B\pi)_{B1}$$

between path-connected spaces. The base space is $BZ\pi$. The fibre consists of some path-components of $\mathrm{map}(BY, BY)^{h\pi}_{B1} = (BZY)^{h\pi}$, the space of self-maps of BG over $B\pi$ with restriction to BY homotopic to the identity map. We have

$$\pi_i((BZY)^{h\pi}) = H^{1-i}(\pi; \pi_0(ZY)) \times H^{2-i}(\pi; \pi_1(ZY))$$

because $BZY = K(\pi_0 Y, 1) \times K(\pi_1 Y, 2)$ is a product of Eilenberg–MacLane spaces [37, 3.1] [18, 1.1]. It follows that $\mathrm{map}(BG, BG)_{B1}$ is an abelian [17, 3.5, 8.6] p-compact toral group described by the data

$$\pi_2((BZY)^{h\pi}) \otimes \mathbf{Q} \cong \pi_2(\mathrm{map}(BG, BG)_{B1}) \otimes \mathbf{Q}$$
$$1 \to \pi_1((BZY)^{h\pi}) \to \pi_1(\mathrm{map}(BG, BG)_{B1}) \to Z\pi \to \pi_0((BZY)^{h\pi})$$

where the second line is an exact sequence is an exact sequence.

Finally, the left commutative diagram of Lie groups

$$\begin{array}{ccccccc}
(ZY)^\pi \times G & \longrightarrow & G & & B(ZY)^\pi & \longrightarrow & (BZY)^{h\pi} \\
\downarrow & & \| & & \downarrow & & \downarrow \\
ZG \times G & \longrightarrow & G & \rightsquigarrow & BZG & \longrightarrow & \mathrm{map}(BG, BG)_{B1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Z\pi \times \pi & \longrightarrow & \pi & & BZ\pi & \longrightarrow & \mathrm{map}(B\pi, B\pi)_{B1}
\end{array}$$

induces the right commutative diagram of mapping space. Compairing the homotopy groups, we see that $BZ(G) \to \mathrm{map}(BG, BG)_{B1}$ is weak homotopy equivalence. □

lemma:ZGLprod

9.18. Lemma. *We have*

$$Z\left(\frac{\mathrm{GL}(i_0, \mathbf{R}) \times \cdots \mathrm{GL}(i_t, \mathbf{R})}{\langle -E \rangle}\right) = \frac{\langle -E \rangle \times \cdots \times \langle -E \rangle}{\langle -E \rangle} \cong C_2^t$$
$$Z\left(\frac{\mathrm{SL}(n, \mathbf{R}) \cap (\mathrm{GL}(i_0, \mathbf{R}) \times \cdots \mathrm{GL}(i_t, \mathbf{R}))}{\langle -E \rangle}\right) = \frac{\mathrm{SL}(n, \mathbf{R}) \cap (\langle -E \rangle \times \cdots \times \langle -E \rangle)}{\langle -E \rangle}$$
$$Z\left(\frac{\mathrm{GL}(i_0, \mathbf{H}) \times \cdots \mathrm{GL}(i_t, \mathbf{H})}{\langle -E \rangle}\right) = \frac{\langle -E \rangle \times \cdots \times \langle -E \rangle}{\langle -E \rangle} \cong C_2^t$$

for all natural numbers $i_0, \ldots, i_t > 0$ with sum n (which, in the second line, is even).

Proof. (For the case where the field is $\mathbf{R}$.) Put $G = \mathrm{GL}(i_0, \mathbf{R}) \times \cdots \mathrm{GL}(i_t, \mathbf{R})$. There is [47, 5.11] a short exact sequence

$$1 \to Z(G)/\langle -E \rangle \to Z(G/\langle -E \rangle) \to \mathrm{Hom}(G, \langle -E \rangle)_{\mathrm{id}} \to 1$$

where the group to the right consists of all homomorphisms $\phi \colon G \to \langle -E \rangle$ such that the map $g \to \phi(g)g$ is conjugate to the identity of G. Let $B \in \mathrm{GL}(i_j, \mathbf{R})$ be any matrix of positive trace. Then $\phi(E, \ldots, E, B, E, \ldots, E) = E$ since the map $g \to \phi(g)g$ preserves trace. It follows that $\phi(g) = E$ for all $g \in G$ since ϕ is constant on the 2^t components of G. Thus the group to the right in the above short exact sequence, $\mathrm{Hom}(G, \langle -E \rangle)_{\mathrm{id}}$, is trivial.

Since

$$Z\big(\mathrm{SL}(n, \mathbf{R}) \cap \prod \mathrm{GL}(i_j, \mathbf{R})\big) \subset C_{\prod \mathrm{GL}(i_j, \mathbf{R})}(\prod \mathrm{SL}(i_j, \mathbf{R})) = \prod Z\mathrm{GL}(i_j, \mathbf{R})$$

we see that $Z\big(\mathrm{SL}(n, \mathbf{R}) \cap \prod \mathrm{GL}(i_j, \mathbf{R})\big) = \mathrm{SL}(n, \mathbf{R}) \cap \prod Z\mathrm{GL}(i_j, \mathbf{R})$. Suppose that the homomorphism $\phi \colon \mathrm{SL}(n, \mathbf{R}) \cap \prod \mathrm{GL}(i_j, \mathbf{R}) \to \langle -E \rangle$ is such that the map $g \to \phi(g)g$ is conjugate to the identity. Let $B_1 \in \mathrm{GL}(i_{j_1}, \mathbf{R})$ and $B_2 \in \mathrm{GL}(i_{j_2}), \mathbf{R})$ be any pair of matrices such that $\mathrm{tr}(B_1) + \mathrm{tr}(B_2) > 0$. Then $\phi(E, \ldots, B_{i_1}, \ldots, B_{i_2}, \ldots, E) = E$ by trace considerations. The short exact sequence from [47, 5.11], similar to (9.16), now yields the formula of the second line.

The formula of the third line has a similar proof. □

It is not true in general that $Z(G)/Z$ is the center of the quotient G/Z of the Lie group G by the central subgroup Z.

sec:centralquot

9.19. Centralizers in quotients. Let G be a Lie group and $Z \subset G$ a central subgroup. Write $N(G)$ for the normalizer of the maximal torus, $T(G)$, and $W = W(G) = N(G)/T(G)$ for the Weyl group. Suppose that $V \subset T(G)/Z$ is a toral subgroup of the quotient Lie group G/Z and let $V^* \subset T(G) \subset G$ be the preimage of V in G.

There is an exact sequence

$$1 \to W(V^*) \to W(V) \to \mathrm{Hom}(V^*, Z)$$

relating the point-wise stabilizer subgroups for the action of the Weyl group W on V^* and V. The image of homomorphism to the right consists of all $\zeta \in \mathrm{Hom}(V^*, Z)$ for which the automorphism of V^* given by $v^* \to \zeta(v^*)v^*$, $v^* \in V^*$, is of the form $v^* \to wv^*$ for some Weyl group element $w \in W$.

Similarly, there is an exact sequence [47, 5.11]

$$1 \to C_G(V^*)/Z \to C_{G/Z}(V) \to \mathrm{Hom}(V^*, Z)$$

relating the centraizers of $V^* \subset G$ and $G \subset G/Z$. The image of homomorphism to the right consists of all $\zeta \in \mathrm{Hom}(V^*, Z)$ for which the automorphism of V^* given by $v^* \to \zeta(v^*)v^*$, $v^* \in V^*$, is of the form $v^* \to g^{-1}v^*g$ for some $g \in G$.

lemma:WVWVast

9.20. LEMMA. $W(V)/W(V^*) = C_{G/Z}(V)/C_G(V)$.

PROOF. Any automorphism of the toral subgroup V^* that is induced by conjugation with an element of G is in fact induced by conjugation with an element of $N(G)$ [7, IV.2.5] and hence agrees with the action of a Weyl group element. □

sec:actLie

9.21. Action on centralizers in Lie case. Let $\nu\colon V \to G$ be a monomorphism of a non-trivial elementary abelian p-group to a compact Lie group G. There is a canonical map $\mathrm{B}C_G(\nu(V)) \to \mathrm{map}(\mathrm{B}V, \mathrm{B}G)_{\mathrm{B}\nu}$ from the classifying space of the Lie theoretic centralizer of $\nu(V)$ to the mapping space component containing $\mathrm{B}\nu$. Write c_g for conjugation with $g \in G$.

9.22. LEMMA. *Suppose that $\nu\alpha = c_g\nu$ for some element $g \in G$ and some automorphism $\alpha \in \mathrm{GL}(V)$. Then conjugation by g takes $C_G(\nu(V))$ to $C_G(c_g\nu(V)) = C_G(\nu\alpha(V)) = C_G(\nu(V))$ and the diagram*

$$\begin{array}{ccc}
\mathrm{B}C_G(\nu(V)) & \longrightarrow & \mathrm{map}(\mathrm{B}V, \mathrm{B}G)_{\mathrm{B}\nu} \\
{\scriptstyle \mathrm{B}c_g}\uparrow\cong & & \cong\downarrow{\scriptstyle (\mathrm{B}\alpha)^*} \\
\mathrm{B}C_G(\nu(V)) & \longrightarrow & \mathrm{map}(\mathrm{B}V, \mathrm{B}G)_{\mathrm{B}\nu}
\end{array}$$

is homotopy commutative.

PROOF. The commutative diagram of Lie group morphisms

$$\begin{array}{ccccc}
V \times C_G(\nu(V)) & \xrightarrow{\nu\times 1} & \nu(V) \times C_G(\nu(V)) & \xrightarrow{\mathrm{mult}} & G \\
{\scriptstyle \alpha\times c_g}\downarrow & & & & \| \\
V \times C_G(\nu(V)) & \xrightarrow[\nu\times 1]{} & \nu(V) \times C_G(\nu(V)) & \xrightarrow[\mathrm{mult}]{} & G
\end{array}$$

induces a commutative diagram

$$\begin{array}{ccc}
\mathrm{B}V \times \mathrm{B}C_G(\nu(V)) & \xrightarrow{\mathrm{B}(\mathrm{mult}\circ(\nu\times 1))} & \mathrm{B}G \\
{\scriptstyle \mathrm{B}\alpha\times\mathrm{B}c_g}\downarrow & & \| \\
\mathrm{B}V \times \mathrm{B}C_G(\nu(V)) & \xrightarrow{\mathrm{B}(\mathrm{mult}\circ(\nu\times 1))} & \mathrm{B}G
\end{array}$$

of classifying spaces. Taking adjoints, we obtain the homotopy commutative diagram

$$\begin{array}{ccc}
\mathrm{B}C_G(\nu(V)) & \longrightarrow & \mathrm{map}(\mathrm{B}V, \mathrm{B}G)_{\mathrm{B}\nu} \\
{\scriptstyle \mathrm{B}c_g}\uparrow & & \downarrow{\scriptstyle (\mathrm{B}\alpha)^*} \\
\mathrm{B}C_G(\nu(V)) & \longrightarrow & \mathrm{map}(\mathrm{B}V, \mathrm{B}G)_{\mathrm{B}\nu}
\end{array}$$

as claimed. □

9.23. Corollary. *Suppose that* $\mu\colon V \to N(G)$ *is a monomorphism and that* $\mu\alpha = c_n\mu$ *for some* $\alpha \in \mathrm{GL}(V)$ *and* $n \in N(G)$*. Then*

$$w^{-1} = \pi_2((\mathrm{B}\alpha)^*)\colon \pi_2(\mathrm{B}T(G))^{\pi_0(\mu)(V)} \to \pi_2(\mathrm{B}T(G))^{\pi_0(\mu)(V)}$$

where $w \in W(G)$ *is the image of* $n \in N(G)$*.*

Proof. There is a commutative diagram

$$\begin{array}{ccccccc}
\pi_2(\mathrm{B}T) & = & \pi_2(\mathrm{B}N(G)) & \hookleftarrow & \pi_2(\mathrm{B}C_{N(G)}(V,\mu)) & \xrightarrow{\cong} & \pi_2(\mathrm{map}(\mathrm{B}V,\mathrm{B}N),\mathrm{B}\mu) \\
\uparrow{\scriptstyle w} & & \uparrow{\scriptstyle \pi_2(\mathrm{B}c_n)} & & \uparrow{\scriptstyle \pi_2(\mathrm{B}c_n)} & & \downarrow{\scriptstyle \pi_2((\mathrm{B}\alpha)^*)} \\
\pi_2(\mathrm{B}T) & = & \pi_2(\mathrm{B}N(G)) & \hookleftarrow & \pi_2(\mathrm{B}C_{N(G)}(V,\mu)) & \xrightarrow{\cong} & \pi_2(\mathrm{map}(\mathrm{B}V,\mathrm{B}N),\mathrm{B}\mu)
\end{array}$$

where $\pi_2(\mathrm{B}C_{N(G)}(V,\mu)) = \pi_2(\mathrm{B}T(G))^{\pi_0(\mu)(V)}$ denotes the fixed point group for the group action $\pi_0(\mu)\colon V \to W(G) \subseteq \mathrm{Aut}(\pi_2(\mathrm{B}T(G)))$. Since $\mathrm{B}c_n\colon \mathrm{B}N \to \mathrm{B}N$ is freely homotopic to the identity along the loop $w \in \pi_1(\mathrm{B}N)$ its effect on the $\mathbf{Z}_p[\pi_1(\mathrm{B}N)]$-module $\pi_2(\mathrm{B}N)$ is multiplication by w. □

lowdegree

9.24. Low degree identifications. There are the following low degree identifications [7, pp. 61, 292] [34, above def 3.3]

eq:lowdegree

$$\begin{aligned}
&\mathrm{Spin}(3) = \mathrm{Sp}(1) = \mathrm{SU}(2), \quad \mathrm{SO}(3) = \mathrm{PSp}(1) = \mathrm{PSU}(2)\\
&\mathrm{Spin}(4) = \mathrm{Spin}(3)\times\mathrm{Spin}(3) = \mathrm{SU}(2)\times\mathrm{SU}(2), \quad \mathrm{PSO}(4) = \mathrm{SO}(3)\times\mathrm{SO}(3)\\
&\mathrm{Spin}(5) = \mathrm{Sp}(2), \quad \mathrm{SO}(5) = \mathrm{PSp}(2)\\
&\mathrm{Spin}(6) = \mathrm{SU}(4), \quad \mathrm{PSO}(6) = \mathrm{PSU}(4)
\end{aligned} \tag{9.25}$$

Bibliography

[1] J. Frank Adams, *Lectures on Lie groups*, W. A. Benjamin, Inc., New York-Amsterdam, 1969. MR MR0252560 (40 #5780)

[2] K. Andersen, J. Grodal, J. Møller, and A. Viruel, *The classification of p-compact groups for p odd*, Ann of Math (to appear).

[3] Kasper K. S. Andersen, *The normalizer splitting conjecture for p-compact groups*, Fund. Math. **161** (1999), no. 1-2, 1–16, Algebraic topology (Kazimierz Dolny, 1997). MR 1 713 198

[4] N. Bourbaki, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines*, Hermann, Paris, 1968. MR 39 #1590

[5] Nicolas Bourbaki, *Éléments de mathématique*, Hermann, Paris, 1975, Fasc. XXXVIII: Groupes et algèbres de Lie. Chapitre VII: Sous-algèbres de Cartan, éléments réguliers. Chapitre VIII: Algèbres de Lie semi-simples déployées, Actualités Scientifiques et Industrielles, No. 1364. MR 56 #12077

[6] ______, *Éléments de mathématique: groupes et algèbres de Lie*, Masson, Paris, 1982, Chapitre 9. Groupes de Lie réels compacts. [Chapter 9. Compact real Lie groups]. MR **84i:**22001

[7] Theodor Bröcker and Tammo tom Dieck, *Representations of compact Lie groups*, Graduate Texts in Mathematics, vol. 98, Springer-Verlag, New York, 1985. MR **86i:**22023

[8] Henri Cartan and Samuel Eilenberg, *Homological algebra*, Princeton University Press, Princeton, N. J., 1956. MR MR0077480 (17,1040e)

[9] Allan Clark and John Ewing, *The realization of polynomial algebras as cohomology rings*, Pacific J. Math. **50** (1974), 425–434. MR 51 #4221

[10] Morton Curtis, Alan Wiederhold, and Bruce Williams, *Normalizers of maximal tori*, Localization in group theory and homotopy theory, and related topics (Sympos., Battelle Seattle Res. Center, Seattle, Wash., 1974), Springer, Berlin, 1974, pp. 31–47. Lecture Notes in Math., Vol. 418. MR 51 #13131

[11] W. G. Dwyer, *Lie groups and p-compact groups*, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), no. Extra Vol. II, 1998, pp. 433–442 (electronic). MR **99h:**55025

[12] W. G. Dwyer and D. M. Kan, *Centric maps and realization of diagrams in the homotopy category*, Proc. Amer. Math. Soc. **114** (1992), no. 2, 575–584. MR **92e:**55011

[13] W. G. Dwyer and C. W. Wilkerson, *Centers and Coxeter elements*, Preprint.

[14] ______, *Normalizers of tori*, Preprint.

[15] ______, *A cohomology decomposition theorem*, Topology **31** (1992), no. 2, 433–443. MR **MR1167181 (93h:**55008)

[16] ______, *A new finite loop space at the prime two*, J. Amer. Math. Soc. **6** (1993), no. 1, 37–64. MR **MR1161306 (93d:**55011)

[17] ______, *Homotopy fixed-point methods for Lie groups and finite loop spaces*, Ann. of Math. (2) **139** (1994), no. 2, 395–442. MR **95e:**55019

[18] ______, *The center of a p-compact group*, The Čech centennial (Boston, MA, 1993) (Providence, RI), Amer. Math. Soc., 1995, pp. 119–157. MR **96a:**55024

[19] ______, *Product splittings for p-compact groups*, Fund. Math. **147** (1995), no. 3, 279–300. MR **96h:**55005

[20] ______, *Diagrams up to cohomology*, Trans. Amer. Math. Soc. **348** (1996), no. 5, 1863–1883. MR **97a:**55020

[21] Leonard Evens, *The cohomology of groups*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1991, Oxford Science Publications. MR **MR1144017 (93i:**20059)

[22] Robert L. Griess, Jr., *Automorphisms of extra special groups and nonvanishing degree* 2 *cohomology*, Pacific J. Math. **48** (1973), 403–422. MR 57 #16429

[23] ______, *Elementary abelian p-subgroups of algebraic groups*, Geom. Dedicata **39** (1991), no. 3, 253–305. MR **92i:**20047

[24] J.-F. Hämmerli, M. Matthey, and U. Suter, *Automorphisms of normalizers of maximal tori and first cohomology of Weyl groups*, J. Lie Theory **14** (2004), no. 2, 583–617. MR MR2066874

[25] Jean-François Hämmerli, *Normalizers of maximal tori and classifying spaces of compact Lie groups*, Ph.D. thesis, Université de Neuchâtel, 2000.

[26] ______, *Some remarks on nonconnected compact Lie groups*, Enseign. Math. (2) **49** (2003), no. 1-2, 67–84. MR 1 998 883

[27] G. Hochschild and J.-P. Serre, *Cohomology of group extensions*, Trans. Amer. Math. Soc. **74** (1953), 110–134. MR 14,619b

[28] Bertram Huppert, *Endliche Gruppen. I*, Springer-Verlag, Berlin, 1967, Die Grundlehren der Mathematischen Wissenschaften, Band 134. MR 37 #302

[29] ———, *Character theory of finite groups*, Walter de Gruyter & Co., Berlin, 1998. MR **99j:**20011

[30] Stefan Jackowski and James McClure, *Homotopy decomposition of classifying spaces via elementary abelian subgroups*, Topology **31** (1992), no. 1, 113–132. MR **92k:**55026

[31] Stefan Jackowski, James McClure, and Bob Oliver, *Self-homotopy equivalences of classifying spaces of compact connected Lie groups*, Fund. Math. **147** (1995), no. 2, 99–126. MR **96f:**55009

[32] Jean Lannes, *Sur les espaces fonctionnels dont la source est le classifiant d'un p-groupe abélien élémentaire*, Inst. Hautes Études Sci. Publ. Math. (1992), no. 75, 135–244, With an appendix by Michel Zisman. MR **93j:**55019

[33] ———, *Théorie homotopique des groupes de Lie (d'après W. G. Dwyer et C. W. Wilkerson)*, Astérisque (1995), no. 227, Exp. No. 776, 3, 21–45, Séminaire Bourbaki, Vol. 1993/94. MR **96b:**55017

[34] M. Matthey, *Computing the second cohomology of Weyl groups acting on maximal tori*, http://www.math.uni-muenster.de/math/inst/sfb/about/publ/heft204.ps.

[35] ———, *Normalizers of maximal tori and cohomology of Weyl groups*, http://www.math.uni-muenster.de/math/inst/sfb/about/publ/index.htm.

[36] Mamoru Mimura, *The characteristic classes for the exceptional Lie groups*, Adams Memorial Symposium on Algebraic Topology, 1 (Manchester, 1990), London Math. Soc. Lecture Note Ser., vol. 175, Cambridge Univ. Press, Cambridge, 1992, pp. 103–130. MR **MR1170574 (93c:**55019**)**

[37] J. M. Møller and D. Notbohm, *Centers and finite coverings of finite loop spaces*, J. Reine Angew. Math. **456** (1994), 99–133. MR **95j:**55029

[38] ———, *Connected finite loop spaces with maximal tori*, Trans. Amer. Math. Soc. **350** (1998), no. 9, 3483–3504. MR **98k:**55008

[39] Jesper M. Møller, *The normalizer of the Weyl group*, Math. Ann. **294** (1992), no. 1, 59–80. MR **94b:**55010

[40] ———, *Completely reducible p-compact groups*, The Čech centennial (Boston, MA, 1993), Amer. Math. Soc., Providence, RI, 1995, pp. 369–383. MR **97b:**55020

[41] ———, *Homotopy Lie groups*, Bull. Amer. Math. Soc. (N.S.) **32** (1995), no. 4, 413–428. MR **96a:**55026

[42] ———, *Extensions of p-compact groups*, Algebraic topology: new trends in localization and periodicity (Sant Feliu de Guíxols, 1994), Birkhäuser, Basel, 1996, pp. 307–327. MR **98c:**55012

[43] ———, *Rational isomorphisms of p-compact groups*, Topology **35** (1996), no. 1, 201–225. MR **97b:**55019

[44] ———, *Deterministic p-compact groups*, Stable and unstable homotopy (Toronto, ON, 1996), Fields Inst. Commun., vol. 19, Amer. Math. Soc., Providence, RI, 1998, pp. 255–278. MR **99b:**55012

[45] ———, *Normalizers of maximal tori*, Math. Z. **231** (1999), no. 1, 51–74. MR **2000i:**55028

[46] ———, *Toric morphisms between p-compact groups*, Cohomological methods in homotopy theory (Bellaterra, 1998), Progr. Math., vol. 196, Birkhäuser, Basel, 2001, pp. 271–306. MR **2002i:**55010

[47] ———, *N-determined p-compact groups*, Fund. Math. **173** (2002), no. 3, 201–300. MR 1 925 483

[48] D. Notbohm, *Unstable splittings of classifying spaces of p-compact groups*, Q. J. Math. **51** (2000), no. 2, 237–266. MR **2001d:**55004

[49] Dietrich Notbohm, *Homotopy uniqueness of classifying spaces of compact connected Lie groups at primes dividing the order of the Weyl group*, Topology **33** (1994), no. 2, 271–330. MR **95e:**55020

[50] ———, *Classifying spaces of compact Lie groups and finite loop spaces*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 1049–1094. MR **96m:**55029

[51] ———, *A uniqueness result for orthogonal groups as 2-compact groups*, Arch. Math. (Basel) **78** (2002), no. 2, 110–119. MR **2003b:**22013

[52] ———, *On the 2-compact group $DI(4)$*, J. Reine Angew. Math. **555** (2003), 163–185. MR 1 956 596

[53] Bob Oliver, *Higher limits via Steinberg representations*, Comm. Algebra **22** (1994), no. 4, 1381–1393. MR **95b:**18007

[54] ———, *p-stubborn subgroups of classical compact Lie groups*, J. Pure Appl. Algebra **92** (1994), no. 1, 55–78. MR **94k:**57055

[55] Wilhelm Plesken, *On absolutely irreducible representations of orders*, Number theory and algebra, Academic Press, New York, 1977, pp. 241–262. MR 57 #6072

[56] Derek J. S. Robinson, *A course in the theory of groups*, second ed., Springer-Verlag, New York, 1996. MR **96f:**20001

[57] G. C. Shephard and J. A. Todd, *Finite unitary reflection groups*, Canadian J. Math. **6** (1954), 274–304. MR MR0059914 (15,600b)

[58] A. Vavpetic and A. Viruel, *Symplectic groups are N-determined 2-compact groups.*

[59] Aleš Vavpetič and Antonio Viruel, *On the homotopy type of the classifying space of the exceptional Lie group F_4*, Manuscripta Math. **107** (2002), no. 4, 521–540. MR **2003d:**55013

[60] Antonio Viruel, *Homotopy uniqueness of BG_2*, Manuscripta Math. **95** (1998), no. 4, 471–497. MR **99e:**55029

[61] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR **95f:**18001

[62] Zdzisław Wojtkowiak, *On maps from $\operatorname{ho}\varinjlim F$ to Z*, Algebraic topology, Barcelona, 1986, Springer, Berlin, 1987, pp. 227–236. MR **89a:**55034