Journal of Algebraic Combinatorics (2021) 54:915–946
https://doi.org/10.1007/s10801-021-01031-z

Equivariant Euler characteristics of unitary buildings

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Received: 9 September 2018 / Accepted: 1 March 2021 / Published online: 24 March 2021

Abstract
The (p-primary) equivariant Euler characteristics of the buildings for the general unitary groups over finite fields are determined.

Keywords Equivariant Euler characteristic · Totally isotropic subspace · General unitary group over a finite field · Generating function · Irreducible polynomial

Mathematics Subject Classification 05E18 · 06A07

1 Introduction

Let G be a finite group, Π a finite G-poset, and $r \geq 1$ a natural number. The *rth equivariant reduced Euler characteristic* of the G-poset Π as defined by Atiyah and Segal [2] is the normalised sum

$$\widetilde{\chi}_r(\Pi, G) = \frac{1}{|G|} \sum_{X \in \mathrm{Hom}(\mathbf{Z}^r, G)} \widetilde{\chi}(C_\Pi(X(\mathbf{Z}^r))) \tag{1.1}$$

of the reduced Euler characteristics of the $X(\mathbf{Z}^r)$-fixed Π-subposets, $C_\Pi(X(\mathbf{Z}^r))$, as X ranges over the set of all homomorphisms of $\mathbf{Z}^r$ to G. For example, when G acts trivially on Π, $\widetilde{\chi}_r(\Pi, G) = \widetilde{\chi}(\Pi)|\mathrm{Hom}(\mathbf{Z}^r, G)|/|G|$ is the reduced Euler characteristic of Π times the number of conjugacy classes of commuting $(r-1)$-tuples of elements of G [10, Lemma 4.13]. Here are three more examples of already known equivariant reduced Euler characteristics:

Supported by the Danish National Research Foundation through the Centre for Symmetry and Deformation (DNRF92).

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(1) The symmetric group Σ_n acts on the poset $B(n)^*$ of non-extreme subsets of the n-set. The generating function for the $(r+1)$th equivariant reduced Euler characteristics of the Σ_n-posets $B(n)^*$ is

$$1+\sum_{n\geq 1}\widetilde{\chi}_{r+1}(B(n)^*,\Sigma_n)x^n=\prod_{n\geq 1}(1-x^n)^{\lambda_r(n)}$$

where $\lambda_r(n)$ is the number of index r-subgroups of $\mathbf{Z}^n$ ([3, Theorem 2.1] with $M=\{0,1\}$).

(2) The symmetric group Σ_n acts on the poset $\Pi(n)^*$ of non-extreme partitions of the n-set. The equivariant reduced Euler characteristics of the Σ_n-posets $\Pi(n)^*$ are

$$\widetilde{\chi}_r(\Pi(n)^*,\Sigma_n)=\frac{1}{n}c_r(n)$$

where c_r is the sequence with Dirichlet convolution $(c_r*\lambda_r)(n)=(-1)^{n+1}$ [20, Theorem 1.3, Corollary 1.4].

(3) The general linear group $\mathrm{GL}_n^+(\mathbf{F}_q)$ acts on the poset $\mathrm{L}_n^+(\mathbf{F}_q)^*$ of non-extreme subspaces of the n-dimensional vector space over the field $\mathbf{F}_q$ of prime power order q. The generating function for the $(r+1)$th equivariant reduced Euler characteristics of the $\mathrm{GL}_n^+(\mathbf{F}_q)$-posets $\mathrm{L}_n^+(\mathbf{F}_q)^*$ is

$$1+\sum_{n\geq 0}\widetilde{\chi}_{r+1}(\mathrm{L}_n^+(\mathbf{F}_q),\mathrm{GL}_n^+(\mathbf{F}_q))x^n=\prod_{0\leq j\leq r}(1-q^jx)^{(-1)^{r-j}\binom{r}{j}}$$

according to [21, Theorem 1.4].

In this article we shall consider the general unitary group $\mathrm{GL}_n^-(\mathbf{F}_q)$, the isometry group of the unitary n-geometry over the field $\mathbf{F}_{q^2}$, acting on the poset $\mathrm{L}_n^-(\mathbf{F}_q)^*=\{0\subsetneq U\subsetneq\mathbf{F}_{q^2}^n\mid U\subseteq U^\perp\}$ of nontrivial totally isotropic subspaces. (See Sect. 2 for more details.) We now emphasise the definition of the equivariant reduced Euler characteristics in this particular case and proceed to present the main results of this paper.

Definition 1.1 [2] The rth equivariant reduced Euler characteristic of the $\mathrm{GL}_n^-(\mathbf{F}_q)$-poset $\mathrm{L}_n^-(\mathbf{F}_q)^*$ is the normalised sum

$$\widetilde{\chi}_r(\mathrm{L}_n^-(\mathbf{F}_q)^*,\mathrm{GL}_n^-(\mathbf{F}_q))=\frac{1}{|\mathrm{GL}_n^-(\mathbf{F}_q)|}\sum_{X\in\mathrm{Hom}(\mathbf{Z}^r,\mathrm{GL}_n^-(\mathbf{F}_q))}\widetilde{\chi}(C_{\mathrm{L}_n^-(\mathbf{F}_q)^*}(X(\mathbf{Z}^r)))$$

of the Euler characteristics of the induced subposets $C_{\mathrm{L}_n^-(\mathbf{F}_q)^*}(X(\mathbf{Z}^r))$ of $X(\mathbf{Z}^r)$-invariant subspaces as X ranges over all homomorphisms of the free abelian group $\mathbf{Z}^r$ on r generators into the general unitary group.

The generating function for the *negative* rth equivariant reduced Euler characteristics is the power series

$$\mathrm{FGL}_r^-(q,x) = 1 - \sum_{n\geq 1} \widetilde{\chi}_r(\mathrm{GL}_n^-(\mathbf{F}_q))x^n \tag{1.2}$$

with coefficients in the ring of integral polynomials in q. (The shortened notation $\widetilde{\chi}_r(\mathrm{GL}_n^-(\mathbf{F}_q))$ is and will be used for the rth equivariant reduced Euler characteristic $\widetilde{\chi}_r(\mathrm{L}_n^-(\mathbf{F}_q)^*, \mathrm{GL}_n^-(\mathbf{F}_q))$ of Definition 1.1.)

Theorem 1.2 $\mathrm{FGL}_{r+1}^-(q,x) = \prod_{0\leq j\leq r}(1+(-1)^{r-j}q^jx)^{(-1)^{r-j}\binom{r}{j}}$ *for all* $r \geq 0$.

The first few instances of the generating function

$$\mathrm{FGL}_{r+1}^-(q,x) = \prod_{0\leq j\leq r}(1+(-1)^{r-j}q^jx)^{(-1)^{r-j}\binom{r}{j}} = \frac{\prod_{j\equiv r \bmod 2}(1+q^jx)^{\binom{r}{j}}}{\prod_{j\not\equiv r \bmod 2}(1-q^jx)^{\binom{r}{j}}}$$

are

$$1+x, \quad \frac{1+qx}{1-x}, \quad \frac{(1+q^2x)(1+x)}{(1-qx)^2}, \quad \frac{(1+q^3x)(1+qx)^3}{(1-q^2x)^3(1-x)},$$
$$\frac{(1+q^4x)(1+q^2x)^6(1+x)}{(1-q^3x)^4(1-qx)^4}$$

for $r+1 = 1, 2, 3, 4$. In particular, $-\widetilde{\chi}_2(\mathrm{GL}_n^-(\mathbf{F}_q)) = q+1$ and $-\widetilde{\chi}_3(\mathrm{GL}_n^-(\mathbf{F}_q)) = nq^{n-1}(q+1)^2$ for $n \geq 1$.

The proofs of Theorem 1.2 and its corollary below are in Sect. 6.

Corollary 1.3 $\mathrm{FGL}_{r+1}^-(q,x) = \exp\left(-\sum_{n\geq 1}(-1)^n(q^n-(-1)^n)^r\frac{x^n}{n}\right)$ *for all* $r \geq 0$.

We also consider, for any prime p, the p-primary equivariant reduced Euler characteristics, $\widetilde{\chi}_r(p, \mathrm{GL}_n^-(\mathbf{F}_q))$, for the $\mathrm{GL}_n^-(\mathbf{F}_q)$-poset $\mathrm{L}_n^-(\mathbf{F}_q)^*$ (Definition 8.1) as defined by Tamanoi [27]. It turns out that the rth p-primary generating function at q, the generating function $\mathrm{FGL}_r^-(p,q,x)$ for the negative rth p-primary equivariant reduced Euler characteristics (8.1), is in some sense a p-local version of the exponential form of $\mathrm{FGL}_r^-(q,x)$ from Corollary 1.3. (We write n_p for the p-part of the natural number n.)

Theorem 1.4 $\mathrm{FGL}_{r+1}^-(p,q,x) = \exp\left(-\sum_{n\geq 1}(-1)^n(q^n-(-1)^n)_p^r\frac{x^n}{n}\right)$ *for all* $r \geq 0$.

The infinite product expansions of the absolute and the p-primary generating functions

$$\mathrm{FGL}_{r+1}^-(q,x) = \prod_{n\geq 1}(1-x^n)^{a_{r+1}^-(q,n)}$$
$$a_{r+1}^-(q,n) = \frac{1}{n}\sum_{d|n}(-1)^d\mu(n/d)(q^d-(-1)^d)^r$$

$$\mathrm{FGL}^{-}_{r+1}(p,q,x) = \prod_{n\geq 1}(1-x^n)^{a^{-}_{r+1}(p,q,n)}$$
$$a^{-}_{r+1}(p,q,n) = \frac{1}{n}\sum_{d|n}(-1)^d\mu(n/d)(q^d-(-1)^d)^r_p$$

follow immediately from Theorems 1.2 and 1.4 using the elementary [21, Lemma 3.7].

More explicitly, the equivariant Euler characteristics and the p-primary Euler characteristics of the general unitary group are

$$-\widetilde{\chi}_{r+1}(\mathrm{GL}^{-}_n(\mathbf{F}_q)) = \frac{1}{|W_n|}\sum_{w\in W_n}\det(w)\det(q+w)^r,$$
$$-\widetilde{\chi}_{r+1}(p,\mathrm{GL}^{-}_n(\mathbf{F}_q)) = \frac{1}{|W_n|}\sum_{w\in W_n}\det(w)\det(q+w)^r_p$$

where the sum ranges over the elements of the standard n-dimensional integral permutation representation W_n of the symmetric group Σ_n (Propositions 6.4, 8.4).

This paper is organised as follows. Section 2 describes the general unitary group $\mathrm{GL}^{-}_n(\mathbf{F}_q)$ as a subgroup of $\mathrm{GL}^{+}_n(\mathbf{F}_{q^2})$. Characteristic polynomials for elements of $\mathrm{GL}^{-}_n(\mathbf{F}_q)$ are self-dual and Sect. 3 is an exposition of the combinatorics of self-dual irreducible monic polynomials over $\mathbf{F}_{q^2}$. The main result of Sect. 4 is the product formula of Lemma 4.2 for equivariant Euler characteristics. Section 5 establishes the key ingredients in the proof of Theorem 1.2: a vanishing result (Lemma 5.2) and a recursion formula (Lemma 5.3). Theorem 1.2 and Corollary 1.3 are proved in Sect. 6. This section also connects the equivariant Euler characteristics of the general unitary group to representation theory and algebraic geometry: Remark 6.3 explains the role of the second equivariant Euler characteristic $\widetilde{\chi}_2(\mathrm{GL}^{-}_n(\mathbf{F}_q))$ in the Knörr-Robinson conjecture for $\mathrm{GL}^{-}_n(\mathbf{F}_q)$ at the defining characteristic [15], and Sect. 6.2 points out a curious coincidence with Hasse–Weil zeta functions of supersingular elliptic curves over $\mathbf{F}_{q^2}$. In Sect. 6.1 we develop more explicit expressions for the equivariant Euler characteristics. (The formulas of Proposition 6.4 may indicate a general description of the equivariant Euler characteristics of finite groups of Lie type.) In Sect. 7 we shortly review the S-transform and use it to (re)prove polynomial identities associated to partitions. Section 8 discusses p-primary equivariant reduced Euler characteristics of general unitary groups for a given prime p. The corresponding *unreduced* Euler characteristics can be interpreted as Euler characteristics computed in Morava K-theories at p of the homotopy orbit spaces $B\mathrm{L}^{-}_n(\mathbf{F}_q)^*_{h\mathrm{GL}^{-}_n(\mathbf{F}_q)}$ for the action of $\mathrm{GL}^{-}_n(\mathbf{F}_q)$ on the classifying space of the poset $\mathrm{L}^{-}_n(\mathbf{F}_q)^*$. The proof of Theorem 1.4 together with more explicit expressions for the p-primary equivariant Euler characteristics $\widetilde{\chi}_r(p,\mathrm{GL}^{-}_n(\mathbf{F}_q))$ can be found here.

The following notation will be used in this paper in addition to notation related to multisets introduced at the beginning of Sect. 6.1:

p	a prime number
$\nu_p(n)$	the p-adic valuation of n
n_p	the p-part of the natural number n ($n_p = p^{\nu_p(n)}$)
$\mathbf{Z}_p$	the ring of p-adic integers
q	a prime power
$\mathbf{F}_q$	the finite field with q elements
s	the characteristic of $\mathbf{F}_q$
$\widetilde{\chi}_r(\mathrm{GL}_n^{\pm}(\mathbf{F}_q))$	equivariant Euler characteristic $\widetilde{\chi}_r(\mathrm{L}_n^{\pm}(\mathbf{F}_q)^*, \mathrm{GL}_n^{\pm}(\mathbf{F}_q))$ (Definition 1.1) [21, Definition 1.2],
$\widetilde{\chi}_r(p, \mathrm{GL}_n^{\pm}(\mathbf{F}_q))$	p-primary Euler characteristic $\widetilde{\chi}_r(p, \mathrm{L}_n^{\pm}(\mathbf{F}_q)^*, \mathrm{GL}_n^{\pm}(\mathbf{F}_q))$ (Definition 8.1) [21, Definition 4.2]
$\binom{m}{-k}$	the signed binomial coefficient $(-1)^k\binom{m}{k}$

2 The general unitary group $\mathrm{GL}_n^-(\mathbf{F}_q)$

Let q be a prime power, $n \geq 1$ a natural number, and $V_n(\mathbf{F}_{q^2}) = \mathbf{F}_{q^2}^n$ the vector space of dimension n over the field $\mathbf{F}_{q^2}$ with q^2 elements. The non-degenerate sesquilinear form

$$\langle u, v\rangle = c \sum_{1\leq i\leq n} (-1)^{i+1} u_i v_{n+1-i}^q \qquad u, v \in V_n(\mathbf{F}_{q^2}) \tag{2.1}$$

is Hermitian ($\langle au, v\rangle = a\langle u, v\rangle$, $\langle u, v\rangle^q = \langle v, u\rangle$, $a \in \mathbf{F}_{q^2}$, $u, v \in V_n(\mathbf{F}_{q^2})$) when the constant $c \in \mathbf{F}_{q^2}$ satisfies $c^{q-1} = (-1)^{n+1}$. The *general unitary group* $\mathrm{GL}_n^-(\mathbf{F}_q)$ [9, §2.7] is the group of all linear automorphisms of $V_n(\mathbf{F}_{q^2})$ preserving the Hermitian bilinear form (2.1). Let $\varphi_q(g)$ denote the matrix obtained from $g \in \mathrm{GL}_n^+(\mathbf{F}_{q^2})$ by raising all entries to the power q. Then g lies in $\mathrm{GL}_n^-(\mathbf{F}_q)$ if and only if $gA(\varphi_q(g))^t = A$ where A is the matrix whose only nonzero entries are a string of alternating $+1$'s and -1's running diagonally from upper right to lower left corner. The order of $\mathrm{GL}_n^-(\mathbf{F}_q)$ is [32, (2.6.1)] [35, (3.25)]

$$\begin{aligned}|\mathrm{GL}_n^-(\mathbf{F}_q)| = (q+1)|\mathrm{SL}_n^-(\mathbf{F}_q)| &= q^{\binom{n}{2}} \prod_{1\leq i\leq n} (q^i - (-1)^i) \\ &= \prod_{0\leq i\leq n-1} (q^n - (-1)^{n-i} q^i)\end{aligned}$$

and there is a short exact sequence

$$1 \longrightarrow \mathrm{SL}_n^-(\mathbf{F}_q) \longrightarrow \mathrm{GL}_n^-(\mathbf{F}_q) \xrightarrow{\det} C_{q+1} \longrightarrow 1 \tag{2.2}$$

where C_{q+1} is the order $q+1$ subgroup of the cyclic unit group $\mathbf{F}_{q^2}^\times$. We have $|\mathrm{GL}_n^-|(q) = (-1)^n|\mathrm{GL}_n^+|(-q)$ where $|\mathrm{GL}_n^+|(q) = \prod_{0\leq i\leq n-1}(q^n - q^i)$ and $|\mathrm{GL}_n^-|(q) = \prod_{0\leq i\leq n-1}(q^n - (-1)^{n-i}q^i)$ are the order polynomials [17, p. 207] for the general linear and unitary groups. The *special unitary group* $\mathrm{SL}_n^-(\mathbf{F}_q)$ is generated by root group elements $x_{\widehat{\alpha}}(t)$ or $x_{\widehat{\alpha}}(t,u)$ of type I, II, and (for odd n) IV [9, Table 2.4] and the general unitary group $\mathrm{GL}_n^-(\mathbf{F}_q)$ by root groups together with the diagonal matrices $\mathrm{diag}(z,1,\dots,1,z^{-q})$ for $z \in \mathbf{F}_{q^2}^\times$.

A subspace of $V_n(\mathbf{F}_{q^2})$ is *totally isotropic* if the Hermitian sesquilinear form (2.1) vanishes completely on it. Let $\mathrm{L}_n^-(\mathbf{F}_q)$ be the poset of totally isotropic subspaces in $V_n(\mathbf{F}_{q^2})$ and $\mathrm{L}_n^-(\mathbf{F}_q)^*$ the subposet of *nontrivial* totally isotropic subspaces. The standard action of $\mathrm{GL}_n^+(\mathbf{F}_{q^2})$ on subspaces of $V_n(\mathbf{F}_{q^2})$ restricts to an action of $\mathrm{GL}_n^-(\mathbf{F}_q)$ on $\mathrm{L}_n^-(\mathbf{F}_q)^*$. The classifying simplicial complex of $\mathrm{L}_n^-(\mathbf{F}_q)^*$, the flag complex of totally isotropic subspaces, is the building for $\mathrm{GL}_n^-(\mathbf{F}_q)$ [1, §6.8]. We may replace the flag complex $\mathrm{L}_n^-(\mathbf{F}_q)^*$ by the Brown subgroup poset $\mathcal{S}^{s+*}_{\mathrm{GL}_n^-(\mathbf{F}_q)}$ of nontrivial s-subgroups of $\mathrm{GL}_n^-(\mathbf{F}_q)$ where s is the defining characteristic [22, Theorem 3.1].

The (non-equivariant) reduced Euler characteristics of the spherical posets $\mathrm{L}_n^\pm(\mathbf{F}_q)^*$ are given by

$$-\widetilde{\chi}(\mathrm{L}_n^+(\mathbf{F}_q)^*) = (-1)^{n-1}q^{\binom{n}{2}}, \qquad -\widetilde{\chi}(\mathrm{L}_n^-(\mathbf{F}_q)^*) = (-1)^{\lfloor n/2\rfloor}q^{\binom{n}{2}}$$

according to the Solomon–Tits theorem [7, Proposition 8.3] (or [26, Example 3.10.2] for the case of $\mathrm{L}_n^+(\mathbf{F}_q)^*$).

3 Self-dual polynomials over $\mathbf{F}_{q^2}$

In the next lemma, we consider field extensions $\mathbf{F}_q \subseteq \mathbf{F}_{q^{m_1}} \subseteq \mathbf{F}_{q^{m_2}}$ where $1 \le m_1 \le m_2$. Let $\sigma_0, \sigma_1, \dots, \sigma_n$ be the elementary symmetric polynomials in $n \ge 1$ variables [16, Example 1.74] (where σ_0 stands for the constant polynomial 1).

Lemma 3.1 *Let $a_1, \dots, a_n$ be n elements of the field $\mathbf{F}_{q^{m_2}}$. Then*

$$\forall i \in \{0,1,\dots,n\}\colon \sigma_i(a_1,\dots,a_n) \in \mathbf{F}_{q^{m_1}} \iff \forall i \in \{0,1,\dots,n\}\colon \sigma_i(a_1^{-q},\dots,a_n^{-q}) \in \mathbf{F}_{q^{m_1}}$$

Proof The nth elementary symmetric function is $\sigma_n(a_1,\dots,a_n) = a_1\cdots a_n$. Observe that

$$\forall i \in \{0,1,\dots,n\}\colon \sigma_i(a_1^{-1},\dots,a_n^{-1})\sigma_n(a_1,\dots,a_n) = \sigma_{n-i}(a_1,\dots,a_n)$$

If all values of $\sigma_i(a_1,\dots,a_n)$ are in the subfield $\mathbf{F}_{q^{m_1}}$, also all values of $\sigma_i(a_1^{-1},\dots,a_n^{-1})$ and $\sigma_i(a_1^{-q},\dots,a_n^{-q}) = \sigma_i(a_1^{-1},\dots,a_n^{-1})^q$ are in this subfield. □

Definition 3.2 (*Dual* polynomial) [32, Notation p. 13] Let $p(x) = a_0x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m \in \mathbf{F}_{q^2}[x]$ be a polynomial of degree $m \ge 1$ with nonzero constant term (so that $a_0 \ne 0$ and $a_m \ne 0$). The dual polynomial to $p(x)$ is

$$\overline{p}(x) = a_0 \prod_{1\le i\le m}(x-\alpha_i^{-q})$$

where $p(x) = a_0 \prod_{1\le i\le m}(x-\alpha_i)$ with $\alpha_1,\ldots,\alpha_m$ in the splitting field for $p(x)$ over $\mathbf{F}_{q^2}$.

If $p(x) = \overline{p}(x)$ we say that $p(x)$ is self-dual.

We note that

- dualization is involutory: $\overline{\overline{p}} = p$
- dualization respects products: $\overline{p_1 p_2} = \overline{p}_1\overline{p}_2$
- dualization respects divisibility: $p_1 \mid p_2 \iff \overline{p}_1 \mid \overline{p}_2$
- a polynomial (with nonzero constant term) is irreducible if and only its dual polynomial is irreducible
- if $p = a_0 \prod r_i^{e_i}$ is the canonical factorisation of the polynomial p [12, Theorem 1.59] then $\overline{p} = a_0 \prod \overline{r}_i^{e_i}$ is the canonical factorisation of the dual polynomial

Although the dual of a polynomial over $\mathbf{F}_{q^2}$ is defined in terms of elements of an extension of $\mathbf{F}_{q^2}$, it is actually again a polynomial over $\mathbf{F}_{q^2}$ as Lemma 3.1 shows that the coefficients of $\overline{p}(x)$ lie in $\mathbf{F}_{q^2}$ if those of $p(x)$ do.

Proposition 3.3 *Let $p(x) = a_0x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m \in \mathbf{F}_{q^2}[x]$ be a polynomial as in Definition 3.2 with $a_0 \neq 0$ and $a_m \neq 0$. The dual polynomial $\overline{p}(x)$ is given by*

$$a_m^q\overline{p}(x) = a_0(a_m^q x^m + a_{m-1}^q x^{m-1} + \cdots + a_1^q x + a_0^q)$$

and $p(x)$ is self-dual if and only if its coefficients satisfy the equation

$$a_m^q(a_0, a_1, \ldots, a_{m-1}, a_m) = a_0(a_m^q, a_{m-1}^q, \ldots, a_1^q, a_0^q)$$

Proof The reciprocal [16, Definition 3.12] to the polynomial $p(x)$ is

$$p^*(x) = x^m p(x^{-1}) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0 = a_m \prod_{1\le i\le m}(x-\alpha_i^{-1})$$

and thus

$$a_m^q \prod_{1\le i\le m}(x-\alpha_i^{-q}) = a_m^q x^m + a_{m-1}^q x^{m-1} + \cdots + a_{1^q} x + a_0^q$$

since the Frobenius map $\sigma_q(x) = x^q$ is a field automorphism of $\mathbf{F}_{q^2}$. The dual polynomial is

$$\overline{p}(x) = a_0 \prod_{1\le i\le m}(x-\alpha_i^{-q}) = \frac{a_0}{a_m^q}(a_m^q x^m + a_{m-1}^q x^{m-1} + \cdots + a_1^q x + a_0^q)$$

and hence

$$\begin{aligned} p(x) = \overline{p}(x) &\iff a_m^q p(x) = a_m^q \overline{p}(x) \\ &\iff a_m^q(a_0x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m) \end{aligned}$$

$$= a_0(a_m^q x^m + a_{m-1}^q x^{m-1} + \cdots + a_1^q x + a_0^q)$$

which is the criterion of the proposition. □

If g is a unitary automorphism of a vector space over $\mathbf{F}_{q^2}$ and $p(x)$ the polynomial of Definition 3.2 then

$$a_0^q \langle p(g)x, y\rangle = a_m \langle g^m(x), \bar{p}(g)(y)\rangle \tag{3.1}$$

for all vectors x, y.

Lemma 3.4 *The characteristic polynomial c_g of any unitary automorphism $g \in \mathrm{GL}_n^-(\mathbf{F}_q)$ is self-dual.*

Proof Let $r \in \mathbf{F}_{q^2}[x]$ be an irreducible polynomial. Then

$$r \nmid c_g \iff r(g) \text{ is invertible} \overset{(3.4)}{\iff} \bar{r}(g) \text{ is invertible} \iff \bar{r} \nmid c_g$$

or, equivalently, $r \mid c_g \iff \bar{r} \mid c_g$. This shows that $c_g = \bar{c}_g$. □

Corollary 3.5 [32, Proof of (ii), p. 35] *The number of self-dual monic polynomials in $\mathbf{F}_{q^2}[x]$ of degree $m > 0$ with nonzero constant term is $q^m + q^{m-1}$.*

Proof A monic polynomial of degree m, $p(x) = x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m \in \mathbf{F}_{q^2}[x]$ with $a_m \neq 0$ is by Proposition 3.3 self-dual if and only if

$$a_m^q(a_1, a_2, \ldots, a_{m-1}, a_m) = (a_{m-1}^q, \ldots, a_1^q, 1)$$

or, equivalently,

$$a_m^{q+1} = 1 \text{ and } (a_1, \ldots, a_{m-1}) = a_m(a_{m-1}^q, \ldots, a_1^q) \tag{3.2}$$

Suppose first that $m = 2k + 1$ is odd. There are $q + 1$ elements a_m in $\mathbf{F}_{q^2}$ such that $a_m^{q+1} = 1$. For $1 \le j \le k$, let a_j be any element of $\mathbf{F}_{q^2}$ and put $a_{m-j} = a_m a_j^q$. Then $a_m a_{m-j}^q = a_j^{q^2} = a_j$. This shows that the self-duality criterion (3.2) has $(q+1)q^{2k} = q^m + q^{m-1}$ solutions. Suppose next that $m = 2k$ is even. The coefficient a_m can again be chosen in exactly $q + 1$ ways. For each j with $1 \le j \le k - 1$, the coefficient a_j can be chosen freely in $\mathbf{F}_{q^2}$ and we let $a_{m-j} = a_m a_j^q$. There are $q = (q - 1) + 1$ possibilities for choosing the coefficient a_k such that $a_k = a_m a_k^q$ as $a_m^{q+1} = 1$. Thus the self-duality criterion (3.2) has $(q + 1)q^{2k-2}q = q^m + q^{m-1}$ solutions. □

Definition 3.6 (*See Fig.* 1) For every integer $d \ge 1$,

- $\mathrm{IM}_d(q)$ is the number of Irreducible Monic polynomials of degree d over $\mathbf{F}_q$ with nonzero constant term

	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$
$d\,\mathrm{IM}_d(q)$	$q-1$	q^2-q	q^3-q	q^4-q^2	q^5-q	$q^6-q^3-q^2+q$
$d\,\mathrm{IM}_d(q^2)$	q^2-1	q^4-q^2	q^6-q^2	q^8-q^4	$q^{10}-q^2$	$q^{12}-q^6-q^4+q^2$
$d\,\mathrm{SDIM}_d^-(q)$	$q+1$	0	q^3-q	0	q^5-q	0
$2d\,\mathrm{SDIM}_d^+(q)$	q^2-q-2	q^4-q^2	$q^6-q^3-q^2+q$	q^8-q^4	$q^{10}-q^5-q^2+q$	$q^{12}-q^6-q^4+q^2$

Fig. 1 The polynomials $\mathrm{IM}_d(q)$, $\mathrm{IM}_d(q^2)$ and $\mathrm{SDIM}_d^{\pm}(q)$ for $d=1,\ldots,6$

- $\mathrm{SDIM}_d^-(q)$ is the number of Self-Dual Irreducible Monic polynomials of degree d over $\mathbf{F}_{q^2}$ with nonzero constant term
- $\mathrm{SDIM}_d^+(q) = \frac{1}{2}(\mathrm{IM}_d(q^2) - \mathrm{SDIM}_d^-(q))$ is the number of unordered pairs of non-self-dual irreducible monic polynomials of degree d over $\mathbf{F}_{q^2}$ with nonzero constant term

For all $d \geq 1$, $\mathrm{IM}_d(q^2) = \sum_{d|n} \mu(n/d)(q^{2d}-1)$ [16, Corollary 3.21] (simplifying to $\mathrm{IM}_d(q^2) = \sum_{d|n} \mu(n/d)q^{2d}$ when $d > 1$). The well-known identities [33, p. 258]

$$\prod_{d\geq 1} \frac{1}{(1-x^d)^{\mathrm{IM}_d(q)}} = \frac{1-x}{1-qx}, \qquad \prod_{d\geq 1} \frac{1}{(1-x^d)^{\mathrm{IM}_d(q^2)}} = \frac{1-x}{1-q^2x} \tag{3.3}$$

are easily proved using [21, Lemma 3.7]. When $d=1$, $\mathrm{IM}_1(q^2) = q^2-1$ (represented by the polynomials $x-\lambda$, $\lambda \in \mathbf{F}_{q^2}^{\times}$), $\mathrm{SDIM}_1^-(q) = q+1$ (represented by the polynomials $x-\lambda$, $\lambda \in \mathbf{F}_{q^2}^{\times}$, $\lambda = \lambda^{-q}$) and $\mathrm{SDIM}_1^+(q) = \frac{1}{2}(q^2-1-(q+1)) = \frac{1}{2}(q+1)(q-2)$ (represented by the pairs $(x-\lambda, x-\lambda^{-q})$, $\lambda \in \mathbf{F}_{q^2}^{\times}$, $\lambda \neq \lambda^{-q}$).

The next proposition shows among other things that self-dual irreducible polynomials have odd degrees, i.e. that $\mathrm{SDIM}_d^-(q) = 0$ for all even d.

Proposition 3.7 *Let $p(x) \in \mathbf{F}_{q^2}[x]$ be a self-dual irreducible monic polynomial of degree $m \geq 1$ over $\mathbf{F}_{q^2}$ with $p(0) \neq 0$. Then m is odd and*

$$p(x) = \prod_{0\leq j\leq m-1} (x - \lambda^{q^{2j}})$$

where $\lambda \in \mathbf{F}_{q^{2m}}$, $\lambda^{q^m+1} = 1$, and all the elements $\lambda, \lambda^{q^2}, \ldots, \lambda^{q^{2m-2}}$ are distinct.

Proof Let $p(x)$ be a monic irreducible polynomial $p(x)$ of degree m over $\mathbf{F}_{q^2}$. The field $\mathbf{F}_{q^{2m}}$ contains an element λ such that

$$p(x) = \prod_{0\leq j\leq m-1} (x - \lambda^{q^{2j}})$$

and all the elements $\lambda, \lambda^{q^2}, \ldots, \lambda^{q^{2m-2}}$ are distinct [16, Theorem 2.14]. By self-duality $\lambda^{-q} = \lambda^{q^{2k}}$ for a unique integer k with $0 \leq k \leq m-1$.

Assume first that $p(x)$ has degree $m=2$. The roots of $p(x)$ are $\{\lambda, \lambda^{q^2}\}$ where λ^{-q} equals λ or λ^{q^2} by self-duality. In the first case, $1 = \lambda\lambda^{-1} = \lambda\lambda^q = \lambda^{q+1}$ and

$\lambda^{q^2-1} = (\lambda^{q+1})^{q-1} = 1$. In the second case, $\lambda^q = \lambda^{-q^2} = (\lambda^{-q})^q = (\lambda^{q^2})^q = \lambda^{q^3}$ and $1 = \lambda^{q^3-q} = (\lambda^{q^2-1})^q$ so $\lambda^{q^2-1} = 1$ also here. In both cases, we have that $\lambda, \lambda^{q^2} \in \mathbf{F}_{q^2}^\times$. Since this contradicts irreducibility of $p(x)$ over $\mathbf{F}_{q^2}$, monic irreducible self-dual polynomials of degree 2 do not exist.

Assume next that $m > 2$. Since $\lambda^{q^2} = (\lambda^{-q})^{-q} = \lambda^{q^{4k}}$ it follows that m divides $2k-1$ and is odd. Furthermore, k equals 1 or $\frac{1}{2}(m+1)$ as k is at most $m-1$. However, $k = 1$ implies $\lambda^{q^2} = \lambda^{q^{4k}} = \lambda^{q^4}$ contradicting that λ^{q^2} and λ^{q^4} are distinct when $m \geq 3$. From $2k = m+1$ we get $\lambda^{-q} = \lambda^{q^{2k}} = \lambda^{q^{m+1}}$, equivalently, $\lambda^{-1} = \lambda^{q^m}$ or $\lambda^{q^m+1} = 1$. $\square$

The next count of self-dual irreducible monic polynomials in $\mathbf{F}_{q^2}[x]$ is closely related to the classical count of irreducible monic polynomials or self-reciprocal irreducible monic polynomials in $\mathbf{F}_q[x]$ [16, Corollary 3.21, Theorem 3.25] [18, Theorem 3]

Lemma 3.8 *Let $m \geq 1$ be an odd integer.*

(1) The self-dual irreducible monic polynomials in $\mathbf{F}_{q^2}[x]$ with nonzero constant term whose degrees divide the odd integer $m \geq 1$ are precisely the irreducible factors of the polynomial $x^{q^m+1} - 1 \in \mathbf{F}_{q^2}[x]$.

(2) $\sum_{d|m} d\,\mathrm{SDIM}_d^-(q) = q^m + 1$ and $m\,\mathrm{SDIM}_m^-(q) = \sum_{d|m} \mu(m/d)(q^d+1)$ for any odd integer $m \geq 1$.

Proof (1) Let $p(x)$ be an irreducible factor of $x^{q^m+1} - 1$. Obviously, $p(0) \neq 0$. If α is a root of $p(x)$ in its splitting field then $\alpha^{q^m+1} = \alpha$. Therefore $\alpha^{-q} = \alpha^{q^m}$ is also a root of $p(x)$. This shows that $p(x)$ is self-dual (Definition 3.2).

Next, let $p(x)$ be a self-dual irreducible monic polynomial with nonzero constant term of degree d dividing m. According to Proposition 3.7, $p(x)$ has a root $\lambda \in \mathbf{F}_{q^{2d}}$ such that $\lambda^{q^{d+1}} - 1 = 0$. Then $p(x)$ divides $x^{q^{d+1}} - 1$ which divides $x^{q^{m+1}} - 1$ as $d \leq m$ [16, Lemma 2.12, Corollary 3.7].

(2) The polynomial $x^{q^{2m}} - x = x(x^{q^{2m-1}} - 1) \in \mathbf{F}_{q^2}[x]$ has no multiple roots according to the standard criterion of [16, Theorem 1.68]. The polynomial $x^{q^{m+1}} - 1$ is a factor of $x^{q^{2m}} - x$ by [16, Corollary 3.7] and hence also has no multiple roots. From (1) it now follows that $x^{q^{m+1}} - 1$ is the product of all self-dual irreducible polynomials with nonzero constant terms of degrees dividing m. The second assertion is the Möbius inversion of the first one which is a count of degrees. $\square$

Corollary 3.9 *The arithmetic functions $\mathrm{IM}_n(q)$ and $\mathrm{SDIM}_n^\pm(q)$ of Definition 3.6 satisfy the relations*

$$\mathrm{SDIM}_n^-(q) = \begin{cases} q+1 & n = 1 \\ \mathrm{IM}_n(q) & n > 1 \text{ odd} \\ 0 & n > 0 \text{ even} \end{cases} \qquad \mathrm{SDIM}_n^+(q) = \begin{cases} \frac{1}{2}q(q-1) - 1 & n = 1 \\ \mathrm{IM}_{2n}(q) & n > 1 \end{cases}$$

Proof For $n = 1$, the $\mathrm{SDIM}_1^-(q) = q+1$ self-dual irreducible monic polynomials are the polynomials $x - \lambda$ with $\lambda \in \mathbf{F}_{q^2}$ such that $\lambda^{q+1} = 1$. For odd $n > 1$, Lemma 3.8.(2) shows that $\mathrm{SDIM}_n^-(q) = \frac{1}{n}\sum_{d|n}\mu(n/d)q^d = \mathrm{IM}_n(q)$, the number of irreducible polynomials of degree n over $\mathbf{F}_q$ [23, Chapter 2, Corollary], [12, Theorem 3.25].

When $n > 1$ is odd

$$\begin{aligned}\mathrm{IM}_{2n}(q) &= \frac{1}{2n}\sum_{D|2n}\mu(2n/D)q^D = \frac{1}{2n}\sum_{d|n}\mu(n/d)q^{2d} + \frac{1}{2n}\sum_{d|n}\mu(2n/d)q^d \\ &= \frac{1}{2n}\sum_{d|n}\mu(n/d)q^{2d} - \frac{1}{2n}\sum_{d|n}\mu(n/d)q^d \\ &= \frac{1}{2}(\mathrm{IM}_n(q^2) - \mathrm{IM}_n(q)) = \mathrm{SDIM}_n^+(q)\end{aligned}$$

where we use that $\mu(2k) = -\mu(k)$ for odd $k \geq 1$. When $n > 0$ is even

$$\begin{aligned}\mathrm{IM}_{2n}(q) &= \frac{1}{2n}\sum_{D|2n}\mu(2n/D)q^D = \frac{1}{2n}\sum_{d|n}\mu(n/d)q^{2d} + \frac{1}{2n}\sum_{\substack{d|n \\ d\text{odd}}}\mu(2n/d)q^d \\ &= \frac{1}{2n}\sum_{d|n}\mu(n/d)q^{2d} = \frac{1}{2}\,\mathrm{IM}_n(q^2) = \mathrm{SDIM}_n^+(q)\end{aligned}$$

where we use that an even divisor of $2n$ has the form $2d$ for a divisor d of n, an odd divisor of $2n$ is a divisor of n, and $\mu(2k) = 0$ even $k \geq 2$. □

Corollary 3.10 $\sum_{d|n} d\,\mathrm{SDIM}_d^-(q) = q^{n/n_2} + 1$ *and* $\sum_{d|n} d\,\mathrm{SDIM}_d^+(q) = \frac{1}{2}(q^{2n} - q^{n/n_2}) - 1$ *for any natural number* $n \geq 1$.

Proof To get the first equation,

$$\sum_{d|n} d\,\mathrm{SDIM}_d^-(q) = 2 + \sum_{d|n/n_2} d\,\mathrm{IM}_d(q) = 2 + q^{n/n_2} - 1 = q^{n/n_2} + 1$$

we use Corollary 3.9 and [16, Corollary 3.21]. The second equation,

$$\begin{aligned}\sum_{d|n} d\,\mathrm{SDIM}_d^+(q) &= \frac{1}{2}(\sum_{d|n} d\,\mathrm{IM}_d(q^2) - \sum_{d|n} d\,\mathrm{SDIM}_d^-(q)) \\ &= \frac{1}{2}(q^{2n} - 1 - q^{n/n_2} - 1) = \frac{1}{2}(q^{2n} - q^{n/n_2}) - 1\end{aligned}$$

follows because $\mathrm{SDIM}_d^+(q) = \frac{1}{2}(\mathrm{IM}_d(q^2) - \mathrm{SDIM}_d^-(q))$ (Definition 3.6). □

4 Equivariant reduced Euler characteristics of products

This short section establishes a multiplicative property of equivariant Euler characteristics for use in the proof of the crucial Lemma 5.3.

Lemma 4.1 $-\widetilde{\chi}(P_1 * \cdots * P_t) = \prod_{1\le i\le t} -\widetilde{\chi}(P_i)$ *for finitely many finite posets* $P_1, \ldots, P_t$.

Proof The join $P * Q$, of the finite posets P and Q, is the poset $P \coprod Q$ where all elements of P are $<$ all elements of Q. The n-simplices of the join are n-simplices of P, i simplices of P joined to j-simplices of Q where $i + j = n - 1$, and n-simplices of Q. Alternatively, when we regard a poset as having a single cell $\emptyset$ in degree -1, the n-simplices of the join are all i-simplices of P joined to all j-simplices of Q where $i + j = n - 1$. In other words $c_n(P * Q) = \sum_{i+j=n-1} c_i(P)c_j(Q)$, where c_n stands for the number of n-simplices. The reduced Euler characteristic of the join is

$$
\begin{aligned}
-\widetilde{\chi}(P * Q) &= \sum_{n\ge -1} (-1)^{n-1} c_n(P * Q) = \sum_{n\ge -1} \sum_{i+j=n-1} (-1)^i c_i(P)(-1)^j c_j(Q) \\
&= \sum_{i\ge -1} c_i(P) \sum_{j\ge -1} c_j(Q) \\
&= \widetilde{\chi}(P)\widetilde{\chi}(Q) = (-\widetilde{\chi}(P))(-\widetilde{\chi}(Q))
\end{aligned}
$$

Proceeding by induction we get the formula for the reduced Euler characteristic of finite joins of finite posets. □

For a finite poset P with a least element $\widehat{0}$, let $P^* = P - \{\widehat{0}\}$ be the induced subposet obtained by removing $\widehat{0}$ from P. Let G_i be finite groups and P_i finite G_i-posets with least elements indexed by the finite set I. The product poset $\prod_{i\in I} P_i$ is a finite $\prod_{i\in I} G_i$-poset with a least element.

Lemma 4.2 *The classical and the equivariant Euler characteristics of the* $\prod_{i\in I} G_i$-*poset* $\left(\prod_{i\in I} P_i\right)^*$ *are given by*

$$
-\widetilde{\chi}\Big(\big(\prod_{i\in I} P_i\big)^*\Big) = \prod_{i\in I} -\widetilde{\chi}(P_i^*), \qquad -\widetilde{\chi}_r\Big(\big(\prod_{i\in I} P_i\big)^*, \prod_{i\in I} G_i\Big) = \prod_{i\in I} -\widetilde{\chi}_r(P_i^*, G_i)
$$

where $r \ge 1$.

Proof If P_1 and P_2 are finite posets with least elements then Lemma 4.1 implies $-\widetilde{\chi}((P_1 \times P_2)^*) = (-\widetilde{\chi}(P_1^*))(-\widetilde{\chi}(P_2^*))$ because $(P_1 \times P_2)^* = (P_1 \times P_2)_{>(\widehat{0},\widehat{0})} = (P_1)_{>0} * (P_2)_{>0} = P_1^* * P_2^*$ by [22, Proposition 1.9], [31, Theorem 5.1.(c)]. The general formula for the classical Euler characteristic follows by induction over the cardinality of the index set I. Proceed exactly as in [21, Lemma 2.3] to obtain the formula for the equivariant Euler characteristics □

5 Semisimple classes of the general unitary group

Conjugacy classes in the general linear group $\mathrm{GL}_n^+(\mathbf{F}_q)$ or the general unitary group $\mathrm{GL}_n^-(\mathbf{F}_q)$ are classified by functions from the set of irreducible polynomials in $\mathbf{F}_q[x]$ or $\mathbf{F}_{q^2}[x]$ to the set of partitions [5] [4, §2.1, §2.2], [8, Proposition 1A].

An element of $\mathrm{GL}_n^-(\mathbf{F}_q)$ is *semisimple* if it is diagonalisable over the algebraic closure of $\mathbf{F}_q$ [6, §1.4]. Alternatively, the semisimple elements of $\mathrm{GL}_n^-(\mathbf{F}_q)$ are precisely the q-regular elements (the elements of order prime to q); see [28, §2]. A *semisimple* or *q-regular* class in $\mathrm{GL}_n^-(\mathbf{F}_q)$ is the conjugacy class of a semisimple (= q-regular) element.

Corollary 5.1 $\mathrm{GL}_n^{\pm}(\mathbf{F}_q)$ *contains exactly* $q^n \mp q^{n-1}$ *semisimple classes for any* $n \geq 1$. *Two semisimple elements of* $\mathrm{GL}_n^{\pm}(\mathbf{F}_q)$ *are conjugate if and only if their characteristic polynomials are identical.*

Proof The number of semisimple classes is given by a general result of Steinberg [6, Theorem 3.7.6]. The second statement is an immediate consequence of the classification of q-regular classes in $\mathrm{GL}_n^-(\mathbf{F}_q)$ mentioned above. □

For a G-poset Π, let $\sim$ be the equivalence relation between G-poset endomorphisms of Π generated by the relation $f_0 \sim f_1$ if $f_0(x) \leq f_1(x)$ for all $x \in \Pi$. We say that Π is *G-poset contractible* if there is a G-fixed point x_0 in Π such that $1_\Pi \sim x_0$ where 1_Π is the identity map of Π and x_0 is the constant map with value x_0 [21, §2]. If Π is G-poset contractible then any subposet $C_\Pi(X)$ fixed by a subset X of G is poset contractible.

Lemma 5.2 *For* $n > 1$, *the poset* $C_{\mathrm{L}_n^-(\mathbf{F}_q)^*}(g)$ *is* $C_{\mathrm{GL}_n^-(\mathbf{F}_q)}(g)$*-poset contractible unless* $g \in \mathrm{GL}_n^-(\mathbf{F}_q)$ *is semisimple.*

Proof This is proved in [34, §4] once we recall Quillen's identification [22] of $\mathrm{L}_n^-(\mathbf{F}_q)^*$ with the Brown poset of nontrivial s-subgroups of $\mathrm{GL}_n^-(\mathbf{F}_q)$ where s is the characteristic of $\mathbf{F}_q$. □

The next lemma facilitates a recursive approach to the equivariant Euler characteristics $\widetilde{\chi}_r(\mathrm{GL}_n^-(\mathbf{F}_q))$. The characteristic polynomial of any unitary automorphism is self-dual by Lemma 3.4 and thus admits an essentially unique factorisation of the form $\prod r_i^{m_i^-} \times \prod_j (s_j\bar{s}_j)^{m_j^+}$ where the r_i are distinct self-dual irreducible monic polynomials and the s_j are distinct non-self-dual irreducible monic polynomials. ($[\mathrm{GL}_n^-(\mathbf{F}_q)]$ denotes the set of conjugacy classes in $\mathrm{GL}_n^-(\mathbf{F}_q)$.)

Lemma 5.3 *For* $n > 1$ *and* $r \geq 1$, *the* $(r+1)$*th equivariant Euler characteristic of the* $\mathrm{GL}_n^-(\mathbf{F}_q)$*-poset* $\mathrm{L}_n^-(\mathbf{F}_q)^*$ *is*

$$\widetilde{\chi}_{r+1}(\mathrm{GL}_n^-(\mathbf{F}_q)) = \sum_{\substack{[g]\in[\mathrm{GL}_n^-(\mathbf{F}_q)] \\ \mathrm{GCD}(q,|g|,=)1}} \widetilde{\chi}_r(C_{\mathrm{L}_n^-(\mathbf{F}_q)^*}(g), C_{\mathrm{GL}_n^-(\mathbf{F}_q)}(g))$$

where the contribution from the semisimple class g *with characteristic polynomial* $\prod r_i^{m_i^-} \times \prod_j (s_j\bar{s}_j)^{m_j^+}$ *is given by*

$$-\widetilde{\chi}_r(C_{\mathrm{L}_n^-(\mathbf{F}_q)^*}(g), C_{\mathrm{GL}_n^-(\mathbf{F}_q)}(g)) = \prod_i -\widetilde{\chi}_r(\mathrm{GL}^-_{m_i^-}(\mathbf{F}_{q^{d_i^-}})) \times \prod_j +\widetilde{\chi}_r(\mathrm{GL}^+_{m_j^+}(\mathbf{F}_{q^{2d_j^+}}))$$

for $\deg r_i = d_i^-$, $\deg s_j = d_j^+$ *and* $\sum_i m_i^- d_i^- + \sum_j 2m_j^+ d_j^+ = n$.

Proof View the n-dimensional unitary geometry V as an $\mathbf{F}_{q^2}[x]$-module via the action of g. Since g is semisimple the $\mathbf{F}_{q^2}[x]$-module V is

$$\begin{aligned} V &= \bigoplus_{r_i=\bar{r}_i} \ker(r_i(g)) \oplus \bigoplus_{s_j\neq\bar{s}_j} \ker(s_j(g)) \oplus \ker(\bar{s}_j(g)) \\ &= \bigoplus_{r_i=\bar{r}_i} (\mathbf{F}_{q^2}[x]/(r_i(x)))^{m_i^-} \oplus \bigoplus_{s_j\neq\bar{s}_j} (\mathbf{F}_{q^2}[x]/(s_j(x)) \oplus \mathbf{F}_{q^2}[x]/(\bar{s}_j(x)))^{m_j^+} \end{aligned}$$

The direct summands, $\ker r_i(g)$ and $\ker(s_j(g)) \oplus \ker(\bar{s}_j(g))$, in this decomposition of V are pairwise orthogonal. For example, let r_{i_1} and r_{i_2} be two distinct self-dual irreducible factors of the characteristic polynomial. For $v_1 \in \ker(r_{i_1}(g))$ and $v_2 \in \ker(r_{i_2}(g))$, the inner products $\langle r_{i_2}^{m_{i_2}^-}(g)v_1, v_2\rangle$ and $\langle g^{d_{i_2}^- m_{i_2}^-} v_1, r_{i_2}^{m_{i_2}^-}(g)v_2\rangle = 0$ agree up to a nonzero scalar by (3.1). Since $r_{i_2}^{m_{i_2}^-}(g)$ defines an automorphism of $\ker(r_{i_1}(g))$, this shows that $\ker(r_{i_1}(g)) \perp \ker(r_{i_2}(g))$. Similarly, $\ker(s_i(g)) \perp (\ker(r_j(g)) \oplus \ker(\bar{r}_j(g)))$ and $\ker(r_{j_1}(g)) \perp (\ker(r_{j_2}(g)) \oplus \ker(\bar{r}_{j_2}(g)))$ for distinct factors r_{j_1} and r_{j_2}. Thus all summands $\ker r_i(g)$ and $\ker(s_j(g)) \oplus \ker(\bar{s}_j(g))$ are non-degenerate unitary geometries.

The centraliser of g in the general unitary group of V is the group [32], [8, Proposition 1A] [4, Lemma 2.3], [30, Lemma 3.3]

$$C_{\mathrm{GL}^-(V)}(g) = \prod_i \mathrm{GL}^-_{m_i^-}(\mathbf{F}_{q^{d_i^-}}) \times \prod_j \mathrm{GL}^+_{m_j^+}(\mathbf{F}_{q^{2d_j^+}})$$

of unitary $\mathbf{F}_{q^2}[x]$-automorphisms and the centraliser of g in the poset of totally isotropic subspaces of V is the poset

$$C_{L^-(V)}(g) = \prod_i L^-(\ker r_i(g)) \times \prod_j L^-(\ker(s_j(g)) \oplus \ker(\bar{s}_j(g)))$$

of totally isotropic $\mathbf{F}_{q^2}[x]$-subspaces. The representation of $\mathrm{GL}^-_{m_i^-}(\mathbf{F}_{q^{d_i^-}})$ in $\ker(s_i(g)) \cong \mathbf{F}^{m_i^-}_{q^{2d_i^-}}$ is standard. We now turn to the representation of $\mathrm{GL}^+_{m_j^+}(\mathbf{F}_{q^{2d_j^+}})$ in $\ker(s_j(g)) \oplus \ker(\bar{s}_j(g)) \cong (\mathbf{F}_{q^{2d_j^+}} \oplus \mathbf{F}_{q^{2d_j^+}})^{m_j^+} = \mathbf{F}^{2m_j^+}_{q^{2d_j^+}}$ described in [8, §1, p 112, 1])

The Kleidman–Liebeck Theorem [14] [35, Theorem 3.9] lists certain natural subgroups of the general unitary groups. The unitary $2m$-geometry $V_{2m}(\mathbf{F}_{q^2})$ over $\mathbf{F}_{q^2}$ has a basis $e_1, \dots, e_m, f_1, \dots, f_m$ such that $\langle e_i, f_i\rangle = 1$, $1 \le i \le m$, are the only nonzero Hermitian inner products between the basis vectors [14, Proposition 2.3.2]. Write $V_{2m}(\mathbf{F}_{q^2}) = V_1 \oplus V_2$ as the direct sum of the two maximal totally isotropic subspaces V_1 and V_2 spanned by $e_1, \dots, e_m$, and $f_1, \dots, f_m$, respectively. The representation of $\mathrm{GL}^+_m(\mathbf{F}_{q^2})$ in $\mathrm{GL}^-_{2m}(\mathbf{F}_q)$ given by

$$\mathrm{GL}^+_m(\mathbf{F}_{q^2}) \ni A \to \begin{pmatrix} A & 0 \\ 0 & A^{-1\alpha t} \end{pmatrix} \in \mathrm{GL}^-_{2m}(\mathbf{F}_q)$$

stabilises the direct sum decomposition $V = V_1 \oplus V_2$ [14, Lemma 4.1.9, Table 4.2.A, Lemma 4.2.3]. (The matrix $A^{-1\alpha t}$ is the conjugate-transpose of the inverse of A so that $\langle Av_1, A^{-1\alpha t}v_2\rangle = \langle A^{-1}Av_1, v_2\rangle = \langle v_1, v_2\rangle$ for $v_1 \in V_1$, $v_2 \in V_2$.) The stabiliser of g in the poset of totally isotropic subspaces of $V_{2m}(\mathbf{F}_{q^2})$ is the $\mathrm{GL}^+_m(\mathbf{F}_q)$-poset of pairs of orthogonal subspaces

$$\Sigma\, \mathrm{L}^+_m(\mathbf{F}_{q^2}) = \{(U_1, U_2) \mid U_1 \le V_1, U_2 \le V_2, U_1 \perp U_2\}$$

The subposet $\Sigma\, \mathrm{L}^+_m(\mathbf{F}_{q^2})^*$, obtained from $\Sigma\, \mathrm{L}^+_m(\mathbf{F}_{q^2})$ by removing the pair $(0, 0)$, is $\mathrm{GL}^+_m(\mathbf{F}_q)$-homotopy equivalent to the suspension [31, §3] of $\mathrm{L}^+_m(\mathbf{F}_{q^2})^*$: Let $\{1, 2\}$ be the discrete poset of two incomparable points. The two $\mathrm{GL}^+_m(\mathbf{F}_q)$-poset morphisms

$$\{1, 2\} * \mathrm{L}^+_m(\mathbf{F}_{q^2})^* \underset{g}{\overset{f}{\rightleftarrows}} \Sigma\, \mathrm{L}^+_m(\mathbf{F}_{q^2})^*$$

given by $f(1, U) = (U, 0)$, $f(2, U) = (0, U)$, and

$$g(U_1, U_2) = \begin{cases} (1, U_1) & U_1 \neq 0 \\ (2, U_2) & U_1 = 0 \end{cases}$$

are homotopy equivalences as gf is the identity of the suspension of $\mathrm{L}^+_m(\mathbf{F}_{q^2})^*$ and fg is homotopic to the identity of $\Sigma\, \mathrm{L}^+_m(\mathbf{F}_{q^2})^*$ as $fg(U_1, U_2) \le (U_1, U_2)$. By the product formula in Lemma 4.1,

$$\begin{aligned} &-\widetilde{\chi}_r(\Sigma\, \mathrm{L}^+_m(\mathbf{F}_{q^2})^*, \mathrm{GL}^+_m(\mathbf{F}_{q^2})) \\ &\quad = -\widetilde{\chi}_r(\{1, 2\} * \mathrm{L}^+_m(\mathbf{F}_{q^2})^*, \mathrm{GL}^+_m(\mathbf{F}_{q^2})) = \widetilde{\chi}_r(\mathrm{L}^+_m(\mathbf{F}_{q^2})^*, \mathrm{GL}^+_m(\mathbf{F}_{q^2})) \end{aligned}$$

and the formula of the lemma is a consequence of the product formula in Lemma 4.2. □

Observe that the contribution of a q-regular class depends only on the multiplicities and degrees of the irreducible factors of its characteristic polynomial.

6 Proofs of Theorem 1.2 and Corollary 1.3

We use Lemma 5.3 in an inductive computation of the generating functions (1.2). The next proposition gives the start of the induction.

Proposition 6.1 *Suppose that $r = 1$ or $n = 1$.*

(1) When $r = 1$, $-\widetilde{\chi}_1(\mathrm{GL}_n^-(\mathbf{F}_q)) = \delta_{1,n}$ *is* 1 *for* $n = 1$ *and* 0 *for all* $n > 1$.
(2) When $n = 1$, $-\widetilde{\chi}_r(\mathrm{GL}_1^-(\mathbf{F}_q)) = (q+1)^{r-1}$ *for all* $r \geq 1$.

Proof When $n = 1$, $\mathrm{L}_1^-(\mathbf{F}_q)^* = \emptyset$ is empty. Since $\widetilde{\chi}(\emptyset) = -1$, the rth equivariant Euler characteristic is

$$-\widetilde{\chi}_r(\mathrm{GL}_1^-(\mathbf{F}_q)) = |\operatorname{Hom}(\mathbf{Z}^r, \mathrm{GL}_1^-(\mathbf{F}_q))|/|\mathrm{GL}_1^-(\mathbf{F}_q)| \\ = |\mathrm{GL}_1^-(\mathbf{F}_q)|^{r-1} = (q+1)^{r-1}$$

for all $r \geq 1$.

The first equivariant reduced Euler characteristic of $\widetilde{\chi}_1(\mathrm{GL}_n^-(\mathbf{F}_q))$, where $n > 1$, is the classical Euler characteristic of the orbit space $\mathrm{BL}_n^-(\mathbf{F}_q)^*/\mathrm{GL}_n^-(\mathbf{F}_q)$ for the $\mathrm{GL}_n^-(\mathbf{F}_q)$-action on the building, the classifying space of the poset $\mathrm{L}_n^-(\mathbf{F}_q)^*$ [19, Proposition 2.13]. According to Quillen we can replace $\mathrm{L}_n^-(\mathbf{F}_q)^*$ by the Brown poset $\mathcal{S}^{s+*}_{\mathrm{GL}_n^-(\mathbf{F}_q)}$ of nontrivial s-subgroups of $\mathrm{GL}_n^-(\mathbf{F}_q)$ [22], Theorem 3.1]. Webb's theorem [34, Proposition 8.2.(i)] applies to this replacement showing $\widetilde{\chi}_1(\mathrm{GL}_n^-(\mathbf{F}_q)) = \widetilde{\chi}\big(\mathrm{BL}_n^-(\mathbf{F}_q)^*/\mathrm{GL}_n^-(\mathbf{F}_q)\big) = 0$. □

Lemma 5.3 can be reformulated succinctly as the recurrence

$$\mathrm{FGL}_{r+1}^-(q,x) = T_{\mathrm{SDIM}^-(q)}(\mathrm{FGL}_r^-(q,x))T_{\mathrm{SDIM}^+(q)}(\mathrm{FGL}_r^+(q^2,x^2)) \tag{6.1}$$

using the power series transform from [21, Definition 3.1] reviewed in Sect. 7 below.

Corollary 6.2 *The following identities hold*

$$T_{\mathrm{SDIM}^-(q)}(1-x)T_{\mathrm{SDIM}^+(q)}(1-x^2) = \frac{1-qx}{1+x} \qquad T_{\mathrm{SDIM}^-(q)}(1+x)T_{\mathrm{SDIM}^+(q)} \\ \times(1-x^2) \\ = \frac{1+qx}{1-x} \\ T_{\mathrm{SDIM}^-(q)}\left(\frac{1+x}{1-x}\right) = \frac{(1+x)(1+qx)}{(1-x)(1-qx)}$$

Proof For the first identity, note that

$$T_{\mathrm{SDIM}^-(q)}(1-x)^{-1}T_{\mathrm{SDIM}^+(q)}(1-x^2)^{-1} = 1 + \sum_{n\geq 1}(q^n + q^{n-1})x^n \\ = 1 + \sum_{n\geq 1}(qx)^n + x\sum_{n\geq 1}(qx)^{n-1} = \frac{1+x}{1-qx}$$

since the coefficient of x^n in this power series is the number of self-dual monic polynomials in $\mathbf{F}_{q^2}[x]$ determined in Corollary 3.5. (An alternative proof,

$$T_{\mathrm{SDIM}^-(q)}(1-x)T_{\mathrm{SDIM}^+(q)}(1-x^2) = \prod_{d\geq 1}(1-x^d)^{\mathrm{SDIM}_d^-(q)}\prod_{d\geq 1}(1-x^{2d})^{\mathrm{SDIM}_d^+(q)}$$

$$= (1-x)^2 \prod_{\substack{d\geq 1\\ d\text{odd}}} (1-x^d)^{\mathrm{IM}_d(q)} (1-x^2)^{-1} \prod_{\substack{d\geq 2\\ d\text{even}}} (1-x^d)^{\mathrm{IM}_d(q)}$$

$$= \frac{1-x}{1+x} \prod_{d\geq 1} (1-x^d)^{\mathrm{IM}_d(q)} \overset{(3.9)}{=} \frac{1-qx}{1+x}$$

follows from Corollary 3.9.) Since $\mathrm{SDIM}_d^-(q)$ is nonzero only for odd d (Proposition 3.7),

$$T_{\mathrm{SDIM}^-(q)}(1+x) = \prod_{d\geq 1} (1+x^d)^{\mathrm{SDIM}_d^-(q)} = \prod_{d\geq 1} (1-(-x)^d)^{\mathrm{SDIM}_d^-(q)}$$

is the $\mathrm{SDIM}^-(q)$-transform of $1-x$ evaluated at $-x$. Obviously, the $\mathrm{SDIM}^+(q)$-transform of $1-x^2$ is an even function of x. Thus $T_{\mathrm{SDIM}^-(q)}(1+x)T_{\mathrm{SDIM}^+(q)}(1-x^2)$ is $T_{\mathrm{SDIM}^-(q)}(1-x)T_{\mathrm{SDIM}^+(q)}(1-x^2)$ evaluated at $-x$. This proves the second identity. The third identity is simply the quotient of the first two. □

Proof of Theorem 1.2 The first generating function (6.3) is $\mathrm{FGL}_1^-(q,x) = 1+x$ by Proposition 6.1.(1). Assume the formula of Theorem 1.2 holds for some $r \geq 1$. Using a consequence of Corollary 6.2,

$$T_{\mathrm{SDIM}^-(q)}(1 \pm q^j x) T_{\mathrm{SDIM}^+(q)}(1-q^{2j}x^2) = \frac{1 \pm q^{j+1}x}{1 \mp q^j x}$$

which follows from the multiplicative property of these power series transforms [21, (3.2)], and recursion (6.1), the computation

$$\begin{aligned}
&\mathrm{FGL}_{r+1}^-(q,x)\\
&= T_{\mathrm{SDIM}^-(q)}(\mathrm{FGL}_r^-(q,x)) T_{\mathrm{SDIM}^+(q)}(\mathrm{FGL}_r^+(q^2,x^2))\\
&= \frac{T_{\mathrm{SDIM}^-(q)}(\prod_{j\equiv r \bmod 2}(1+q^j x)^{\binom{r}{j}})}{T_{\mathrm{SDIM}^-(q)}(\prod_{j\not\equiv r \bmod 2}(1-q^j x)^{\binom{r}{j}})} \frac{T_{\mathrm{SDIM}^+(q)}(\prod_{j\equiv r \bmod 2}(1-q^{2j}x^2)^{\binom{r}{j}})}{T_{\mathrm{SDIM}^+(q)}(\prod_{j\not\equiv r \bmod 2}(1-q^{2j}x^2)^{\binom{r}{j}})}\\
&= \frac{\prod_{j\equiv r \bmod 2}(1+q^{j+1}x)^{\binom{r}{j}}}{\prod_{j\equiv r \bmod 2}(1-q^j x)^{\binom{r}{j}}} \frac{\prod_{j\not\equiv r \bmod 2}(1+q^j x)^{\binom{r}{j}}}{\prod_{j\not\equiv r \bmod 2}(1-q^{j+1}x)^{\binom{r}{j}}}\\
&= \frac{\prod_{j\equiv r+1 \bmod 2}(1+q^j x)^{\binom{r}{j-1}}}{\prod_{j\not\equiv r+1 \bmod 2}(1-q^j x)^{\binom{r}{j}}} \frac{\prod_{j\equiv r+1 \bmod 2}(1+q^j x)^{\binom{r}{j}}}{\prod_{j\not\equiv r+1 \bmod 2}(1-q^j x)^{\binom{r}{j-1}}}\\
&= \frac{\prod_{j\equiv r+1 \bmod 2}(1+q^j x)^{\binom{r+1}{j}}}{\prod_{j\not\equiv r+1 \bmod 2}(1-q^j x)^{\binom{r+1}{j}}}
\end{aligned}$$

shows that the formula holds also for $r+1$. □

Proof of Corollary 1.3 The logarithm of the $(r+1)$th generating function $\mathrm{FGL}^-_{r+1}(q,x)$ is

$$\begin{aligned}
&\log \mathrm{FGL}^-_{r+1}(q,x)\\
&= \sum_{0\le j\le r} (-1)^{r-j}\binom{r}{j}\log(1+(-1)^{r-j}q^jx)\\
&= \sum_{0\le j\le r} (-1)^{r-j}\binom{r}{j}\sum_{n\ge 1}(-1)^{n+1}(-1)^{n(r-j)}q^{nj}\frac{x^n}{n}\\
&= \sum_{n\ge 1}(-1)^{n+1}\sum_{0\le j\le r}\binom{r}{j}(-1)^{(n+1)(r-j)}q^{nj}\frac{x^n}{n}\\
&= \sum_{n\ge 1}(-1)^{n+1}(q^n+(-1)^{n+1})^r\frac{x^n}{n}\\
&= -\sum_{n\ge 1}(-1)^{n}(q^n-(-1)^{n})^r\frac{x^n}{n}
\end{aligned}$$

□

Remark 6.3 (The Knörr-Robinson conjecture) The (non-block-wise form of the) the Knörr-Robinson conjecture for the general unitary group $\mathrm{GL}^-_n(\mathbf{F}_q)$ relative to the characteristic s of $\mathbf{F}_q$ asserts that [15,29] [22, Theorem 3.1]

$$-\widetilde{\chi}_2(\mathrm{GL}^-_n(\mathbf{F}_q)) = z_s(\mathrm{GL}^-_n(\mathbf{F}_q))$$

where $z_s(\mathrm{GL}^-_n(\mathbf{F}_q)) = |\{\chi \in \mathrm{Irr}_{\mathbf{C}}(\mathrm{GL}^-_n(\mathbf{F}_q)) \mid |\mathrm{GL}^-_n(\mathbf{F}_q)|_s \mid \chi(1)\}|$ is the number of irreducible complex representations of $\mathrm{GL}^-_n(\mathbf{F}_q)$ of s-defect 0 [13, p. 134]. As $\mathrm{FGL}^-_2(q,x) = \frac{1+qx}{1-x}$, the left side is $q+1$ and so is the right side [11, Remark, p. 69]. This confirms the Knörr–Robinson conjecture for $\mathrm{GL}^-_n(\mathbf{F}_q)$ relative to the defining characteristic.

6.1 Alternative presentations of the equivariant reduced Euler characteristics

The binomial formula applied to the right hand side of Theorem 1.2 gives the more direct expression

$$-\widetilde{\chi}_{r+1}(\mathrm{GL}^-_n(\mathbf{F}_q)) = \sum_{n_0+\cdots+n_r=n}\prod_{0\le j\le r}(-1)^{jn_j}\binom{(-1)^j\binom{r}{j}}{n_j}q^{n_j(r-j)} \tag{6.2}$$

where the sum ranges over all $\binom{n+r}{n}$ weak compositions of n into $r+1$ parts [26, p. 15]. This is also a consequence of [21, Corollary 3.10] and 'Ennola duality',

$$\mathrm{FGL}^-_r(q,x) = \mathrm{FGL}^+_r(-q,(-1)^rx), \qquad r\ge 1 \tag{6.3}$$

which follows by comparing the expressions of [21, Theorem 1.4] and Theorem 1.2.

We shall next relate the equivariant Euler characteristics more directly to the structure of the general linear and unitary groups. Recall that a (finite) multiset λ is a (finite) base set $B(\lambda)$ with a multiplicity function assigning a natural number $E(\lambda, b)$ to all $b \in B(\lambda)$. Representing the multiset as $\lambda = \{b^{E(\lambda,b)} \mid b \in B(\lambda)\}$ and assuming the base $B(\lambda)$ consists of natural numbers, we let

$$|\lambda| = \sum_{b\in B(\lambda)} E(\lambda, b) \qquad n(\lambda) = \sum_{b\in B(\lambda)} bE(\lambda, b)$$

$$T(\lambda) = \frac{n(\lambda)!}{\prod_{b\in B(\lambda)} E(\lambda, b)! b^{E(\lambda,b)}} \qquad U(\lambda, q) = \prod_{b\in B(\lambda)} (q^b - 1)^{E(\lambda,b)}$$

so that $|\lambda|$ is the cardinality or number of parts of λ, λ partitions n, $\lambda \vdash n$, if $n(\lambda) = n$, $T(\lambda)$ is the number of elements in the symmetric group $\Sigma_{n(\lambda)}$ of cycle type λ [24, Proposition 1.1.1], and $U(\lambda, q)$ is an integral polynomial in q. With this notation, the coefficients of x^n in the power series of [21, Corollary 1.5] and Corollary 1.3 are

$$\begin{aligned} \widetilde{\chi}_{r+1}(\mathrm{GL}_n^+(\mathbf{F}_q)) &= \frac{1}{n!}\sum_{\lambda\vdash n}(-1)^{|\lambda|}T(\lambda)U(\lambda, q)^r, \\ -\widetilde{\chi}_{r+1}(\mathrm{GL}_n^-(\mathbf{F}_q)) &= (-1)^{n(r+1)}\frac{1}{n!}\sum_{\lambda\vdash n}(-1)^{|\lambda|}T(\lambda)U(\lambda, -q)^r \end{aligned} \tag{6.4}$$

with summation over all partitions λ of n.

Let F_q denote the standard Frobenius endomorphism of the algebraic group $\mathrm{GL}_n(\overline{\mathbf{F}}_s)$, $s = \mathrm{char}(\mathbf{F}_q)$, with fixed points $\mathrm{GL}_n(\overline{\mathbf{F}}_s)^{F_q} = \mathrm{GL}_n^+(\mathbf{F}_q)$. The standard maximal torus $T_n(\overline{\mathbf{F}}_s) \cong \overline{\mathbf{F}}_s^\times \times \cdots \times \overline{\mathbf{F}}_s^\times$ consisting of the diagonal matrices in $\mathrm{GL}_n(\overline{\mathbf{F}}_s)$ is maximally split with respect to F_q [17, Definition 21.13, Example 21.14]. The Weyl group W_n of $T_n(\overline{\mathbf{F}}_s)$ acts as the standard permutation representation of the symmetric group Σ_n in the n-dimensional real vector space $X(T_n(\overline{\mathbf{F}}_s)) \otimes \mathbf{R}$ spanned by the character group $X(T_n(\overline{\mathbf{F}}_s))$. As usual, $T_n(\overline{\mathbf{F}}_s)_w$ denotes the F_q-stable maximal torus of $\mathrm{GL}_n(\overline{\mathbf{F}}_s)$ corresponding to the Weyl group element $w \in W_n$ [17, Proposition 25.1].

Let σ be the graph automorphism of $\mathrm{GL}_n(\overline{\mathbf{F}}_s)$ given by $\sigma(M) = A^{-1}(M^t)^{-1}A$, $M \in \mathrm{GL}_n(\overline{\mathbf{F}}_s)$, where A is the involutory permutation $A(i) = n + 1 - i$, $1 \leq i \leq n$. The fixed points for the Steinberg endomorphism $F_q\sigma$ are $\mathrm{GL}_n(\overline{\mathbf{F}}_s)^{F_q\sigma} = \mathrm{GL}_n^-(\mathbf{F}_q)$, $T_n(\overline{\mathbf{F}}_s)$ is a maximally split torus also with respect to $F_q\sigma$, and σ acts on $X(T_n(\overline{\mathbf{F}}_s))\otimes\mathbf{R}$ as $-A$ [17, Examples 21.14.(2), 22.11.(2)].

Proposition 6.4 *The equivariant Euler characteristics of the* $\mathrm{GL}_n^\pm(\mathbf{F}_q)$*-posets* $\mathrm{L}_n^\pm(\mathbf{F}_q)^*$, $n \geq 1$, *are*

$$\begin{aligned} \widetilde{\chi}_{r+1}(\mathrm{GL}_n^+(\mathbf{F}_q)) &= \frac{(-1)^n}{|W_n|}\sum_{w\in W_n} \det(w)|T_n(\overline{\mathbf{F}}_s)_w^{F_q}|^r \\ &= \frac{(-1)^n}{|W_n|}\sum_{w\in W_n} \det(w)\det(q - w)^r \end{aligned}$$

$$
\begin{aligned}
-\widetilde{\chi}_{r+1}(\mathrm{GL}_n^-(\mathbf{F}_q)) &= \frac{(-1)^{\binom{n}{2}}}{|W_n|} \sum_{w \in W_n} \det(w) |T_n(\overline{\mathbf{F}}_s)_w^{F_q \sigma}|^r \\
&= \frac{1}{|W_n|} \sum_{w \in W_n} \det(w) \det(q+w)^r
\end{aligned}
$$

Proof The number of elements of $T_n(\overline{\mathbf{F}}_s)_w$ that are fixed by the Frobenius endomorphism F_q is

$$
|T_n(\overline{\mathbf{F}}_s)_w^{F_q}| = U(\lambda(w), q) = \det(q - w^{-1})
$$

where $\lambda(w)$ is the cycle type of the permutation w and determinants are computed in the real vector space $X(T_n(\overline{\mathbf{F}}_s)) \otimes \mathbf{R}$ [17, Proposition 25.3, Example 25.4]. Equation (6.4) now takes the form

$$
\begin{aligned}
&\widetilde{\chi}_{r+1}(\mathrm{GL}_n^+(\mathbf{F}_q)) \\
&\quad = \frac{(-1)^n}{|W_n|} \sum_{w \in W_n} \det(w) U(\lambda(w), q)^r \\
&\quad = \frac{(-1)^n}{|W_n|} \sum_{w \in W_n} \det(w) |T_n(\overline{\mathbf{F}}_s)_w^{F_q}|^r = \frac{(-1)^n}{|W_n|} \sum_{w \in W_n} \det(w) \det(q - w^{-1})^r \\
&\quad = \frac{(-1)^n}{|W_n|} \sum_{w \in W_n} \det(w) \det(q - w)^r
\end{aligned}
$$

since $(-1)^{\lambda(w)} = (-1)^n \det(w)$ and $\det(w) = \det(w^{-1})$ for all $w \in W_n$.

The number of elements of $T_n(\overline{\mathbf{F}}_s)_w$ that are fixed by Steinberg endomorphism $F_q\sigma$ is [17, Proposition 25.3.(c)]

$$
\begin{aligned}
|T_n(\overline{\mathbf{F}}_s)_w^{F_q \sigma}| &= \det(q - (-wA)^{-1}) = (-1)^n \det(-q - (wA)^{-1}) \\
&= (-1)^n U(\lambda(wA), -q)
\end{aligned}
$$

Using Ennola duality (6.3) combined with (6.4), and [17, Proposition 25.3], the calculation

$$
\begin{aligned}
&-\widetilde{\chi}_{r+1}(\mathrm{GL}_n^-(\mathbf{F}_q)) \\
&\quad = (-1)^{n(r+1)} \frac{(-1)^n}{|W_n|} \sum_{w \in W_n} \det(w) U(\lambda(w), -q)^r \\
&\quad = \frac{(-1)^{nr}}{|W_n|} \sum_{w \in W_n} \det(w) U(\lambda(w), -q)^r \\
&\quad = \frac{1}{|W_n|} \sum_{w \in W_n} \det(w) ((-1)^n U(\lambda(w), -q))^r = \frac{1}{|W_n|} \sum_{w \in W_n} \det(wA) ((-1)^n U(\lambda(wA), -q))^r
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{|W_n|}\sum_{w\in W_n}\det(wA)|T_n(\overline{\mathbf{F}}_s)_w^{F_q\sigma}|^r = \frac{1}{|W_n|}\sum_{w\in W_n}\det(wA)\det(q-(-wA)^{-1})^r \\
&= \frac{1}{|W_n|}\sum_{w\in W_n}\det(wA)\det(q+(wA)^{-1})^r = \frac{1}{|W_n|}\sum_{w\in W_n}\det(w)\det(q+w^{-1})^r \\
&= \frac{1}{|W_n|}\sum_{w\in W_n}\det(w)\det(q+w)^r
\end{aligned}
$$

finishes the proof. Here, $\det(wA) = \det(A)\det(w)$ where $\det(A) = (-1)^{\binom{n}{2}}$ is $+1$ for $n \equiv 0, 1 \bmod 4$ and -1 for $n \equiv 2, 3 \bmod 4$. □

Let $\widetilde{\chi}_{r+1}(\mathrm{GL}_n^{\pm}(\mathbf{F}_q))^{-1}$ denote the coefficient of x^n in the reciprocal power series $\mathrm{FGL}_{r+1}^{\pm}(q,x)^{-1}$. The proof of the next result is similar to that of Proposition 6.4 except that it is based on the identities

$$
\begin{aligned}
\widetilde{\chi}_{r+1}(\mathrm{GL}_n^{+}(\mathbf{F}_q))^{-1} &= \frac{1}{n!}\sum_{\lambda\vdash n} T(\lambda)U(\lambda,q)^r, \\
\widetilde{\chi}_{r+1}(\mathrm{GL}_n^{-}(\mathbf{F}_q))^{-1} &= \frac{(-1)^n}{n!}\sum_{\lambda\vdash n} T(\lambda)((-1)^n U(\lambda,-q))^r,
\end{aligned}
\tag{6.5}
$$

rather than (6.4). The right hand sides of these identities are the coefficients of x^n in the reciprocal of the power series of Corollary 1.3 and [21, Corollary 1.5].

Proposition 6.5 *The reciprocal equivariant Euler characteristics of the* $\mathrm{GL}_n^{\pm}(\mathbf{F}_q)$*-posets* $\mathrm{L}_n^{\pm}(\mathbf{F}_q)^*$*,* $n \geq 1$*, are*

$$
\begin{aligned}
\widetilde{\chi}_{r+1}(\mathrm{GL}_n^{+}(\mathbf{F}_q))^{-1} &= \frac{1}{|W_n|}\sum_{w\in W_n}|T_n(\overline{\mathbf{F}}_s)_w^{F_q}|^r = \frac{1}{|W_n|}\sum_{w\in W_n}\det(q-w)^r \\
(-1)^n\widetilde{\chi}_{r+1}(\mathrm{GL}_n^{-}(\mathbf{F}_q))^{-1} &= \frac{1}{|W_n|}\sum_{w\in W_n}|T_n(\overline{\mathbf{F}}_s)_w^{F_q\sigma}|^r = \frac{1}{|W_n|}\sum_{w\in W_n}\det(q+w)^r
\end{aligned}
$$

Again, the case $r = 1$ has special significance in that $\widetilde{\chi}_2(\mathrm{GL}_n^{+}(\mathbf{F}_q))^{-1}$ is the number of semisimple classes in $\mathrm{GL}_n^{+}(\mathbf{F}_q)$ and $(-1)^n\widetilde{\chi}_2(\mathrm{GL}_n^{-}(\mathbf{F}_q))^{-1}$ the number of semisimple classes in $\mathrm{GL}_n^{-}(\mathbf{F}_q)$ [6, Proposition 3.7.4].

Example 6.6 The polynomial identities

$$
\begin{aligned}
&\frac{1}{|W_n|}\sum_{w\in W_n}\det(1-qw) = 1-q \\
&\frac{1}{|W_n|}\sum_{w\in W_n}\det(w)\det(q-w)^2 = (-1)^{n+1}n(q-1)^2q^{n-1} \\
&\frac{1}{|W_n|}\sum_{w\in W_n}\det(1+qw) = 1+q
\end{aligned}
$$

$$\frac{1}{|W_n|}\sum_{w\in W_n}\det(w)\det(q+w)^2 = n(q+1)^2q^{n-1}$$

are the instances $r = 1, 2$ of Proposition 6.4. The right hand sides of the equations in the left column, where $r = 1$, are the negative of the number of irreducible complex representations of s-defect 0. (Indeed, $|W_n|(1+q) = -|W_n|\widetilde{\chi}_2(\mathrm{GL}_n^-(\mathbf{F}_q)) = \sum_{w\in W_n}\det(w)\det(q+w) = \sum_{w\in W_n}\det(w^{-1})\det(q+w^{-1}) = \sum_{w\in W_n}\det(w)\det(q+w^{-1}) = \sum_{w\in W_n}\det(1+qw)$.)

The polynomial identities

$$\frac{1}{|W_n|}\sum_{w\in W_n}\det(q-w) = q^n - q^{n-1} \qquad \frac{1}{|W_n|}\sum_{w\in W_n}\det(q-w)^2 = \frac{q-1}{q+1}(q^{2n}-1)$$

$$\frac{1}{|W_n|}\sum_{w\in W_n}\det(q+w) = q^n + q^{n-1} \qquad \frac{1}{|W_n|}\sum_{w\in W_n}\det(q+w)^2 = \frac{q+1}{q-1}(q^{2n}-1)$$

are the instances $r = 1, 2$ of Proposition 6.5. The right hand sides of the equations in the left column, where $r = 1$, count semisimple classes.

The next corollary, an immediate consequence of (6.4) and Proposition 6.4, lists the generating functions for the equivariant Euler characteristics $\widetilde{\chi}_{r+1}(\mathrm{GL}^{\pm}(n, \mathbf{F}_q))$, $r \geq 0$, for a fixed n. (The first part is [21, Proposition 4.19].)

Corollary 6.7 *For any fixed $n \geq 1$,*

$$\sum_{r\geq 0}\widetilde{\chi}_{r+1}(\mathrm{GL}_n^+(\mathbf{F}_q))x^r = \frac{1}{n!}\sum_{\lambda\vdash n}(-1)^{|\lambda|}\frac{T(\lambda)}{1-U(\lambda,q)x}$$
$$= \frac{(-1)^n}{|W_n|}\sum_{w\in W_n}\frac{\det(w)}{1-x\det(q-w)}$$
$$\sum_{r\geq 0}-\widetilde{\chi}_{r+1}(\mathrm{GL}_n^-(\mathbf{F}_q))x^r = \frac{(-1)^n}{n!}\sum_{\lambda\vdash n}(-1)^{|\lambda|}\frac{T(\lambda)}{1-(-1)^nU(\lambda,-q)x}$$
$$= \frac{1}{|W_n|}\sum_{w\in W_n}\frac{\det(w)}{1-x\det(q+w)}$$

For example, the power series $n!\sum_{r\geq 0}-\widetilde{\chi}_{r+1}(\mathrm{GL}_n^-(\mathbf{F}_q))x^r$ is

$$\frac{1}{1-(q+1)x}, \qquad \frac{1}{1-(q+1)^2x} - \frac{1}{1-(q^2-1)x}, \qquad \frac{1}{1-(q+1)^3x}$$
$$-\frac{3}{1-(q^2-1)(q+1)x} + \frac{2}{1-(q^3+1)x}$$

for $n = 1, 2, 3$.

6.2 Hasse–Weil zeta functions and equivariant Euler characteristics

The Hasse–Weil zeta function for a projective variety V defined over $\mathbf{F}_q$,

$$Z(V/\mathbf{F}_q; T) = \exp\Big(\sum_{n\geq 1} |V(\mathbf{F}_{q^n})| \frac{T^n}{n}\Big)$$

encodes the number of points on V over $\mathbf{F}_{q^n}$ for all $n \geq 1$ [25, V.2–V.3].

Proposition 6.8 *For any $m \geq 1$*

$$\mathrm{FGL}^-_{2m+1}(q, -T) = Z(E^m/\mathbf{F}_{q^2}; T)^{-1}$$

is the reciprocal of the Hasse–Weil zeta function of the m-fold self-product $E^m = E \times \cdots \times E$ of any supersingular elliptic curve E defined over $\mathbf{F}_{q^2}$.

Proof Let E be any supersingular elliptic curve defined over $\mathbf{F}_{q^2}$ [25, Definition, p. 145]. We note that

$$Z(E/\mathbf{F}_{q^2}; T) = \frac{(1+qT)^2}{(1-T)(1-q^2T)} = \mathrm{FGL}^-_3(q, -T)^{-1} \overset{\text{Cor.1.5}}{=} \exp\Big(\sum_{n\geq 1} (q^n - (-1)^n)^2 \frac{T^n}{n}\Big) \tag{6.6}$$

and hence

$$Z(E^m/\mathbf{F}_{q^2}; T) = \exp\Big(\sum_{n\geq 1} (q^n - (-1)^n)^{2m} \frac{T^n}{n}\Big) \overset{\text{Cor.1.5}}{=} \mathrm{FGL}^-_{2m+1}(q, -T)^{-1}$$

as $|E(\mathbf{F}_{q^2})| = (q^n - (-1)^n)^2$ by (6.6) and $E^m(\mathbf{F}_{q^{2n}}) = E(\mathbf{F}_{q^{2n}})^m$ for general reasons. □

7 Transforms of polynomial power series and polynomial identities

Let $F(q, x) = 1 + \sum_{n\geq 1} a(n)(q)x^n \in 1 + (x) \subseteq \mathbf{Q}[q][[x]]$ be a power series with leading term 1 in the power series ring over the ring of rational polynomials in q. Given a sequence $S = (S(n)(q))_{n\geq 1}$ of rational numbers defined for each prime power q, the S-transform of $F(q, x)$ is the power series [21, Definition 3.1]

$$T_S(F(q, x)) = \prod_{d\geq 1} F(q^d, x^d)^{S(d)(q)}$$

The transformation $T_S \colon 1 + (x) \to 1 + (x)$ is multiplicative in F and exponential in S the sense that

$$T_S(1) = 1, \qquad T_S(F_1(q, x)F_2(q, x)) = T_S(F_1(q, x)T_S(F_2(q, x)),$$

$$T_{mS}(F(q,x)) = T_S(F(q,x))^m$$

for all $F_1(q,x), F_2(q,x) \in 1+(x) \subseteq \mathbf{Q}[q][[x]]$ and rational numbers $m \in \mathbf{Q}$ [21, §3.2].

For example, the S-transform of $1 \pm x^k$ is easily determined by evaluating the coefficient of x^{kn} in the infinite product expansion $T_S(1 \pm x^k) = \prod_{d\geq 1}(1 \pm x^{kd})^{S(d)(q)}$. (See the beginning of Sect. 6.1 for multiset notation. We use the convention that the binomial coefficient $\binom{m}{-k} = (-1)^k\binom{m}{k}$ for all natural numbers k.)

Lemma 7.1 *For any rational number m and natural number k, the mS-transform of the power series $1 \pm x^k$ is*

$$T_{mS}(1 \pm x^k) = 1 + \sum_{n\geq 1}\Big[\sum_{\lambda\vdash n}\prod_{d\in B(\lambda)}\binom{mS(d)(q)}{\pm E(\lambda,d)}\Big]x^{kn}$$

The below corollary is Lemma 7.1 applied to the classical identity $T_{\mathrm{IM}(q)}(1-x) = \frac{1-qx}{1-x}$ (3.3), while the theorem is the lemma applied to the identities

$$T_{a^{\pm}_{r+1}(q)}(1-x) = \mathrm{FGL}^{\pm}_{r+1}(q,x),$$
$$a^{\pm}_{r+1}(q,n) = \frac{1}{n}\sum_{d|n}(\pm 1)^d\mu(n/d)(q^d - (\pm 1)^d)^r$$

found below Theorem 1.4 or below [21, Theorem 1.7], and to the power series identities of Corollary 6.2.

Corollary 7.2 *For any rational number m,*

$$1 + \sum_{n\geq 1}\Big[\sum_{\lambda\vdash n}\prod_{d\in B(\lambda)}\binom{m\,\mathrm{IM}_d(q)}{-E(\lambda,d)}\Big]x^n = \left(\frac{1-qx}{1-x}\right)^m$$

Theorem 7.3 *For any rational number m and natural number $r \geq 0$*

$$1 + \sum_{n\geq 1}\Big[\sum_{\lambda\vdash n}\prod_{d\in B(\lambda)}\binom{ma^{\pm}_{r+1}(q,d)}{-E(\lambda,d)}\Big]x^n = \mathrm{FGL}^{\pm}_{r+1}(q,x)^m$$

$$\Big(1 + \sum_{n^-\geq 1}\Big[\sum_{\lambda^-\vdash n^-}\prod_{d^-\in B(\lambda^-)}\binom{m\,\mathrm{SDIM}^-_{d^-}(q)}{-E(\lambda^-,d^-)}\Big]x^{n^-}\Big)$$
$$\Big(1 + \Big[\sum_{\lambda^+\vdash n^+}\prod_{d^+\in B(\lambda^+)}\binom{m\,\mathrm{SDIM}^+_{d^+}(q)}{-E(\lambda^+,d^+)}\Big]x^{2n^+}\Big) = \Big(\frac{1-qx}{1+x}\Big)^m$$

$$\Big(1 + \sum_{n^-\geq 1}\Big[\sum_{\lambda^-\vdash n^-}\prod_{d^-\in B(\lambda^-)}\binom{m\,\mathrm{SDIM}^-_{d^-}(q)}{-E(\lambda^-,d^-)}\Big]x^{n^-}\Big)$$
$$\Big(1 + \Big[\sum_{\lambda^+\vdash n^+}\prod_{d^+\in B(\lambda^+)}\binom{-m\,\mathrm{SDIM}^-_{d^+}(q)}{E(\lambda^+,d^+)}\Big]x^{n^+}\Big) = \Big(\frac{(1-qx)(1-x)}{(1+x)(1+qx)}\Big)^m$$

Thévenaz' polynomial identities for partitions [28], Theorem A, Theorem B] are the cases $m = \pm 1$ of Corollary 7.2. The purely combinatorial proof of a generalised version of Thévenaz' polynomial identities presented here may qualify as an answer to question (1) on p. 129 of [28]. Corollary 7.2 is the special case $r = 1$ of the first equation of Theorem 7.3 as $a_2^+(q, d) = \mathrm{IM}_d(q)$.

These polynomial identities are examples of Corollary 7.2 and Theorem 7.3 at $n = 3$

$$
\begin{aligned}
&\binom{m\,\mathrm{IM}_3(q)}{-1} + \binom{m\,\mathrm{IM}_2(q)}{-1}\binom{m\,\mathrm{IM}_1(q)}{-1} + \binom{m\,\mathrm{IM}_1(q)}{-3} \\
&\quad = \begin{cases} (q-1)q^2 & m = -1 \\ \frac{1}{16}(q-1)(5q^2+2q+1) & m = -\frac{1}{2} \\ \frac{1}{16}(1-q)(q^2+2q+5) & m = \frac{1}{2} \\ 1-q & m = 1 \end{cases} \\
&\binom{m\,\mathrm{SDIM}_3^-(q)}{-1} + \binom{m\,\mathrm{SDIM}_1^-(q)}{-3} + \binom{m\,\mathrm{SDIM}_1^-(q)}{-1}\binom{m\,\mathrm{SDIM}_1^+(q)}{-1} \\
&\quad = \begin{cases} -q^3-q^2 & m = -1 \\ \frac{1}{16}(q+1)(5q^2-2q+1) & m = -\frac{1}{2} \\ -\frac{1}{16}(q+1)(q^2-2q+5) & m = \frac{1}{2} \\ 1+q & m = +1 \end{cases} \\
&\binom{m\,\mathrm{SDIM}_3^-(q)}{-1} + \binom{m\,\mathrm{SDIM}_1^-(q)}{-3} + \binom{m\,\mathrm{SDIM}_1^-(q)}{-2}\binom{-m\,\mathrm{SDIM}_1^-(q)}{1} \\
&\quad + \binom{m\,\mathrm{SDIM}_1^-(q)}{-1}\binom{-m\,\mathrm{SDIM}_1^-(q)}{2} \\
&\quad\quad + \binom{-m\,\mathrm{SDIM}_3^-(q)}{1} + \binom{-m\,\mathrm{SDIM}_1^-(q)}{3} \\
&\quad = \begin{cases} \frac{1}{2}(q^3+q^2+q+1) & m = -\frac{1}{2} \\ 2(q+1)(q^2+q+1) & m = -1 \\ 4(q+1)(3q^2+5q+3) & m = -2 \end{cases}
\end{aligned}
$$

The terms on the left side correspond to the three partitions $\{3^1\}, \{2^1 1^1\}, \{1^3\}$, of 3 in the first, to (n^-, n^+) in $\{(3, 0), (1, 1)\}$ in the second, and to (n^-, n^+) in $\{(3, 0), (2, 1), (1, 2), (0, 3)\}$ in the third example.

8 Primary equivariant reduced Euler characteristics

Let p be a prime and, as in the previous sections, q a prime power. (The prime p may or may not divide the prime power q.) In this section we discuss the p-primary equivariant reduced Euler characteristics of the $\mathrm{GL}_n^-(\mathbf{F}_q)$-poset $\mathrm{L}_n^-(\mathbf{F}_q)^*$.

Definition 8.1 [27, (1-5)] The rth p-primary equivariant reduced Euler characteristic of the $\mathrm{GL}_n^-(\mathbf{F}_q)$-poset $\mathrm{L}_n^-(\mathbf{F}_q)^*$ is the normalised sum

$$\widetilde{\chi}_r(p, \mathrm{GL}_n^-(\mathbf{F}_q)) = \frac{1}{|\,\mathrm{GL}_n^-(\mathbf{F}_q)|} \sum_{X \in \mathrm{Hom}(\mathbf{Z}\times\mathbf{Z}_p^{r-1}, \mathrm{GL}_n^-(\mathbf{F}_q))} \widetilde{\chi}(C_{\mathrm{L}_n^-(\mathbf{F}_q)^*}(X(\mathbf{Z}\times\mathbf{Z}_p^{r-1})))$$

of reduced Euler characteristics of fixed sub-posets.

In this definition, $\mathbf{Z}_p$ denotes the ring of p-adic integers and the sum ranges over all homomorphisms of $\mathbf{Z}\times\mathbf{Z}_p^{r-1}$ into $\mathrm{GL}_n^-(\mathbf{F}_q)$ or, equivalently, over all commuting r-tuples $(X_1, X_2, \dots, X_r)$ of elements of $\mathrm{GL}_n^-(\mathbf{F}_q)$ where $X_2, \dots, X_r$ have p-power order. The *first* p-primary equivariant reduced Euler characteristic is independent of p and agrees with the first equivariant reduced Euler characteristic. If p divides q, then $\widetilde{\chi}_r(p, \mathrm{GL}_n^-(\mathbf{F}_q)) = 0$ for all $r, n > 1$ by Lemma 5.2.

The rth p-primary equivariant *unreduced* Euler characteristic $\chi_r(p, \mathrm{GL}_n^-(\mathbf{F}_q))$, obtained by replacing the reduced Euler characteristics with Euler characteristics in Definition 8.1, agrees with the Euler characteristic computed in Morava $K(r)$-theory at p of the homotopy orbit space $\mathrm{BL}_n^-(\mathbf{F}_q)^*_{h\mathrm{GL}_n^-(\mathbf{F}_q)}$ for the action of $\mathrm{GL}_n^-(\mathbf{F}_q)$ on the classifying space for the poset $\mathrm{L}_n^-(\mathbf{F}_q)^*$ [10], [27, 2-3, 5-1] , [20, Remark 7.2].

The rth p-primary generating function at q is the integral power series

$$\mathrm{FGL}_r^-(p, q, x) = 1 - \sum_{n\geq 1} \widetilde{\chi}_r(p, \mathrm{GL}_n^-(\mathbf{F}_q))x^n \in \mathbf{Z}[[x]] \tag{8.1}$$

associated to the sequence $(-\widetilde{\chi}_r(p, \mathrm{GL}_n^-(\mathbf{F}_q)))_{n\geq 1}$ of the *negative* of the p-primary equivariant reduced Euler characteristics. For $r = 1$, $\mathrm{FGL}_1^-(p, q, x) = \mathrm{FGL}_1^-(q, x) = 1 + x$, and when $p \mid q$, $\mathrm{FGL}_r^-(p, q, x) = 1 + x$ for all $r \geq 1$. The interesting case is when the characteristic of $\mathbf{F}_q$ is different from p.

Definition 8.2 For every integer $d \geq 1$,

- $\mathrm{IM}_d(p, q)$ is the number of p-power order Irreducible Monic polynomials of degree d over $\mathbf{F}_q$ with nonzero constant term
- $\mathrm{SDIM}_d^-(p, q)$ is the number of p-power order Self-Dual Irreducible Monic polynomials of degree d over $\mathbf{F}_{q^2}$ with nonzero constant term
- $\mathrm{SDIM}_d^+(p, q) = \frac{1}{2}(\mathrm{IM}_d(p, q^2) - \mathrm{SDIM}_d^-(p, q))$ is the number of unordered pairs of p-power order non-self-dual irreducible monic polynomials of degree d over $\mathbf{F}_{q^2}$ with nonzero constant term

The next lemma follows from Lemma 3.8 combined with the fact from [16, Lemma 3.6] that $x^a - 1$ divides $x^b - 1$ in $\mathbf{F}_{q^2}[x]$ if and only if a divides b.

Lemma 8.3 *Assume $p \nmid q$ and let $m \geq 1$ be an odd integer.*

(1) The p-power order self-dual irreducible monic polynomials of degree dividing m are precisely the irreducible factors of $x^{(q^m+1)_p} - 1 \in \mathbf{F}_{q^2}[x]$.

(2) $\sum_{d|m} d\,\mathrm{SDIM}_d^-(p, q) = (q^m + 1)_p$ and $m\,\mathrm{SDIM}_m^-(p, q) = \sum_{d|m} \mu(m/d)(q^d + 1)_p$.

The p-primary version of Lemma 5.3 states that for $p \nmid q$, $r \geq 1$, and $n > 1$,

$$\widetilde{\chi}_{r+1}(p, \mathrm{GL}_n^-(\mathbf{F}_q)) = \sum_{[g] \in [\mathrm{GL}_n^-(\mathbf{F}_q)_p]} \widetilde{\chi}_r(p, C_{\mathrm{L}_n^-(\mathbf{F}_q)^*}(g), C_{\mathrm{GL}_n^-(\mathbf{F}_q)}(g))$$

where the sum ranges over the set $[\mathrm{GL}_n^-(\mathbf{F}_q)_p]$ of conjugacy classes of p-elements. The point here is that a semisimple element of $\mathrm{GL}_n^-(\mathbf{F}_q)$ has p-power order if and only all irreducible factors of its characteristic polynomial have p-power order [21, Lemma 4.4]. In terms of generating functions we get the p-primary version

$$\mathrm{FGL}_{r+1}^-(p, q, x) = T_{\mathrm{SDIM}^-(p,q)}(\mathrm{FGL}_r^-(p, q, x)) T_{\mathrm{SDIM}^+(p,q)}(\mathrm{FGL}_r^+(p, q^2, x^2)) \tag{8.2}$$

of (6.1). In the following we prefer to work with the equivalent relation

$$\begin{aligned} a_{r+1}^-(p, q, N) = \sum_{d|N} & a_r^-(p, q^d, N/d)\, \mathrm{SDIM}_d^-(p, q) \\ &+ \sum_{2d|N} a_r^+(p, q^{2d}, N/2d)\, \mathrm{SDIM}_d^+(p, q) \end{aligned} \tag{8.3}$$

where

$$\begin{aligned} a_r^-(p, q, n) &= \frac{1}{n} \sum_{d|n} (-1)^d \mu(n/d)(q^d - (-1)^d)_p^{r-1} \\ a_r^+(p, q, n) &= \frac{1}{n} \sum_{d|n} \mu(n/d)(q^d - 1)_p^{r-1} \end{aligned}$$

To go from (8.2) to (8.3) we use the infinite product expansions

$$\begin{aligned} &\mathrm{FGL}_{r+1}^-(p, q, x) = \prod_{N \geq 1} (1 - x^N)^{a_{r+1}^-(p,q,N)} \\ &T_{\mathrm{SDIM}^-(p,q)}(\mathrm{FGL}_r^-(p, q, x)) = \prod_{d \geq 1} \mathrm{FGL}_r^-(p, q^d, x^d)^{\mathrm{SDIM}_d^-(p,q)} \\ &= \prod_{n,d \geq 1} (1 - x^{dn})^{a_r^-(p,q^d,n)\, \mathrm{SDIM}_d^-(p,q)} \\ &T_{\mathrm{SDIM}^+(p,q)}\, \mathrm{FGL}_r^+(p, q^2, x^2) = \prod_{d \geq 1} \mathrm{FGL}_r^+(p, q^{2d}, x^{2d})^{\mathrm{SDIM}_d^+(p,q)} \\ &= \prod_{n,d \geq 1} (1 - x^{2dn})^{a_r^+(p,q^{2d},n)\, \mathrm{SDIM}_d^+(p,q)} \end{aligned}$$

of the three factors in (8.2) obtained by applying [21, Lemma 3.7] to the expressions of Theorem 1.4 and [21, Theorem 1.7].

Proof of Theorem 1.4 We must show that the functions $a_r^{\pm}(p,q,n)$ satisfy recurrence relation (8.3). The right side of (8.3) multiplied by N is

$$\begin{aligned}
&\sum_{d|N} d\,\mathrm{SDIM}_d^-(p,q) \sum_{e|(N/d)} (-1)^e \mu(N/de)(q^{de}-(-1)^e)_p^{r-1} \\
&\quad + \sum_{2d|N} 2d\,\mathrm{SDIM}_d^+(p,q) \sum_{e|(N/2d)} \mu(N/2de)(q^{2de}-1)_p^{r-1} \\
&= \sum_{d|N} d\,\mathrm{SDIM}_d^-(p,q) \sum_{e|(N/d)} (-1)^e \mu(N/de)(q^{de}-(-1)^e)_p^{r-1} \\
&\quad - \sum_{2d|N} d\,\mathrm{SDIM}_d^-(p,q) \sum_{e|(N/2d)} \mu(N/2de)(q^{2de}-1)_p^{r-1} \\
&\quad + \sum_{2d|N} d\,\mathrm{IM}_d(p,q^2) \sum_{e|(N/2d)} \mu(N/2de)(q^{2de}-1)_p^{r-1}
\end{aligned}$$

When N is odd, we are left with

$$\begin{aligned}
&-\sum_{d|N} d\,\mathrm{SDIM}_d^-(p,q) \sum_{e|(N/d)} \mu(N/de)(q^{de}+1)_p^{r-1} \\
&= -\sum_{f|d_1|d_2|N} \mu(d_1/f)\mu(N/d_2)(q^f+1)_p(q^{d_2}+1)_p^{r-1} \\
&= -\sum_{f|d_2|N} \mu(N/d_2)(q^f+1)_p(q^{d_2}+1)_p^{r-1} \sum_{\{d_1:\, f|d_1|d_2\}} \mu(d_1/f) \\
&= \quad -\sum_{d|N} \mu(N/d)(q^d+1)_p^r = Na_{r+1}^-(p,q,N)
\end{aligned}$$

where we first use Lemma 8.3.(2)and next observe that the sum

$$\sum_{\{d_1:\, f|d_1|d_2\}} \mu(d_1/f) = \begin{cases} 1 & f = d_2 \\ 0 & f < d_2 \end{cases}$$

contributes only when $f = d_2$. Thus (8.3) holds under the assumption that N is odd.

When $N = 2N_1$ is even we have

$$\begin{aligned}
&\sum_{2d|N} d\,\mathrm{IM}_d(p,q^2) \sum_{e|(N/2d)} \mu(N/2de)(q^{2de}-1)_p^{r-1} \\
&= \sum_{d|N_1} d\,\mathrm{IM}_d(p,q^2) \sum_{de|N_1} \mu(N_1/de)(q^{2de}-1)_p^{r-1} \\
&= \sum_{d_1|d_2|N_1} d_1\,\mathrm{IM}_{d_1}(p,q^2)\mu(N_1/d_2)(q^{2d_2}-1)_p^{r-1} \\
&= \sum_{f|d_2|N} \mu(N/d_2)(q^{2f}-1)_p(q^{2d_2}-1)_p^{r-1} \sum_{\{d_1:\, f|d_1|d_2\}} \mu(d_1/f)
\end{aligned}$$

$$= \sum_{d_1|N_1} \mu(N_1/d_1)(q^{2d_1}-1)^r_p = \sum_{2d|N} \mu(N/2d)(q^{2d}-1)^r_p$$

which is the part of $Na^-_{r+1}(p,q,N)$ defined by the *even* divisors of N. Remember that $\mathrm{SDIM}^-_d(q)$, and then also $\mathrm{SDIM}^-_d(p,q)$, is nonzero only for odd d (Proposition 3.7). Thus the claim for even $N = 2N_1$ is

$$\begin{aligned} &-\sum_{\substack{d|N\\ d\,\mathrm{odd}}} \mu(N/d)(q^d+1)^r_p \\ &= \sum_{\substack{d_1|d_2|N\\ d_1\,\mathrm{odd}}} (-1)^{d_2/d_1} d_1\, \mathrm{SDIM}^-_{d_1}(p,q)\mu(N/d_2)(q^{d_2}-(-1)^{d_2/d_1})^{r-1}_p \\ &\quad - \sum_{\substack{d_1|d_2|N_1\\ d_1\,\mathrm{odd}}} d_1\, \mathrm{SDIM}^-_{d_1}(p,q)\mu(N_1/d_2)(q^{2d_2}-1)^{r-1}_p \end{aligned}$$

Note that for every $j = 1,\ldots,k$, where $N_2 = 2^k$ is the highest power of 2 dividing N, the part of the first sum with $2^j \parallel d_2$ is annihilated by the part of the second sum with $2^{j-1} \parallel d_2$. Thus the right hand side reduces to the part of the first sum where d_2 is odd. By the computation just done for odd N, that sum equals the left hand side. Thus (8.3) holds also for even N. □

When p does not divide q, the sequences $(\mathrm{SDIM}^{\pm}_d(p,q))_{d\geq 1}$ and hence the generating functions $\mathrm{FGL}^-_{r+1}(p,q,x)$, $r \geq 1$, depend only on the closure $\overline{\langle q\rangle}$ of the cyclic subgroup generated by q in the topological group $\mathbf{Z}_p^\times$ of p-adic units [21, Lemma 4.9]. For instance, the 2-primary power series $\mathrm{FGL}^-_{r+1}(2,q,x)$ are identical for $q = 3, 11, 19, 27, \ldots$, with $\log \mathrm{FGL}^-_{r+1}(2,3,x) = \sum_{n\geq 1}(-1)^{n+1}(4n)^r_2 x^n/n$, and the 3-primary power series $\mathrm{FGL}^-_{r+1}(3,q,x)$ are identical for $q = 2, 5, 11, 23, \ldots$ with $\log \mathrm{FGL}^-_{r+1}(3,2,x) = \sum_{n\geq 1}(-1)^{n+1}(3n)^r_3 x^n/n$ [21, Figure 3, Example 4.16].

8.1 Alternative presentations of *p*-primary equivariant reduced Euler characteristics

It is immediate from [21, Theorem 1.7] and Theorem 1.4 that there is 'Ennola duality'

$$\mathrm{FGL}^-_r(p,q,x) = \mathrm{FGL}^+_r(p,-q,(-1)^r x), \qquad r \geq 1 \tag{8.4}$$

between the p-primary generating functions for $\mathrm{GL}^{\pm}_n(\mathbf{F}_q)$.

We can now proceed exactly as in Sect. 6.1 to prove the next two propositions. In Proposition 8.5, $\widetilde{\chi}_{r+1}(p,\mathrm{GL}^{\pm}_n(\mathbf{F}_q))^{-1}$ denotes the coefficient of x^n in the reciprocal power series $\mathrm{FGL}^{\pm}_{r+1}(p,q,x)^{-1}$.

Proposition 8.4 *The p-primary equivariant Euler characteristics of the $\mathrm{GL}_n^{\pm}(\mathbf{F}_q)$-posets $\mathrm{L}_n^{\pm}(\mathbf{F}_q)^*$, $n \geq 1$, are*

$$\begin{aligned}
\widetilde{\chi}_{r+1}(p, \mathrm{GL}_n^{+}(\mathbf{F}_q)) &= \frac{1}{n!}\sum_{\lambda \vdash n}(-1)^{|\lambda|}T(\lambda)U(\lambda,q)_p^r \\
x &= \frac{(-1)^n}{|W_n|}\sum_{w\in W_n}\det(w)|T_n(\overline{\mathbf{F}}_s)_w^{F_q}|_p^r \\
&= \frac{(-1)^n}{|W_n|}\sum_{w\in W_n}\det(w)\det(q-w)_p^r \\
-\widetilde{\chi}_{r+1}(p, \mathrm{GL}_n^{-}(\mathbf{F}_q)) &= (-1)^{n(r+1)}\frac{1}{n!}\sum_{\lambda \vdash n}(-1)^{|\lambda|}T(\lambda)U(\lambda,-q)_p^r \\
&= \frac{(-1)^{\binom{n}{2}}}{|W_n|}\sum_{w\in W_n}\det(w)|T_n(\overline{\mathbf{F}}_s)_w^{F_q\sigma}|_p^r \\
&= \frac{1}{|W_n|}\sum_{w\in W_n}\det(w)\det(q+w)_p^r
\end{aligned}$$

Proposition 8.5 *The reciprocal p-primary equivariant Euler characteristics of the $\mathrm{GL}_n^{\pm}(\mathbf{F}_q)$-posets $\mathrm{L}_n^{\pm}(\mathbf{F}_q)^*$, $n \geq 1$, are*

$$\begin{aligned}
\widetilde{\chi}_{r+1}(p, \mathrm{GL}_n^{+}(\mathbf{F}_q))^{-1} &= \frac{1}{|W_n|}\sum_{w\in W_n}|T_n(\overline{\mathbf{F}}_s)_w^{F_q}|_p^r = \frac{1}{|W_n|}\sum_{w\in W_n}\det(q-w)_p^r \\
(-1)^n\widetilde{\chi}_{r+1}(p, \mathrm{GL}_n^{-}(\mathbf{F}_q))^{-1} &= \frac{1}{|W_n|}\sum_{w\in W_n}|T_n(\overline{\mathbf{F}}_s)_w^{F_q\sigma}|_p^r = \frac{1}{|W_n|}\sum_{w\in W_n}\det(q+w)_p^r
\end{aligned}$$

A slight modification of [6, Proposition 3.7.4] shows that $(\pm 1)^n\widetilde{\chi}_2(p, \mathrm{GL}_n^{\pm}(\mathbf{F}_q))^{-1}$ equals the number of semisimple p-classes in $\mathrm{GL}_n^{\pm}(\mathbf{F}_q)$.

The next corollary, the p-primary version of Corollary 6.7, is an immediate consequence of Proposition 8.4 and it specifies the generating functions for the p-primary equivariant Euler characteristics expanded after the parameter r and with fixed n.

Corollary 8.6 *For any fixed $n \geq 1$,*

$$\begin{aligned}
\sum_{r\geq 0}\widetilde{\chi}_{r+1}(p, \mathrm{GL}_n^{+}(\mathbf{F}_q))x^r &= \frac{1}{n!}\sum_{\lambda\vdash n}\frac{(-1)^{|\lambda|}T(\lambda)}{1-U(\lambda,q)_p x} \\
&= \frac{(-1)^n}{|W_n|}\sum_{w\in W_n}\frac{\det(w)}{1-x\det(q-w)_p} \\
\sum_{r\geq 0}-\widetilde{\chi}_{r+1}(p, \mathrm{GL}_n^{-}(\mathbf{F}_q))x^r &= \frac{(-1)^n}{n!}\sum_{\lambda\vdash n}\frac{(-1)^{|\lambda|}T(\lambda)}{1-(-1)^n U(\lambda,-q)_p x}
\end{aligned}$$

$$= \frac{1}{|W_n|} \sum_{w \in W_n} \frac{\det(w)}{1 - x\det(q+w)_p}$$

For example, when $n = 3$, $p = 2$, and $q = 3, 11, 19, 27, \ldots$ is any prime power with $q \equiv 3 \bmod 8$, $(q^2-1)_2 = (3^2-1)_2$, the generating function (times 3!) for the 2-primary equivariant reduced Euler characteristics of $\mathrm{GL}_3^-(\mathbf{F}_q)$ is

$$3! \sum_{r \geq 0} -\widetilde{\chi}_{r+1}(2, \mathrm{GL}_3^-(\mathbf{F}_q))x^r = \frac{1}{1 - x(q+1)_2^3} - \frac{3}{1 - x(q^2-1)_2(q+1)_2}$$
$$+ \frac{2}{1 - x(q^3+1)_2} = \frac{1}{1-64x} + \frac{2}{1-4x} - \frac{3}{1-32x}$$

with the three terms corresponding to the three partitions $\{1^3\}$, $\{1^1 2^1\}$, $\{3^1\}$ of 3.

Acknowledgements I wish to thank Lars Halvard Halle for pointing out the connection to Hasse–Weil zeta-functions, and Jean Michel for a helpful remark, used in Sect. 6.1, that, to him but not to me, was completely obvious.

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